

On the General Theory of Collisions for Particles with Spin¹

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The general analysis of binary reactions involving particles with arbitrary spin is reformulated in such a way, that it applies equally well to relativistic particles (including photons). This is achieved by using longitudinal spin components (“helicity states”) not only in the initial and final states, but also in the angular momentum states which are employed as usual to reduce the S -matrix to a simpler form. Expressions for the scattering and reaction-amplitude, intensity and polarization are given. They involve fewer vector-addition coefficients than the customary formulas, and no recoupling coefficients. The application to some examples is sketched, and in the Appendix some formulas are given that may be of use in the applications.

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I. INTRODUCTION

The general theory of collisions of the type $a + b \rightarrow c + d$ for polarized particles has been discussed several times [1–5], but until recently not much attention has been paid to the relativistic features of the problem. At first sight, the principles employed in the customary discussion (unitarity, conservation laws, composition law for angular momenta) are so general that nothing further has to be said. The relativistic description of the spin-states of a particle, however, involves certain well-known complications, which must be handled carefully. This has been done by Stapp [6] for collisions between spin-1/2 particles and by Chao and Shirokov [7]³ for particles of arbitrary spin. In either case, the authors assume that the particles concerned have a nonvanishing rest-mass and then show that for suitably defined (but somewhat complicated) spin operators the customary analysis of the initial (or final) state in states of definite orbital momentum L and resultant spin S can be carried out in the usual manner, the only departure from the nonrelativistic case being in a somewhat subtle difference in the interpretation of the spin direction of a particle. We shall examine here the same problem by a different method, which

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in our opinion is somewhat simpler. This method also has the advantage that it applies equally well to massless particles, e.g., photons. The interpretation of the spin direction for a massive particle is, however, the same as in the method of Chao and Shirokov.⁴

In the customary treatment, one uses the component of the spin along a fixed direction, say the z -axis, to classify the possible polarized states of a particle. The states used in our treatment are labeled with the component of the spin along the direction of motion of the particle, which may be called the helicity quantum number (λ) or briefly helicity. Thus we avoid the problem of the separation of the angular momentum operator into a spin- and an orbital-part (λ may also be defined as the component, in the direction of motion, of the total angular momentum of the particle) which leads to complications in the relativistic case. In particular, λ has a well-defined meaning also for massless particles.

Independent of any relativistic considerations, λ as a quantum number has also another convenient property; it is invariant under ordinary rotations, so that it is possible to construct states of definite angular momentum J , in which all particles involved have definite helicities. Thus in describing a reaction $a + b \rightarrow c + d$ in a center-of-mass frame, we may use $E, J, M (=J_z)$ together with the helicities λ_a, λ_b as quantum numbers for the initial state, and similarly $E, J, M, \lambda_c, \lambda_d$ for the final state. E is of course the total energy. The S -matrix for the process has then the form

$$\langle E' J' M' \lambda_c \lambda_d | S | E J M \lambda_a \lambda_b \rangle = \delta(E - E') \delta_{JJ'} \delta_{MM'} \langle \lambda_c \lambda_d | S^J(E) | \lambda_a \lambda_b \rangle. \quad (1)$$

The reduction of the S -matrix to submatrices $S^J(E)$, belonging to definite values of E and J , is of course an essential step in any general analysis of a collision. In the ordinary scheme, the diagonalization of J and J_z is achieved by successive addition of the spin vectors and relative orbital momentum vector. As a result an element of the submatrix $S^J(E)$ is labeled

$$\langle L' S' | S^J | L S \rangle, \quad (2)$$

where LS ($L'S'$) are orbital and total spin quantum-numbers for the initial (final) state. Since the total spin S (S') is not a suitable quantity to describe any possible polarization measurements on the initial (final) state, the ordinary scheme must shuttle back and forth between two different representations, one being the representation in which the simple form (2) obtains, and the other being a representation in which the states are labeled with individual spin components. Our scheme avoids this complication, since the helicity quantum numbers appearing in Eq. (1) are also directly related to individual polarization properties of the particles. This leads to a certain simplicity and neatness of the formulae for the scattering and reaction amplitudes, which may well constitute a practical advantage over the conventional formulae.

⁴ Note added in proof. We have recently received a copy of a paper by Chou Kuang-Chao [*J. Exp. Theoret. Phys.* (U.S.S.R.) **36**, 909 (1959)] in which a treatment is given which applies when one of the incident particles has zero mass.

The general plan of the paper is as follows. In Section 2, we give some explanations about helicity states and in particular about the conventions adopted about the relative phases of the various spin components. These conventions, of course, become important in the calculation of transverse polarizations. The reader who is not too interested in mathematical detail may, however, skip the first part of this section entirely. Next we give the transformation formulas Eqs. (18) and (24) from plane-wave states to states of definite J and J_z ($=M$). In Section 3 we then introduce the states just described into the general S -matrix formalism. The main result is the formula (31) for the scattering- or reaction-amplitude in the c.m. system. Like the formulas usually given,⁵ Eq. (31) is a simple and obvious generalization of the elementary scattering formula for spinless particles

$$f(\theta) = (1/2ip) \sum_J (2J+1)(S^J - 1) P_J(\cos \theta). \quad (3)$$

Our formula, however, contains no Clebsch–Gordan coefficients and a single summation index, instead of the five of the usual formula. It may be argued, of course, that some of the complexity is hidden in the presence of the functions $d_{mm'}^J(\theta)$ in place of the more elementary $P_{Jm}(\theta)$. Some of the d^J -functions most likely to occur are tabulated in the Appendix for greater convenience. Actually, the appearance of these functions in our formulas has a simple intuitive meaning. As is well known, the function $d_{mm'}^J(\theta)$ can be regarded as the wave function of a symmetrical top, m being the component of angular momentum along a fixed (“ z ”) axis and m' being the component along the symmetry (“ z' ”) axis of the top. The presence of a resultant spin-component $\mu = \lambda_c - \lambda_d$ along the direction of motion of the two final particles relative to each other imparts to the two particles properties somewhat similar to those of a symmetrical top spinning about its symmetry axis. The identification $m' = \mu$ is therefore obvious. Similarly $\lambda = \lambda_a - \lambda_b = m$ follows from the fact that the J_z value of the outgoing wave is determined by the initial resultant spin along the z -axis.

In Section 4, we give some general intensity and polarization formulas derived from Eq. (31). The calculation is no more difficult than in the usual case; in fact the formulas now contain some Clebsch–Gordan, but no recoupling, coefficients. The point is that this slight gain in simplicity is not only attained at no cost, but on the contrary while achieving a somewhat wider range of application of the formulas.

The simplifications arising from parity conservation and other symmetry considerations are discussed in Section 5. Finally, in Section 6 we consider some examples.

II. ANGULAR MOMENTUM STATES

In order to describe the states of a free particle of arbitrary spin s and mass m it is fortunately unnecessary, for our purposes, to write down an explicit relativistic

⁵ See for example Ref. [3, Eq. (3.14)].

wave equation. It is instead sufficient [8] to know that such equations exist, and that their plane-wave solutions, representing states of definite linear momentum p and corresponding positive energy $\omega = (m^2 + p^2)^{1/2}$, have the following properties:

(a) for each p , and assuming $m \neq 0$, there are $2s + 1$ linearly independent solutions (states), which may be chosen to be solutions of definite helicity:

$$\lambda = s, s - 1, \dots, -s. \quad (4)$$

If $m = 0$, and if the wave equation is "irreducible," the independent solutions are only two, corresponding to

$$\lambda = \pm s, \quad (4')$$

e.g., for a photon $\lambda = \pm 1$.

(b) When the transformation corresponding to an ordinary rotation of the xyz system of axes is applied to one of the above solutions, one obtains a state with a different *direction* of p , but λ remains unchanged.

(b') When a space reflection is applied, λ changes sign.

(c) When a Lorentz-transformation in the direction of p is applied, one obtains a state with the same (or the opposite) *direction* but a different *magnitude* of p . Assuming the direction of p is *not* reversed, again λ remains unchanged.

(d) The states referred to under (a), and characterized by p and λ , form a complete orthogonal set of states for a free particle. The transformations induced within the set by rotations, reflections and proper Lorentz transformations are represented by certain operators or matrices, which must obey well-known commutation relations; the explicit form of these operators needs not be specified here, except that certain specifications are implied in the statements (b), (b'), and (c).

