CHAPTER 13

THE HELICITY FORMALISM

13.1. The Helicity States

The component of spin s along the direction of motion of a particle is known as its helicity and the helicity quantum number is usually denoted by the symbol λ. It is also the component of total angular momentum $\boldsymbol{J}$ along the direction of motion since the orbital angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is perpendicular to the direction of motion $\hat{\boldsymbol{p}}$ and consequently its projection m_l on the momentum axis is zero.

The helicity formalism has been developed by Jacob and Wick (1959) for relativistic description of scattering of particles with spin and the decay of particles and resonant states. It is equally applicable to massless particles. The helicity formalism leads to simpler intensity and polarization formula over the conventional method in the study of scattering and reaction of particles. The advantages of using the helicity states are many.

1. There is no need to separate the total angular momentum $\boldsymbol{J}$ into orbital and spin parts and hence avoid the difficulties and complications that arise in the treatment of relativistic particles.
2. The helicity λ is invariant under rotations and so states can be constructed with definite $\boldsymbol{J}$ and helicities.
3. The helicity λ is well defined also for massless particles and so there is no need for separate treatment for massless particles.
4. The helicity states are directly related to individual polarization properties of the particles and hence convenient for the polarization study over the conventional formalism of choosing a reference frame with a fixed quantization axis, say z-axis. In the conventional scheme, one has to shuttle back and forth between two representations, one in which the scattering or reaction is conveniently described and the other in which the states are labeled with individual spin components.

In order to specify the helicity states of a particle of mass m and spin s, it is not necessary to know the relativistic wave equation for such a particle. It is enough to know that such a wave equation exists and their plane wave solutions, representing states of definite linear momentum p

and corresponding positive energy $E = (m^2 + p^2)^{1/2}$, have the following properties:

1. For each p, there are $2s + 1$ linearly independent solutions which can be characterized as states of definite helicity λ.

$$\lambda = s, s-1, \ldots, -s. \tag{13.1}$$

   These states characterized by p and λ form a complete set of orthogonal states for a free particle of mass m. If $m = 0$, the number of independent solutions reduces to two: $\lambda = \pm s$. For example, a photon has only two independent helicity states $\lambda = \pm 1$.
2. In the case of ordinary rotation in three dimensional space, the direction of p changes but the helicity λ remains unchanged.
3. Under space reflection about the origin (i.e. parity operation), the helicity λ of a moving particle changes sign.
4. When a Lorentz transformation is applied in the direction of p, the magnitude of p changes and in some cases, the direction of p also, if $m \neq 0$. If the direction of p is not reversed, the helicity λ remains unchanged under Lorentz transformation.

Let $\psi_{p,\lambda}$ denote the state of a particle with momentum p in the positive z-direction. By Lorentz transformation, all states $\psi_{p,\lambda}$ with fixed λ and variable p can be generated. If $m \neq 0$, it is possible to reach the rest state with $p = 0$ by Lorentz transformation. In the rest state, since the total angular momentum of the particle is equal to its spin, it is possible to obtain the relative phases of the states $\psi_{0,\lambda}$ by the requirement

$$(J_x \pm iJ_y)\psi_{0\lambda} = [(s \mp \lambda)(s \pm \lambda + 1)]^{1/2}\psi_{0\lambda\pm 1}. \tag{13.2}$$

In the above equation, J_x, J_y, J_z are the standard spin matrices. For a massless particle, no finite Lorentz transformation can reduce p to zero. For this, we have only two helicity states with $\lambda = \pm s$ and it is possible to go from one state to another by means of a reflection,

$$Y = \mathrm{e}^{-i\pi J_y}\mathcal{P}, \tag{13.3}$$

where $\mathcal{P}$ denotes the parity operator corresponding to reflection with respect to the origin ($x, y, z \to -x, -y, -z$), the operator $\mathrm{e}^{-i\pi J_y}$ denotes a rotation about the y axis through an angle π and Y, the reflection in the xz plane. The operator Y transforms the state $\psi_{p,s}$ into $\psi_{p,-s}$ apart from a phase factor.

$$Y\psi_{p,s} = \eta\psi_{p,-s}. \tag{13.4}$$

Since Y commutes with a Lorentz transformation in the z direction, η should be independent of p. It is therefore a constant which we shall call the "parity factor" of the particle. For example, the $\lambda = \pm 1$ solutions for a photon are $A_\pm = \mp\sqrt{2}(e_x \pm ie_y)\exp(ipz)$ such that $YA_\pm = -A_\mp$. Comparing this with Eq. (13.4), we obtain $\eta = -1$.