A minor complication involved in working without an explicit wave equation is that the relative phases of the basic states defined above have to be specified by some special convention. This is easily done, however. In the first place, according to statement (b), the states with momentum \mathbf{p}' in an arbitrary direction specified by polar angles θ, ϕ , may be *defined* by means of a suitable rotation applied to states $\psi_{p\lambda}$ having a momentum \mathbf{p} in the positive z -direction. The rotation is conveniently defined to be a rotation through an angle θ in the positive direction about the axis $\mathbf{p} \times \mathbf{p}'$. If J_x, J_y, J_z are the angular momentum operators for the particle, a finite rotation with Euler angles α, β, γ is given by the operator

$$R_{\alpha\beta\gamma} = e^{-i\alpha J_x} e^{-i\beta J_y} e^{-i\gamma J_z}. \quad (5)$$

One easily sees that the state with momentum \mathbf{p}' ($|\mathbf{p}'| = |\mathbf{p}| \equiv p$) and helicity λ as defined above is given by

$$|p\theta\phi; \lambda\rangle = R_{\phi, \theta, -\phi} \psi_{p\lambda} = e^{i\lambda\phi} R_{\phi\theta 0} \psi_{p\lambda}. \quad (6)$$

We have now only to specify the relative phases of the states $\psi_{p\lambda}$ with momentum in the positive z -direction. Owing to (c) and in analogy to Eq. (6) we may generate all the states $\psi_{p\lambda}$ with a fixed λ and variable p by applying Lorentz transformations in the z -direction ($x' = x$, $y' = y$, $z' = z - \beta t$, $t' = t - \beta z$) to a fixed state $\psi_{p_0\lambda}$. If $m \neq 0$, we may moreover go to the limit $p_0 = 0$; even in this limit the quantization axis for angular momentum remains, of course, the z -axis. Since for a particle at rest the angular momentum is equal to the spin we may specify the relative phases of the states $\psi_{0\lambda}$ by the requirement

$$(J_z \pm iJ_y) \psi_{0\lambda} = [(s \mp \lambda)(s \pm \lambda + 1)]^{1/2} \psi_{0\lambda \pm 1} \quad (7)$$

meaning that in this case $J_z J_y J_z$ reduce to the standard spin matrices. For a massless particle, no finite Lorentz transformation can reduce p_0 to zero, but we have only two values of λ , Eq. (4'), to compare. Owing to (b') this may be done by means of a reflection. If for example P is the "parity" operator, corresponding to the reflection in the origin ($xyz \rightarrow -x, -y, -z$), then the reflection in the xz plane

$$Y = e^{-i\pi J_y} P \quad (8)$$

transforms ψ_{ps} into $\psi_{p, -s}$ apart from a phase factor η

$$Y\psi_{p,s} = \eta\psi_{p,-s}. \quad (9)$$

Since Y commutes with a Lorentz transformation in the z -direction, it is easy to see that owing to our previous conventions, η is independent of p . It is therefore a constant which we shall call the "parity factor" of the particle. For example it is customary to define the $\lambda = \pm 1$ solutions for a photon as having a vector potential $A = \mp 2^{-1/2}(e_x \pm ie_y) \exp(ipz)$. Furthermore Y applied to A reverses the sign of A_y . One sees that with these conventions $\eta = -1$.

We may further check that Eq. (9) is compatible with the conventions (7) for $m \neq 0$. In this case we know that P transforms $\psi_{0\lambda}$ into itself apart from a phase-factor which must be independent of λ (P commutes with J). Hence we write

$$P\psi_{0\lambda} = \eta\psi_{0\lambda}. \quad (10)$$

Furthermore

$$e^{-i\pi J_y} \psi_{0\lambda} = \sum_{\lambda'} d_{\lambda'\lambda}^s(\pi) \psi_{0\lambda'}, \quad (11)$$

where the matrix $d_{\lambda'\lambda}(\beta)$ is well known. In particular

$$d_{\lambda'\lambda}^s(\pi) = (-1)^{s-\lambda} \delta_{\lambda', -\lambda}. \quad (12)$$

Combining (10) and (11) and applying a Lorentz transformation in the z -direction on both sides we get

$$Y\psi_{p\lambda} = \eta(-1)^{s-\lambda} \psi_{p, -\lambda}, \quad (9')$$

which for $\lambda = s$ reduces to (9). Equations (9) (9') will turn out to be extremely useful later.

We now go over to the description of the states of two free particles 1 and 2 with masses m_1 and m_2 and spins s_1 and s_2 . These states may, of course, be constructed as direct products

$$R_{\phi_1\theta_1-\phi_1}^{(1)} \psi_{p_1\lambda_1}(1) R_{\phi_2\theta_2-\phi_2}^{(2)} \psi_{p_2\lambda_2}(2)$$

of individual states of the above described type for particles 1 and 2. We are, however, mostly interested in states of zero total linear momentum, for which say $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}$ (direction θ, ϕ). Then $\theta_1 = \theta, \phi_1 = \phi$ and $\theta_2 = \pi - \theta, \phi_2 = \phi \pm \pi$ and the two rotations $R^{(1)}$ and $R^{(2)}$ may be replaced by a single rotation R involving the total angular momentum $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ of the two particles. To this end we first define for particle 2, states $\chi_{p\lambda_2}$ with momentum in the negative z -direction as follows:

$$\chi_{p\lambda_2} = (-1)^{s_2-\lambda_2} e^{-i\pi J_y^{(2)}} \psi_{p\lambda_2}. \quad (13)$$

The phase factor in front is not really necessary; it is introduced for convenience in such a way that for a particle at rest ($p=0$), $\chi_{0\lambda}$ reduces simply to $\psi_{0, -\lambda}$ (the change of sign of the helicity may be understood if one assumes that the direction of p is reversed *before* going to the limit $p=0$). We then define a product state for the two particles

$$\psi_{p\lambda_1\lambda_2} = \psi_{p\lambda_1}(1) \chi_{p\lambda_2}(2), \quad (14)$$

in which state the relative momentum p is in the positive z -direction. States with other directions of p may be obtained, as in Eq. (6), by applying a rotation

$$|p\theta\phi; \lambda_1\lambda_2\rangle = R_{\phi, \theta, -\phi} \psi_{p\lambda_1\lambda_2} = e^{i\lambda\phi} R_{\phi, \theta, 0} \psi_{p, \lambda_1\lambda_2}, \quad (15)$$

where $\lambda = \lambda_1 - \lambda_2$ is the resultant angular momentum of the two-particle system in the direction (θ, ϕ) of the relative momentum.

It is now easy to construct states with zero linear momentum and definite total angular momentum J and component $J_z = M$. Since the magnitude p of the relative momentum and the helicities λ_1 and λ_2 are invariant against rotations, one can assign definite values to them, together with J and M . Let $|p; JM; \lambda_1\lambda_2\rangle$ be the ket-symbol for such a state. The prescription we employ is well known [9, 10]; introducing a suitable normalization factor \mathfrak{N} , we write

$$|p; JM; \lambda_1\lambda_2\rangle = \frac{\mathfrak{N}}{2\pi} \int dU \mathcal{D}_{M, \lambda}^*(\alpha\beta\gamma) R_{\alpha\beta\gamma} \psi_{p\lambda_1\lambda_2}, \quad (16)$$

$$dU = \sin \beta \, d\alpha \, d\beta \, d\gamma,$$

where the integral extends to the region⁶ $0 < \alpha < 2\pi$, $0 < \beta < \pi$; $0 < \gamma < 2\pi$ and where the star means complex conjugation. Furthermore

$$\mathcal{D}_{MM'}(\alpha\beta\gamma) = e^{-iM\alpha} d_{MM'}^J(\beta) e^{-iM'\gamma} \quad (17)$$

is the matrix corresponding to $R_{\alpha\beta\gamma}$ in the irreducible representation \mathcal{D}^J . Alternatively $\mathcal{D}_{MM'}^*$ may be described as the wave function of a symmetrical top, whereby M' is the angular momentum of the top with respect to the symmetry axis. Two particles moving in opposite directions and spinning with a resultant angular momentum $\lambda = \lambda_1 - \lambda_2$ about the direction of their relative momentum p , present certain kinematical features in common with a symmetrical top, the symmetry axis being of course replaced by p . Therefore the appearance of $\mathcal{D}_{M\lambda}$ in Eq. (16) is hardly surprising.

The dependence of the integrand in (16) on γ is simply a factor $\exp i \times (\lambda - \lambda_1 + \lambda_2) \gamma$. The integration over γ is therefore trivial (hence also the condition $M' = \lambda \equiv \lambda_1 - \lambda_2$). Remembering (15) we can write

$$|p; JM; \lambda_1 \lambda_2\rangle = \mathfrak{R} \int \mathcal{D}_{M\lambda}^*(\phi, \theta, -\phi) |p\theta\phi; \lambda_1 \lambda_2\rangle d\Omega, \quad (18)$$

where $d\Omega = \sin \theta d\theta d\phi$. This exhibits our angular momentum states as superpositions of the plane-wave states (15).

Let us now examine some questions of normalization. For a product state $|\mathbf{p}_1 \mathbf{p}_2 \lambda_1 \lambda_2\rangle$ with independent momenta for the two particles, we assume at first a conventional normalization

$$\langle \mathbf{p}_1' \mathbf{p}_2' \lambda_1' \lambda_2' | \mathbf{p}_1 \mathbf{p}_2 \lambda_1 \lambda_2 \rangle = (2\pi)^6 \delta(\mathbf{p}_1 - \mathbf{p}_1') \delta(\mathbf{p}_2 - \mathbf{p}_2') \delta_{\lambda_1 \lambda_1'} \delta_{\lambda_2 \lambda_2'} \quad (19)$$

then introduce as new variables the total energy-momentum four-vector

$$P_\mu = (\mathbf{P}, P_0),$$

with components $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$, $P_0 = \omega_1 + \omega_2$, and two polar variables θ, ϕ to specify the direction⁷ of the relative momentum $\mathbf{p} = (m_1 + m_2)^{-1} (m_2 \mathbf{p}_1 - m_1 \mathbf{p}_2)$. Then on the right-hand side of Eq. (19), and more generally in the matrix-element of any operator A which commutes with the four components of P_μ , we can separate out a "center-of-mass factor"

$$\delta_4(P_\mu - P_\mu') \equiv \delta_3(\mathbf{P} - \mathbf{P}') \delta(P_0 - P_0').$$

⁶ The reader can easily verify that this region of integration is adequate also in the case of a two-valued representation.

⁷ The magnitude of \mathbf{p} is, of course, in general a function of θ, ϕ , and P_μ , and in particular, in a center-of-mass frame, a function of P_0 . In the following, p and v will always stand for the magnitudes of relative momentum and velocity, both measured in a c.m. frame. Therefore

$$v = p[(p^2 + m_1^2)^{-1/2} + (p^2 + m_2^2)^{-1/2}].$$

Introducing some additional normalization factors for convenience, we shall write

$$\langle \mathbf{p}_1' \mathbf{p}_2' \lambda_1' \lambda_2' | A | \mathbf{p}_1 \mathbf{p}_2 \lambda_1 \lambda_2 \rangle = (2\pi)^4 \delta_4(P_\mu - P_\mu') \cdot (2\pi)^2 v p^{-2} \\ \times \langle \theta' \phi' \lambda_1' \lambda_2' | A(P_\mu) | \theta \phi \lambda_1 \lambda_2 \rangle, \quad (20)$$

where $A(P_\mu)$ represents, of course, the submatrix of A belonging to a given four-momentum P_μ . The left-hand side of (20) reduces to (19) when A is the unit operator, in which case $A \equiv 1$ and $A(P_\mu)$ may be omitted on the two sides. The additional normalization factors are chosen in such a way that in the special case $\mathbf{P} = 0$ (center-of-mass frame of reference) one has, as one easily verifies,

$$\langle \theta' \phi' \lambda_1' \lambda_2' | \theta \phi \lambda_1 \lambda_2 \rangle = \delta_2(\theta \phi; \theta' \phi') \delta_{\lambda_1 \lambda_1'} \delta_{\lambda_2 \lambda_2'}, \quad (21)$$

where δ_2 is the two-dimensional δ -function on the unit sphere:

$$\delta(\cos \theta - \cos \theta') \delta(\phi - \phi').$$

With a slight modification (see the discussion preceding Eq. (15)) amounting only to a phase factor, which does not affect the normalization, the ket-symbol $|\theta \phi \lambda_1 \lambda_2\rangle$ of Eqs. (20) and (21) may be identified with the states of Eqs. (15) (18). If then Eq. (21) is assumed for the normalization, and if we choose

$$\mathfrak{N}_J = [(2J + 1)/4\pi]^{1/2} \quad (22)$$

and pay attention to the orthogonality relations

$$\int_0^\pi d^j_{m\mu}(\beta) d^j_{m\mu'}(\beta) \sin \beta d\beta = \delta_{jj'} 2/(2j + 1), \quad (23a)$$

$$\frac{1}{2} \sum_j (2j + 1) d^j_{m\mu}(\beta) d^j_{m\mu'}(\beta') = \delta(\cos \beta - \cos \beta'), \quad (23b)$$

we easily find that the angular momentum states Eq. (18) are normalized in a standard fashion

$$\langle J' M' \lambda_1' \lambda_2' | J M \lambda_1 \lambda_2 \rangle = \delta_{JJ'} \delta_{MM'} \delta_{\lambda_1 \lambda_1'} \delta_{\lambda_2 \lambda_2'}$$

and moreover that the transformation matrix

$$\langle \theta \phi; \lambda_1 \lambda_2 | J M; \lambda_1' \lambda_2' \rangle = \mathfrak{N}_J \delta_{\lambda_1 \lambda_1'} \delta_{\lambda_2 \lambda_2'} \mathcal{D}_{M\lambda}^*(\phi, \theta, -\phi) \quad (24)$$

satisfies the unitarity conditions

$$\int d\Omega \langle \theta \phi \lambda_1 \lambda_2 | J M \lambda_1 \lambda_2 \rangle \langle \theta \phi \lambda_1 \lambda_2 | J' M' \lambda_1 \lambda_2 \rangle^* = \delta_{JJ'} \delta_{MM'}, \quad (25)$$

$$\sum_{JM} \langle \theta \phi \lambda_1 \lambda_2 | J M \lambda_1 \lambda_2 \rangle \langle \theta' \phi' \lambda_1 \lambda_2 | J M \lambda_1 \lambda_2 \rangle^* = \delta_2(\theta \phi; \theta' \phi').$$

These conditions are, of course, useful in the discussion of the S -matrix, to which we now turn our attention.

III. S -MATRIX

An element of the S -matrix for the reaction $a + b \rightarrow c + d$ can be designated by

$$\langle \mathbf{p}_c \mathbf{p}_d; \lambda_c \lambda_d | S | \mathbf{p}_a \mathbf{p}_b; \lambda_a \lambda_b \rangle \quad (26)$$

in terms of initial and final states normalized as in Eq. (19). Let us designate by P_μ and P'_μ the initial and final momentum four-vectors

$$P_\mu = (p_a + p_b)_\mu; \quad P'_\mu = (p_c + p_d)_\mu \quad (27)$$

and similarly by p, p', v, v' the initial and final relative momenta and velocities measured, as we have said, in a c.m. frame ($\mathbf{P} = \mathbf{P}' = 0$).

Apart from the introduction of helicity quantum numbers, the discussion proceeds along conventional lines. Owing to energy- and momentum-conservation, one can split out a "center-of-mass factor," introducing polar coordinates θ_0, ϕ_0 for the initial- and θ, ϕ for the final-relative momentum, and one can write the matrix-element (26) in analogy to Eq. (20)

$$\begin{aligned} \langle \mathbf{p}_c \mathbf{p}_d; \lambda_c \lambda_d | S | \mathbf{p}_a \mathbf{p}_b; \lambda_a \lambda_b \rangle &= (2\pi)^6 \delta_4(P_\mu - P'_\mu) (vv')^{1/2} (pp')^{-1} \\ &\quad \times \langle \theta \phi \lambda_c \lambda_d | S(P_\mu) | \theta_0 \phi_0 \lambda_a \lambda_b \rangle, \end{aligned} \quad (28)$$

whereby the normalization of the states of relative motion has been properly taken into account, in such a way that the unitarity of the S -matrix takes a simple form, which the reader can easily write down.

In practice, we are interested in the case: $\mathbf{P} = 0$ and we shall write $S(E)$ instead of $S(P_\mu)$, E being the invariant $(-P_\mu P_\mu)^{1/2}$ or the value of the total energy P_0 in a center-of-mass frame. Moreover we assume that the initial direction of relative motion is along the positive z -axis; we set then $\theta_0 = 0$ and ϕ_0 arbitrary, for example: $\phi_0 = 0$. A familiar calculation then gives for the differential cross section

$$d\sigma = (2\pi/p)^2 |\langle \theta \phi \lambda_c \lambda_d | T(E) | 00 \lambda_a \lambda_b \rangle|^2 d\Omega, \quad (29)$$

where $d\Omega = \sin \theta d\theta d\phi$ is the final solid angle in a c.m. frame and the operator T is related to S by the usual equation

$$S - 1 = iT, \quad (29')$$

where 1 is the unit operator, which, of course, contributes to the matrix element (28) only in the case of elastic scattering ($a = c, b = d$). The cross section as given refers to given values of the initial and final helicities, but can be easily generalized to the case of arbitrary polarization (see Section 4).

We now transform, as usual, to a J, M representation, which in our case may be done by means of the transformation matrix (24). We notice that, since $\theta_0 = 0$,

$$\langle JM; \lambda_a \lambda_b | \theta_0 \phi_0; \lambda_a \lambda_b \rangle = \mathfrak{R}_J \mathcal{D}_{M\lambda}^*(\phi_0, 0, -\phi_0) = \mathfrak{R}_J e^{i(M-\lambda)\phi_0} d_{M\lambda}^J(0) = \mathfrak{R}_J \delta_{M\lambda}$$

(independent of ϕ_0 !). Furthermore recalling the structure of S in the J, M representation, Eq. (1), we find, using (22) and (24)

$$\begin{aligned} \langle \theta \phi \lambda_c \lambda_d | S(E) | 00 \lambda_a \lambda_b \rangle &= \sum_{J, M} \langle \theta \phi \lambda_c \lambda_d | JM \lambda_c \lambda_d \rangle \\ &\quad \times \langle JM \lambda_c \lambda_d | S(E) | J' M' \lambda_a \lambda_b \rangle \langle J' M' \lambda_a \lambda_b | 00 \lambda_a \lambda_b \rangle \\ &= (1/4\pi) \sum_J (2J+1) \langle \lambda_c \lambda_d | S^J(E) | \lambda_a \lambda_b \rangle \mathcal{D}_{\lambda\mu}^*(\phi, \theta, -\phi), \end{aligned} \quad (30)$$

where $\lambda = \lambda_a - \lambda_b$, $\mu = \lambda_c - \lambda_d$. Inserting (30) into (29) (29') we have

$$d\sigma = |f_{\lambda_c \lambda_d; \lambda_a \lambda_b}(\theta, \phi)|^2 d\Omega;$$

$$f_{\lambda_c \lambda_d; \lambda_a \lambda_b}(\theta, \phi) = (1/p) \sum_J (J+1/2) \langle \lambda_c \lambda_d | T^J(E) | \lambda_a \lambda_b \rangle e^{i(\lambda-\mu)\phi} d_{\lambda\mu}^J(\theta); \quad (31)$$

$$(\lambda = \lambda_a - \lambda_b; \mu = \lambda_c - \lambda_d).$$

$T^J(E)$ is the same as $-iS^J(E)$ except in the case of elastic scattering ($a=c, b=d$), when in view of Eq. (29')

$$\langle \lambda_c \lambda_d | S^J(E) | \lambda_a \lambda_b \rangle - \delta_{\lambda_a \lambda_c} \delta_{\lambda_b \lambda_d} = i \langle \lambda_c \lambda_d | T^J(E) | \lambda_a \lambda_b \rangle. \quad (31')$$

IV. GENERAL INTENSITY AND POLARIZATION FORMULAS

When only small J values contribute, it may be convenient to calculate the *amplitudes* f directly, by means of the d^J functions given in the Appendix. Just as in former treatments, it is possible, however, to square the amplitudes directly, expressing the product of two d -functions by means of the Clebsch–Gordan series,

$$d_{\lambda\mu}^J d_{\lambda'\mu'}^{J'} = \sum_l C(JJ'l; \lambda, -\lambda') C(JJ'l; \mu, -\mu') (-1)^{\lambda'-\mu'} d_{\lambda-\lambda', \mu-\mu'}^l, \quad (32)$$

where we use the notation

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 jm \rangle = C(j_1 j_2 j; m_1 m_2) \delta_{m, m_1+m_2}$$

for the Clebsch–Gordan coefficients.

For example the unpolarized cross section is obtained by squaring the amplitude, summing over the final- and averaging over the initial-helicity quantum numbers. In this case one uses (32) with $\lambda = \lambda'$, $\mu = \mu'$ so that

$$d_{\lambda\mu}^l \rightarrow d_{00}^l(\theta) = P_l(\cos \theta).$$

One finds⁸

$$(d\sigma)_{\text{unpol}} = I(\theta) d\Omega;$$

$$\begin{aligned} I(\theta) = & [(2s_a + 1)(2s_b + 1) p^2]^{-1} \sum_{(\lambda)} \sum_{JJ'} (J + 1/2)(J' + 1/2)(-1)^{\lambda - \mu} \quad (33) \\ & \times \langle \lambda_c \lambda_d | T^J | \lambda_a \lambda_b \rangle^* \langle \lambda_c \lambda_d | T^{J'} | \lambda_a \lambda_b \rangle \\ & \times \sum_l C(JJ'l; \lambda, -\lambda) C(JJ'l; \mu, -\mu) P_l(\cos \theta), \end{aligned}$$

where $\sum_{(\lambda)} = \sum_{\lambda_a \lambda_b \lambda_c \lambda_d}$. In this formula as well as in subsequent ones, the statistical weight $(2s + 1)$ must be replaced by 2 for a massless particle.

In a similar way we may compute cross sections for arbitrary polarization of the particles. It is quite easy to write down "the most general cross section" involving density matrices ρ_a, \dots, ρ_d or the corresponding statistical tensors. Even in these general formulas no elimination of magnetic sums by means of recoupling of angular momenta occurs, because all the simplifications arising from rotational invariance have already been taken into account implicitly.

We shall, however, limit ourselves with a single exception to the cases of longitudinal (circular) and transverse polarization of an initial (say a) or final (say c) particle. Longitudinal polarization (total or partial) can obviously be introduced by attributing different weights to the positive and negative values of the helicity quantum number. The ensuing modification of Eq. (33) is immediate and need not be discussed in detail. In particular it is easy to see that, if parity is conserved in the reaction (see Section 5), and if the initial particles are unpolarized, then in the final state values of λ_c (or λ_d) differing only in sign are equally probable. No circular polarization is therefore produced in the reaction, as one expects.

If parity nonconserving reactions of practical interest exist, then the present formalism is obviously well suited to their discussion.

Transverse polarization is usually defined by means of the expectation value of a transverse component of the spin. As we have mentioned earlier, the definition of transverse components of the spin is somewhat arbitrary in the relativistic case. In fact, in the case of a massless particle, transverse polarization cannot be defined in

⁸ Since $I(\theta)$ and the C-G coefficients are real, the product

$$\langle \dots | T^J | \dots \rangle^* \langle \dots | T^{J'} | \dots \rangle$$

may obviously be replaced by its real part. This also follows from the symmetry property $C(JJ'l; \lambda, -\lambda) = C(J'J'l; \lambda, -\lambda)$. A corresponding simplification has been included in the later formulas (38) and (39).

this way at all. The following formulas are derived, therefore, for a particle of finite mass. Consider for example the assumption that particle a is initially in a pure state

$$\psi = \sum_{\lambda} a_{\lambda} \psi_{p\lambda} \quad (34)$$

(we write here λ instead of λ_a for simplicity). A convenient way to define the transverse polarization is to assume that the particle is first “transformed to rest” by means of a Lorentz transformation along the z -axis. That is we assume that the transverse components of the spin s_x and s_y are measured in a Lorentz system of reference, obtained from the original system xyz by the above mentioned Lorentz transformation. According to Section 1, statement (c), λ remains unchanged in the transformation, hence also the probability $|a_{\lambda}|^2$. In fact one can see that, with our phase conventions, the amplitudes a_{λ} also are unchanged. Hence with the above definition and using the known form of the nonrelativistic spin-matrices, see Eq. (7), we can write, after an obvious simplification

$$\langle s_y \rangle = \sum_{\lambda\lambda'} (s_y)_{\lambda\lambda'} a_{\lambda}^* a_{\lambda'} = \sum_{\lambda} [(s + \lambda)(s - \lambda + 1)]^{1/2} \text{Im}(a_{\lambda-1} a_{\lambda}^*), \quad (35)$$

where $\text{Im}(\dots)$ means imaginary part of \dots , and where, following usage, one can recognize a Clebsch–Gordan coefficient

$$\begin{aligned} [(s + \lambda)(s - \lambda + 1)]^{1/2} &= [2s(1 + s)]^{1/2} C(s \ 1 \ s; \lambda, -1) \\ &= -[2s(1 + s)]^{1/2} C(s \ 1 \ s; \lambda - 1, +1). \end{aligned} \quad (36)$$

Conversely if a particle is partially polarized in the y -direction the density matrix has the form

$$(\rho)_{\lambda\lambda'} = (2s + 1)^{-1} \{ \delta_{\lambda\lambda'} + 3[\langle s_y \rangle / s(s + 1)](s_y)_{\lambda\lambda'} + \dots \}, \quad (37)$$

where \dots represents the contribution of higher order statistical tensors, which we neglect in the following. It is now an easy matter to calculate the “polarized cross section” i.e., the part of the cross section $d\sigma/d\Omega$ which is proportional to $\langle s_{ay} \rangle$; it is

$$\begin{aligned} &3 \sqrt{2} \langle s_{ay} \rangle [s_a(s_a + 1)]^{-1/2} [p^2(2s_a + 1)(2s_b + 1)]^{-1} \sin \theta \\ &\quad \times \sum_{J'} (J + 1/2)(J' + 1/2) \\ &\quad \times \sum_{(\lambda)} \text{Im} \{ \langle \lambda_c \lambda_d | T^J | \lambda_a \lambda_b \rangle \langle \lambda_c \lambda_d | T^{J'} | \lambda_a - 1, \lambda_b \rangle^* e^{i\phi} \} \\ &\quad \times \sum_l C(JJ'l; \lambda, 1 - \lambda) C(JJ'l; \mu, -\mu) (-1)^{\lambda - \mu} C(s_a \ 1 \ s_a; \lambda_a, -1) \\ &\quad \cdot [l(l + 1)]^{-1/2} P_l'(\cos \theta). \end{aligned} \quad (38)$$

TABLE I
 $j = l$ (Integer)^a

$$d_{m0}^{(l)}(\theta) = (-1)^m d_{0m}^{(l)}(\theta) = [4\pi/(2l+1)]^{1/2} \mathcal{P}_m(\theta)$$

Examples:

$$d_{00}^{(l)}(\theta) = P_l(\cos \theta)$$

$$d_{10}^{(l)}(\theta) = -[l(l+1)]^{-1/2} \sin \theta P_l'(\cos \theta)$$

$$d_{20}^{(l)}(\theta) = [(l-1)l(l+1)(l+2)]^{-1/2} \{2P_{l-1}'(\cos \theta) - l(l-1)P_l(\cos \theta)\}$$

$$d_{m1}^{(l)}(\theta) = [l(l+1)]^{-1/2} \{-m[(1+\cos \theta)/\sin \theta] d_{m0}^{(l)}(\theta) - [(l-m)(l+m+1)]^{1/2} d_{m+1,0}^{(l)}(\theta) \\ + \frac{1}{2}[l(l+1)]^{-1/2} \{-[(l-m)(l+m+1)]^{1/2} d_{m+1,0}^{(l)} + [(l+m)(l-m+1)]^{1/2} d_{m-1,0}^{(l)} \\ + [(l-m)(l-m-1)]^{1/2} d_{m+1,0}^{(l-1)} + [(l+m)(l+m-1)]^{1/2} d_{m-1,0}^{(l-1)}\}.$$

^a P_l' means $dP_l/d(\cos \theta)$.

Here Eq. (32) has been used, and a function $d_{10}^{(l)}(\theta)$ has been expressed as P_l' ; see Table I. Furthermore it will be shown in Section V that owing to parity conservation the $e^{i\phi}$ may be replaced by $\cos \phi$.

Turning now to the polarization of the final particles, we notice that the basic states of our representation Eqs. (6) and (15) are most obviously interpreted as the analogs of the states $\psi_{p\lambda}$ and $\psi_{p\lambda_1\lambda_2}$ with regard to a system of axes $x'y'z'$, which is obtained from the original axes xyz by a rotation with Euler angles $(\phi, \theta, -\phi)$ or alternatively $(\phi, \theta, 0)$ if one suppresses a factor $e^{i\lambda\phi}$ in the definition of the basic states. The second choice is somewhat preferable, because in that case the y' axis is perpendicular to both z and z' , i.e., to the reaction plane.

Let us therefore compute the expectation value $s_{y'}$ for particle c , assuming unpolarized incident particles. If c was in a pure state $\sum_{\gamma} c_{\gamma} R_{\phi\theta 0} \psi_{p\gamma}$, the desired expectation value would, of course, be given by (35) with s, λ and a_{λ} replaced by s_c, γ and c_{γ} , respectively. Our final state is similarly expanded in terms of $R_{\phi\theta 0} \psi_{p\lambda_c\lambda_d}$, the coefficient being $f_{\lambda_c\lambda_d; \lambda_a\lambda_b}(\theta, \phi) e^{i\mu\phi}$ (where $\mu = \lambda_c - \lambda_d$). Hence one easily calculates for the outgoing state

$$I(\theta) \langle s_{c y'} \rangle = -\sqrt{2} \sin \theta [s_c(s_c+1)]^{1/2} [p^2(2s_a+1)(2s_b+1)]^{-1} \\ \times \sum_{JJ'} (J+1/2)(J'+1/2) \\ \times \sum_{(\lambda)} \text{Im}[\langle \lambda_c \lambda_d | T^J | \lambda_a \lambda_b \rangle^* \langle \lambda_c - 1, \lambda_d | T^{J'} | \lambda_a \lambda_b \rangle] \\ \times \sum_l C(JJ'l; \lambda, -\lambda) C(JJ'l; \mu, 1-\mu) C(s_c 1 s_c; \lambda_c, -1) (-1)^{\lambda-\mu} \\ \times [l(l+1)]^{-1/2} P_l'(\cos \theta). \quad (39)$$

A similar formula may be obtained for $\langle s_{c x'} \rangle$ and may be shown to vanish (as one expects) if the scattering matrix satisfies the symmetry condition for parity-conservation discussed in the next section.

Finally, as an example of more general formulas, we may write the density matrix $\rho_{\gamma\gamma'}^c$ for outgoing particle c , assuming no initial polarization, and using the “ $R_{\phi, \theta, -\phi}$ representation,” in the form

$$\begin{aligned} \rho_{\gamma\gamma'}^c &= [p^2(2s_a + 1)(2s_b + 1)]^{-1} (-1)^{\gamma-\gamma'} \sum_{JJ'} (J + 1/2) [\pi(2l + 1)]^{1/2} \\ &\times \sum_{\lambda_a \lambda_b \lambda_d} \langle \gamma \lambda_d | T^J | \lambda_a \lambda_b \rangle \langle \gamma' \lambda_d | T^{J'} | \lambda_a \lambda_b \rangle^* \\ &\times C(JJ'; \lambda_0) C(JJ'; \mu, \gamma' - \gamma) Y_{l, \gamma' - \gamma}(\theta, \phi), \end{aligned} \quad (39')$$

where $\mu = \gamma - \lambda_d$, and where some trivial rearrangements of the Clebsch–Gordan coefficients have been made. Some properties of this formula will be discussed later.

V. SYMMETRY PROPERTIES OF THE SCATTERING MATRIX AND SELECTION RULES

We must now examine the restrictions to the form of the S -matrix and the simplifications in the formulas (33) etc., which arise from various symmetry considerations, and in the first place from parity conservation, if it applies.

The states of each of our particles a, \dots, d will obey an equation (9') with intrinsic phase-factors η_a, \dots, η_d . From (8) and (9') one sees that the corresponding equation for the parity operator P is

$$P \psi_{p\lambda} = e^{i\pi J_y} Y \psi_{p\lambda} = \eta (-1)^{s-\lambda} e^{i\pi J_y} \psi_{p, -\lambda}. \quad (40)$$

A similar equation may be derived for $\chi_{p\lambda}$ and combining the two we get, for a two-particle state,

$$P \psi_{p\lambda_1 \lambda_2} = \eta_1 \eta_2 (-1)^{s_1 + s_2 - \lambda_1 + \lambda_2} e^{i\pi J_y} \psi_{p, -\lambda_1, -\lambda_2}. \quad (40')$$

We now apply the operator P to an angular-momentum state, Eq. (16), remembering that P commutes with the rotation operators

$$P |JM\lambda_1 \lambda_2\rangle = \eta_1 \eta_2 (-1)^{s_1 + s_2 - \lambda_1 + \lambda_2} \int dU \mathcal{D}_{M\lambda}^*(\alpha\beta\gamma) R_{\alpha\beta\gamma} R_{0, -\pi, 0} \psi_{p, -\lambda_1, -\lambda_2}.$$

We notice that the equation

$$R_{\alpha\beta\gamma} R_{0, -\pi, 0} = R_{\alpha'\beta'\gamma'}$$

defines an element $\alpha'\beta'\gamma'$ in such a way that the transformation from $\alpha\beta\gamma$ to $\alpha'\beta'\gamma'$ preserves the volume element in group space,⁹ i.e., $dU = dU'$. Furthermore, using the known value of $d^J(\pi)$, one has

$$\mathcal{D}_{M\lambda}(\alpha\beta\gamma) = \sum_{\mu} \mathcal{D}_{M\mu}(\alpha'\beta'\gamma') \mathcal{D}_{\mu\lambda}(0 \pi 0) = (-1)^{J-\lambda} \mathcal{D}_{M, -\lambda}(\alpha'\beta'\gamma').$$

With these substitutions one easily gets

$$P |JM; \lambda_1 \lambda_2\rangle = \eta_1 \eta_2 (-1)^{J-s_1-s_2} |JM; -\lambda_1, -\lambda_2\rangle, \quad (41)$$

where it should be noticed that $J - s_1 - s_2$ is necessarily an integer.

If parity is conserved

$$P^{-1}SP = S, \quad (42)$$

and applying this to the submatrix $S(E)$ in the JM -representation and using (41) one finds

$$\begin{aligned} \langle -\lambda_c, -\lambda_d | S^J | -\lambda_a, -\lambda_b \rangle &= \eta_g \langle \lambda_c \lambda_d | S^J | \lambda_a \lambda_b \rangle; \\ \eta_g &= (\eta_c \eta_d / \eta_a \eta_b) (-1)^{s_c + s_d - s_a - s_b}, \end{aligned} \quad (43)$$

i.e., apart from a constant phase factor, an element of the submatrix S_J does not change if one reverses the sign of all the helicity quantum numbers. It is easy to see what this means for the reaction-amplitude (31). Using the symmetry property (A.1) of the Appendix, we find easily

$$f_{-\lambda_c - \lambda_d; -\lambda_a - \lambda_b}(\theta\phi) = \eta_g f_{\lambda_c \lambda_d; \lambda_a \lambda_b}(\theta, \pi - \phi) \quad (44)$$

which can also be obtained directly by a reflection in the xz plane.

Equation (43) means that the submatrix S_J has, roughly speaking, only half as many independent elements as it could otherwise have;¹⁰ the equation plays the same role as the L -selection rule of the usual formulation. For example in Eq. (33) we may notice that changing $\lambda_a, \dots, \lambda_d$ to $-\lambda_a, \dots, -\lambda_d$ does not affect the product of the two C-G coefficients. Hence Eq. (43) tells us that, in the sum $\sum_{(\lambda)}$, two terms with opposite helicities give the same contribution to the sum. The sum may thus be reduced to roughly half as many terms.

⁹ See for example Ref. [9, Chapter X].

¹⁰ It is perhaps worth noticing that in the present formalism it is especially easy to count the number of a priori independent coefficients in the submatrix S^J , taking due account of the existing symmetries, such as Eqs. (43), (47), and (55). This number is independent of J (except for small values of J , when the conditions $|\lambda| \leq J$, $|\mu| \leq J$ may reduce the dimensionality of the submatrix) and is therefore clearly equal to the number of independent functions of θ (or of the momentum transfer) which must appear in the general reaction formula.

The simplification that applies to Eq. (38) is the following. Rewrite (38) interchanging J and J' and replacing $\lambda_a, \lambda_b, \lambda_c, \lambda_d$ by $-\lambda_a + 1, -\lambda_b, -\lambda_c, -\lambda_d$. Then using (43) and the symmetry properties of the C-G coefficients with respect to the interchange of two angular momentum states one sees that $\sum_{(\lambda)} \dots$ in (38) is replaced by

$$-\sum_{(\lambda)} \text{Im}\{\langle \lambda_c \lambda_d | T^J | \lambda_a \lambda_b \rangle^* \langle \lambda_c \lambda_d | T^{J'} | \lambda_a - 1, \lambda_b \rangle e^{i\phi}\} \dots,$$

the dots indicating that the remainder of the expression is unchanged. This shows that expanding $e^{i\phi} = \cos \phi + i \sin \phi$ the terms in $\sin \phi$ cancel out, as mentioned before.

No change occurs in (39), the simplification in the polarization of the outgoing particle being expressed instead by the condition $\langle s_{cx'} \rangle = 0$. The generalization of these considerations to polarization tensors of higher order is immediate.

For example if in Eq. (39') we set $\phi = 0$ for simplicity, then applying condition (43) together with well-known properties of the C-G coefficients we see that

$$\rho_{-\gamma, -\gamma'}^c = (-1)^{\gamma-\gamma'} \rho_{\gamma, \gamma'}^c. \tag{44'}$$

This means, of course, that certain statistical tensors for particle c vanish. In order to see this more precisely, we recall that according to our conventions if we compute statistical tensors by the standard formulas [11] they will be referred to a rotated system of axes in which the z -axis is parallel to $\mathbf{p}_c = -\mathbf{p}_d$ while (since $\phi = 0$) the y -axis is unchanged and perpendicular to the reaction plane. Let us now perform a further rotation of the statistical tensors by means of the formula¹¹

$$T'_{q, \kappa} = \sum_{\lambda} i^{\lambda} d_{\kappa\lambda}^q(\pi/2) T_{q, \lambda},$$

which corresponds to the replacement, in any tensor operator, of s_x, s_y, s_z , with s_z, s_x, s_y . In other words we refer the tensors to a system of axes in which the z -axis is perpendicular to the reaction plane. One sees easily by means of Eq. (A-2) and the Wigner-Eckart theorem that the tensors T' have the symmetry property

$$(T'_{q, \kappa})_{-\gamma, -\gamma'} = (-1)^{\gamma-\gamma'+\kappa} (T'_{q, \kappa})_{\gamma, \gamma'}.$$

Then Eq. (44') shows that

$$T'_{q, \kappa} = (-1)^{\kappa} T'_{q, \kappa}, \tag{44''}$$

i.e., *all tensors with odd κ vanish*. This is one of the selection rules of Shirokov [5].

¹¹ In $T_{q, \kappa}$, q and κ are, respectively, rank and magnetic quantum number of the tensor.

Identical Particles

Let us now see how the symmetry or antisymmetry of the wave functions for identical particles is expressed in our formalism. Indicating by P_{12} the operator which interchanges particles 1 and 2, we may cast the equation which expresses the effect of P_{12} on the basic state (14) in a form quite similar to Eq. (40') for the operator P . To this end one must use simply the definition (13) and the identities $e^{-i\pi J^{(1)}_y} = (-1)^{2s} e^{i\pi J^{(1)}_y}$ and $(-1)^{\lambda_1 - \lambda_2} = (-1)^{\lambda_2 - \lambda_1}$ (since $\lambda_1 - \lambda_2$ is an integer in this case). One easily sees that

$$P_{12} \psi_{p\lambda_1\lambda_2} = (-1)^{2s - \lambda_1 + \lambda_2} e^{i\pi J_y} \psi_{p\lambda_2\lambda_1}, \quad (45)$$

whereupon, remembering that P_{12} commutes with the rotation operators, the same calculation that follows (40') will obviously give for the angular momentum states

$$P_{12} |JM; \lambda_1\lambda_2\rangle = (-1)^{J-2s} |JM; \lambda_2\lambda_1\rangle. \quad (46)$$

The well-known connection between spin and statistics now tells us that the state of the correct symmetry with respect to P_{12} is

$$\{1 + (-1)^{2s} P_{12}\} |JM; \lambda_1\lambda_2\rangle = |JM; \lambda_1\lambda_2\rangle + (-1)^J |JM; \lambda_2\lambda_1\rangle. \quad (47)$$

It should be noticed that, owing to the particular conventions we have used, the \pm factor which distinguishes Bose-Einstein and Fermi-Dirac statistics has been canceled by a spin-factor $(-1)^{2s}$!

One should notice that, according to Eq. (47), for odd J only $\lambda_1 \neq \lambda_2$ is allowed. Another restriction to be remembered in counting possible states is, of course, $J \geq |\lambda_1 - \lambda_2|$. A by-product of Eq. (47), for example, is a very easy derivation of the Landau-Yang enumeration of the possible states for two photons [12, 13]. The corresponding parities, of course, are immediately obtained from (41).

A simpler case is that of two protons. Writing for brevity ψ_{RR} for $|JM; +1/2 + 1/2\rangle$ and so on (R or L means $+1/2$ or $-1/2$) we find the following possible states

$$\begin{aligned} \psi_{RR}, \psi_{LL} & \quad (J \text{ even}), \\ \psi_{RL} + \psi_{LR} & \quad (J \text{ even } 1), \\ \psi_{RL} - \psi_{LR} & \quad (J \text{ odd}). \end{aligned} \quad (48)$$

The states of the third and fourth type have odd parity (assuming $\eta^2 = 1$), while the first two must combine in the form $\psi_{RR} \pm \psi_{LL}$ if one wants states of (odd/even) parity. Finally the states thus obtained may be expressed in terms of the customary $^1S_0, ^3P_{0,1,2}, \dots$ states in the nonrelativistic limit by means of the expansion coefficients of Appendix B. The result for the first few states is given in Table III.

Time Reversal

In order to examine the consequences of time-reversal invariance, let us assume as usual that our theory admits an antiunitary (Wigner) time reversal operator T . When T operates on our basic one-particle state $\psi_{p\lambda}$, it obviously generates a state with the same helicity and the same momentum, but in the negative z -direction. Since such a state can also be generated by a rotation about the y -axis, we can write

$$T\psi_{p\lambda} = \epsilon e^{-inJ_y} \psi_{p\lambda}, \quad (49)$$

where ϵ is a phase factor, which may depend on p and λ . It is easy to see that it does not depend on p . Consider a Lorentz-transformation \mathcal{L} along the z -direction, which changes p to p' : $\mathcal{L}\psi_{p\lambda} = \psi_{p'\lambda}$. Multiplying (49) on the left by \mathcal{L}^{-1} , and noticing that according to well-known commutation relations

$$\mathcal{L}^{-1}T = T\mathcal{L}; \quad \mathcal{L}^{-1}e^{-inJ_y} = e^{-inJ_y}\mathcal{L}, \quad (50)$$

one finds immediately that $\psi_{p'\lambda}$ obeys the same equation (49) with the same value of ϵ , and since p' is arbitrary, this proves the assertion. It is also easy to see that ϵ does not depend on λ . If the mass is not zero, we may consider the limiting case $p \rightarrow 0$, in which case (49) must reproduce the well-known time-reversal properties of the spin-functions of a particle at rest [14, 15]. Using Eq. (14) one sees that

$$T\psi_{0\lambda} = \epsilon(-1)^{s-\lambda} \psi_{0,-\lambda} = \epsilon i^{2(\lambda-s)} \psi_{0,-\lambda} \quad (51)$$

has the required form, if ϵ is independent of λ . In particular if $\epsilon = -i^{2s}$ one gets the Wigner–Eisenbud prescription, for $\epsilon = (-1)^{2s}$ one gets the Coester prescription, etc. The value of ϵ may of course (since T is antiunitary) be altered at will, by multiplying all our states by a suitable common phase-factor. Let us assume for example $\epsilon = 1$. We still have to show that the same conclusion applies to a massless particle. We use then Eq. (9) assuming for simplicity that η is real (for a photon $\eta = -1$ as we have seen). Furthermore $TY = YT$. Then applying Y on both sides of Eq. (51) for $\lambda = +s$, one finds the same equation for $\lambda = -s$. Q.E.D.

From Eq. (13) and (50) one finds that $\chi_{p\lambda}$ obeys the same T -transformation as $\psi_{p\lambda}$, Eq. (49). Hence the two-particle state also

$$T\psi_{p\lambda_1\lambda_2} = e^{-inJ_y} \psi_{p\lambda_1\lambda_2}, \quad (52)$$

where we have set $\epsilon = 1$. We can now easily calculate, since T commutes with $R_{\alpha\beta\gamma}$

$$T|JM; \lambda_1 \lambda_2\rangle = (\mathfrak{R}/2\pi) \int dU \mathcal{D}_{M,\lambda}(\alpha\beta\gamma) R_{\alpha\beta\gamma} T\psi_{p\lambda_1\lambda_2}. \quad (53)$$

Using (52) and writing $R_{\alpha\beta\gamma}R_{0\pi 0} = R_{\alpha'\beta'\gamma'}$ we proceed like in the derivation of (41) and transform (53) into

$$(\mathfrak{R}/2\pi) \int dU' (-1)^{J+\lambda} \mathcal{D}_{M, -\lambda}(\alpha'\beta'\gamma') R_{\alpha'\beta'\gamma'} \psi_{p\lambda_1\lambda_2}.$$

Furthermore $\mathcal{D}_{M, -\lambda}(\alpha'\beta'\gamma') = (-1)^{-\lambda-M} \mathcal{D}_{-M, \lambda}^*(\alpha'\beta'\gamma')$ and finally

$$T |JM; \lambda_1 \lambda_2\rangle = (-1)^{J-M} |J-M; \lambda_1 \lambda_2\rangle. \quad (54)$$

As is well known, from (54) and the time-reversal property of the S -matrix $T^{-1}ST = S^{-1}$, one easily gets the result that the S^J -matrix is symmetric

$$\langle \lambda_c \lambda_d | S^J | \lambda_a \lambda_b \rangle = \langle \lambda_a \lambda_b | S^J | \lambda_c \lambda_d \rangle, \quad (55)$$

where the matrix element on the right refers of course to the inverse transition $c + d \rightarrow a + b$. Since (55) by the customary form, there is no problem in applying to our formalism the customary conclusions about phases of the matrix elements in photomeson production and so on.

VI. ILLUSTRATION OF THE METHOD BY SOME SIMPLE EXAMPLES

Elastic Scattering of a Spin-One-Half by a Spin-Zero Particle

If particle b ($= d$) has zero spin, the indices λ_b, λ_d may be suppressed. For λ_a and λ_c we use the abbreviation \pm for $\pm 1/2$. Assuming conservation of parity, Eq. (43), two of the matrix-elements S_{++}, S_{+-}, \dots of the submatrix S^J can be expressed in terms of the other two, and it is easy to express all four in terms of the two eigenvalues of S^J . Since the submatrix is unitary the eigenvalues are of the form $e^{2i\delta}$ where δ is a *real* phase. A commonly used notation¹² is δ_{l+} or δ_{l-} , depending on whether $J = l + 1/2$ or $l - 1/2$, l being the orbital quantum number. Although l has no place in our description, we may identify these states by their parity. For simplicity we consider in the following the "orbital" parity (i.e., computed omitting the intrinsic parity factors $\eta_1\eta_2$ in Eq. (41)) which in the usual notation is $(-1)^l = (-1)^{J \mp 1/2}$. From (41) we see that the state

$$|JM; +\rangle \pm |JM; -\rangle \quad (56)$$

has parity $\pm(-1)^{J-1/2} = (-1)^{J \mp 1/2}$. The state with the upper sign in (56) has therefore the phase-shift δ_{l+} with $l = J - 1/2$, and the state with the lower sign has $\delta_{l' -}$, with $l' = J + 1/2 = l + 1$. One sees then that

$$\begin{aligned} S_{++}^J &= S_{--}^J = 1/2(e^{2i\delta_{l+}} + e^{2i\delta_{(l+1)-}}), \\ S_{+-}^J &= S_{-+}^J = 1/2(e^{2i\delta_{l+}} - e^{2i\delta_{(l+1)-}}). \end{aligned} \quad (57)$$

¹² A recent example is Chew *et al.* [16]. The formulas in question are, of course, much older.

TABLE II

$$j = l + \frac{1}{2} \text{ (Half-Integer)}^a$$

$$d_{1/2, 1/2}^{(j)} = (l+1)^{-1} \cos \frac{\theta}{2} (P'_{l+1} - P'_l)$$

$$d_{-1/2, 1/2}^{(j)} = (l+1)^{-1} \sin \frac{\theta}{2} (P'_{l+1} + P'_l)$$

$$d_{1/2, 3/2}^{(j)} = (l+1)^{-1} \sin \frac{\theta}{2} \left\{ \sqrt{\frac{l}{l+2}} P'_{l+1} + \sqrt{\frac{l+2}{l}} P'_l \right\}$$

$$d_{-1/2, 3/2}^{(j)} = (l+1)^{-1} \cos \frac{\theta}{2} \left\{ -\sqrt{\frac{l}{l+2}} P'_{l+1} + \sqrt{\frac{l+2}{l}} P'_l \right\}$$

^a P'_l means $dP_l/d(\cos \theta)$.

Inserting into (31) and with the abbreviation

$$f_{l\pm} = p^{-1} e^{i\delta_{l\pm}} \sin \delta_{l\pm}, \quad (58)$$

we find, using $d_{1/2, 1/2}^J$, etc., from Table II, after a slight rearrangement

$$\begin{aligned} f_{++}(\theta\phi) &= \cos \frac{\theta}{2} (f_1 + f_2), \\ f_{+-}(\theta\phi) &= e^{-i\phi} \sin \frac{\theta}{2} (f_1 - f_2), \end{aligned} \quad (59)$$

where f_1 and f_2 are defined as in [16],

$$\begin{aligned} f_1 &= \sum_l (f_{l+} P'_{l+1} - f_{l-} P'_{l-1}), \\ f_2 &= \sum_l (f_{l-} - f_{l+}) P'_l. \end{aligned} \quad (60)$$

The amplitudes f_{--} and f_{-+} are, of course, given by the symmetry relation (44). The ensuing form of the 2×2 scattering-amplitude-matrix can be written

$$(f_1 + f_2) \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} (f_1 - f_2) (\cos \phi \sigma_y - \sin \phi \sigma_x),$$

which is easily seen to be equivalent to the ordinary form if one bears in mind the rotation of the system of axes, to which the final spin-state is referred.

Neutron-Proton Scattering

We shall only point out some of the possible simplifications. Of the 4 states $|JM; \pm \pm \rangle$ the linear combinations

$$|JM; ++ \rangle \pm |JM; -- \rangle \quad (61a)$$

$$|JM; +- \rangle \pm |JM; -+ \rangle \quad (61b)$$

may be formed; those with the upper sign have "orbital" parity $(-1)^{J-1}$, the others parity $(-1)^J$. The former correspond therefore to the customary triplet states, with $J = L \pm 1$. The submatrix S^J splits correspondingly into two submatrices 2×2 . If furthermore the neutron-proton interaction is assumed symmetric in the two particles (neglecting small electromagnetic effects), then the fact, that the state (61b) with the lower sign has the opposite symmetry to the state (61a) of the same parity, implies a further splitting¹³ of the submatrix for parity $(-1)^J$. Thus finally S^J is reducible by symmetry considerations only, to one 2×2 submatrix and two 1×1 submatrices. This is, of course, the usual result.

Photomeson Production

$$p + \gamma \rightarrow n + \pi^+$$

The elements of the reaction matrix may be labeled $f_{\mu\lambda}$ since the π -meson has no spin, and the value of $\lambda = \lambda_p - \lambda_\gamma$ is sufficient to determine λ_p and λ_γ separately. Assuming parity-conservation, Eq. (43), and assuming $\eta_p = \eta_n$, $\eta_\gamma = \eta_\pi = -1$ (these conventions are the usual ones for nucleons and π -meson, for the photon see earlier) so that $\eta_g = -1$ we may designate the elements of the submatrix T^J as follows

$$p^{-1}T^J = \begin{pmatrix} C & A & B & D \\ -D & -B & -A & -C \end{pmatrix}, \quad (62)$$

where the rows correspond to $\mu = +1/2$ and $-1/2$ and the columns to $\lambda = 3/2, 1/2, -1/2, -3/2$ in this order. Each element, say A , will receive a subscript $2J = 1, 3, 5, \dots$

We notice that, for $J = 1/2$, $\lambda = \pm 3/2$ is impossible and the elements C and D do not exist.

Then by means of Table III we easily find

$$e^{-i\phi} f_{1/2, 3/2} = -\sqrt{3} C_3 \cos \frac{\theta}{2} \sin \theta + \dots,$$

$$f_{1/2, 1/2} = \cos \frac{\theta}{2} \{A_1 - A_3 + 3A_3 \cos \theta + \dots\},$$

¹³ This is the customary remark that 1P_1 and 3P_1 , or respectively, 1D_2 and $^3D_2, \dots$ etc., do not mix. Compare Table III.

TABLE III
Helicity States vs Ordinary States for Two Protons

Helicity states	<i>J</i>				
	0	1	2	3	4
$\frac{1}{\sqrt{2}}(\psi_{RR} + \psi_{LL})$	3P_0	—	$\sqrt{2/5} {}^3P_2 - \sqrt{3/5} {}^3F_2$	—	$\sqrt{4/9} {}^3F_4 - \sqrt{5/9} {}^3H_4$
$\frac{1}{\sqrt{2}}$	1S_0	—	1D_2	—	1G_4
$\frac{1}{\sqrt{2}}(\psi_{RL} + \psi_{LR})$	—	—	$\sqrt{3/5} {}^3P_2 + \sqrt{2/5} {}^3F_2$	—	$\sqrt{5/9} {}^3F_4 + \sqrt{4/9} {}^3H_4$
$\frac{1}{\sqrt{2}}(\psi_{RL} - \psi_{LR})$	—	3P_1	—	3F_3	—

$$e^{i\phi} f_{1/2, -1/2} = \sin \frac{\theta}{2} \{ B_1 + B_3 + 3B_3 \cos \theta + \dots \},$$

$$e^{2i\phi} f_{1/2, -3/2} = \sqrt{3} D_3 \sin \frac{\theta}{2} \sin \theta + \dots, \quad (63)$$

where the dots indicate terms with $J > 3/2$. The correspondence between the coefficients A_1, B_1, \dots , etc. and those customarily employed is easily traced by means of the parity of the final states, using Eq. (41). The states of the pion nucleon system ordinarily indicated as $s_{1/2}(p_{1/2})$, for example, correspond, in our $JM\mu$ notation, to the linear combinations

$$|JM, 1/2\rangle + (-) |JM, -1/2\rangle. \quad (64)$$

The transitions $E1 \rightarrow s_{1/2}$ and $M1 \rightarrow p_{1/2}$ are therefore associated to the coefficients $A_1 - B_1$ and $A_1 + B_1$, respectively.

Similarly, $d_{3/2}$ requires a minus sign in Eq. (64), i.e., the transitions to $d_{3/2}$ depend on the difference between the two rows of the submatrix $T^{3/2}$, Eq. (62). The often made assumption that these transitions are negligible at low energies can therefore be expressed by writing

$$A_3 + B_3 = C_3 + D_3 = 0. \quad (65)$$

This reduces the number of independent coefficients in Eq. (63) to the usual four.

APPENDIX A

We collect here, for convenience, some formulas, which are useful in the evaluation of the $d^j(\theta)$ matrices. We refer the reader to Wigner's [9] and other well-known books [17, 18]¹⁴ for more extensive information.

Symmetry Properties

The number of functions to be computed is considerably reduced by the symmetry relations

$$d_{\lambda\mu}^j(\theta) = d_{-\mu, -\lambda}^j(\theta) = (-1)^{\lambda-\mu} d_{\mu\lambda}^j(\theta). \quad (\text{A1})$$

One also has

$$d_{\lambda\mu}(\theta) = (-1)^{j+\lambda} d_{\lambda, -\mu}(\pi - \theta). \quad (\text{A2})$$

Evaluation

The $d_{\lambda\mu}^j$ functions can be expressed in various ways as hypergeometric functions, Jacobi polynomials, etc. Some convenient procedures for the computation, due to Wigner, are described by Edmonds [18]. A useful formula is the recursion relation¹⁵

$$\begin{aligned} (j+\mu)^{1/2} d_{\lambda\mu}^{(j)}(\theta) &= (j+\lambda)^{1/2} d_{\lambda-1/2, \mu-1/2}^{(j-1/2)}(\theta) \cos \frac{\theta}{2} \\ &+ (j-\lambda)^{1/2} d_{\lambda+1/2, \mu-1/2}^{(j-1/2)}(\theta) \sin \frac{\theta}{2}, \end{aligned} \quad (\text{A3})$$

which may be iterated to give

$$\begin{aligned} &2[(j+\mu)(j+\mu-1)]^{1/2} d_{\lambda\mu}^{(j)}(\theta) \\ &= [(j+\lambda)(j+\lambda-1)]^{1/2} (1 + \cos \theta) d_{\lambda-1, \mu-1}^{(j-1)} \\ &+ 2(j^2 - \lambda^2)^{1/2} \sin \theta d_{\lambda, \mu-1}^{(j-1)} + [(j-\lambda)(j-\lambda-1)]^{1/2} (1 - \cos \theta) d_{\lambda+1, \mu-1}^{(j-1)}. \end{aligned} \quad (\text{A4})$$

The following procedure may be adequate in many cases. The values of λ, μ needed are usually small. If $j=l$, an integer, one can start from d_{00}^l , or more generally d_{m0}^l , see Table I, and calculate $d_{m1}^l, d_{m2}^l, \dots$, by means of the recursion formula (A4) or alternatively by using

¹⁴ Our notations follow Rose's book; in particular $\mathcal{D}_m(\Theta) = e^{-im\phi} Y_{lm}(\Theta\phi)$. $\mathcal{D}(\alpha\beta\gamma)$ and $d(\beta)$ in Edmonds' book correspond to $\mathcal{D}(-\alpha, -\beta, -\gamma)$ and $d(-\beta)$ here.

¹⁵ See for example Rose [17], equation preceding (4.26) or Edmonds [18, Eq. (4.4.1)].

$$d_{\lambda, \mu \pm 1}^j = [(j \pm \mu + 1)(j \mp \mu)]^{-1/2} \left\{ \frac{-\lambda}{\sin \theta} + \mu \cot \theta \mp \frac{\partial}{\partial \theta} \right\} d_{\lambda \mu}^j(\theta). \quad (\text{A5})$$

A simple proof of this formula is as follows. From

$$e^{-i\theta J_y} J_z e^{i\theta J_y} = \cos \theta J_z + \sin \theta J_x,$$

we obtain

$$(J_x \pm iJ_y) e^{-i\theta J_y} = (\sin \theta)^{-1} e^{-i\theta J_y} J_z - \cot \theta J_z e^{-i\theta J_y} \mp \frac{\partial}{\partial \theta} e^{-i\theta J_y}. \quad (\text{A6})$$

Formula (A5) is then obtained by taking the matrix element of (A6) between $\langle j\lambda |$ and $|j\mu\rangle$, remembering Eq. (7).

If j is a half-integer, the same recursion procedure can be applied, starting from $d_{\lambda 1/2}^{(j)}$.

$$d_{\lambda 1/2}^{(j)} = (j + 1/2)^{-1/2} \left[(j + \lambda)^{1/2} d_{\lambda - 1/2, 0}^{(j-1/2)}(\theta) \cos \frac{\theta}{2} + (j - \lambda)^{1/2} d_{\lambda + 1/2, 0}^{(j-1/2)}(\theta) \sin \frac{\theta}{2} \right]. \quad (\text{A7})$$

By means of the above formulas the special values in the following tables have been obtained.

APPENDIX B

In the nonrelativistic case, one may wish to know what the connection is between the states $|JM; \lambda_1 \lambda_2\rangle$ and the customarily employed states

$$|JM; LS\rangle = \sum_{m_1 m_2} C(LSJ; m, m_1 + m_2) C(J_1 J_2 S; m_1 m_2) Y_{Lm} u_{m_1} v_{m_2}, \quad (\text{B1})$$

where the spherical-harmonic symbol Y_{Lm} includes for simplicity also the radial part, u_{m_1} , v_{m_2} are spin states for spin s_1 and s_2 , respectively, and

$$m = M - m_1 - m_2.$$

At the same time one can write for our states Eq. (14), remembering the remark following Eq. (13)

$$\psi_{p\lambda_1\lambda_2} = e^{ip(z_1 - z_2)} u_{\lambda_1} v_{-\lambda_2}. \quad (\text{B2})$$

The rotation operator R of Eqs. (15) and (16) acts on each factor of the right-hand side of (B2) separately and gives for example

$$R_{\phi\theta 0} \psi_{p\lambda_1\lambda_2} = \sum_{m_1 m_2} \mathcal{D}_{m_1\lambda_1}^{(s_1)} \mathcal{D}_{m_2-\lambda_2}^{(s_2)} u_{m_1} v_{m_2} R_{\phi\theta 0} e^{ip(z_1-z_2)}, \quad (\text{B3})$$

where the \mathcal{D} 's, here and in the following equations, are functions of the arguments $\phi\theta 0$.

The Y_{Lm} -function of Eq. (B1) can be expressed in a manner similar to Eq. (18). Remembering (15) and (22) we shall write

$$Y_{Lm} = \mathfrak{N}_L \int \mathcal{D}_{m0}^{L*}(\phi\theta 0) R_{\phi\theta 0} e^{ip(z_1-z_2)} d\Omega. \quad (\text{B4})$$

By means of Eqs. (17) and (A1) and (32) it is easy to expand (18) in terms of (B1) or vice versa. The result is

$$JM; LS |JM; \lambda_1\lambda_2\rangle = \left(\frac{2L+1}{2J+1}\right)^{1/2} C(LSJ; 0, \lambda) C(s_1 s_2 s; \lambda_1, -\lambda_2). \quad (\text{B5})$$

Finally as an application of this formula we give in Table III the connection between our states for two protons, Eq. (48), and the customary singlet and triplet states.

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