It is instructive to check the consistency of Eq. (13.4) with Eq. (13.2) for $m \neq 0$. In this case, $\mathcal{P}$ transforms $\psi_{0\lambda}$ into itself apart from a phase-factor which must be independent of λ ($\mathcal{P}$ commutes with J). Hence

$$\mathcal{P}\,\psi_{0\lambda} = \eta\,\psi_{0\lambda}. \tag{13.5}$$

Furthermore

$$e^{-i\pi J_y}\,\psi_{0\lambda} = \sum_{\lambda'} d^s_{\lambda',\lambda}(\pi)\,\psi_{0\lambda'}, \tag{13.6}$$

where the matrix element $d^s_{\lambda',\lambda}(\pi)$ is given by

$$d^s_{\lambda',\lambda}(\pi) = (-1)^{s-\lambda}\delta_{\lambda',-\lambda}. \tag{13.7}$$

Comparing Eqs. (13.5) and (13.6) and applying a Lorentz transformation in the z direction on both sides, we get

$$Y\,\psi_{p\lambda} = \eta(-1)^{s-\lambda}\psi_{p,-\lambda}, \tag{13.8}$$

which for $\lambda = s$ reduces to (13.4).

If $\psi_{p\lambda}$ denotes a state with momentum in the positive z direction, how can we define a state $\chi_{p\lambda}$ with momentum in the negative z direction? We will have occasion to use the state $\chi_{p,\lambda}$ in the treatment of two-particle scattering in centre of momentum frame wherein one particle moves in the positive direction while the other particle moves in the negative direction. A rotation through an angle π about the y axis corresponds to a transformation $x, y, z \rightarrow -x, y, -z$ and hence

$$\chi_{p\lambda} = (-1)^{s-\lambda}\,e^{-i\pi J_y}\,\psi_{p\lambda}. \tag{13.9}$$

The phase factor $(-1)^{s-\lambda}$ is introduced such that

$$\chi_{0\lambda} = \psi_{0,-\lambda}. \tag{13.10}$$

The result (13.10) is obtained from Eqs. (13.6) and (13.7).

It is possible to generate states $|p\,\theta\,\phi;\lambda\rangle$ with momentum $\boldsymbol{p}'(=p\,\theta\,\phi)$ in an arbitrary direction specified by polar angles $\theta\phi$ by means of a suitable

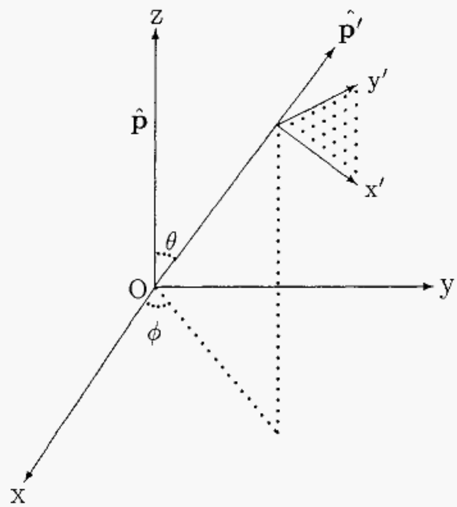

Figure 13.1. The fixed frame of reference x, y, z and the helicity frame x',y',z' (z' coinciding with the direction $\hat{\mathbf{p}}'$).

rotation $R(\alpha,\beta,\gamma)$ applied to states $\psi_{p\lambda}$ having a momentum $\boldsymbol{p}$ in the positive z-direction.

$$|p\,\theta\,\phi;\lambda\rangle = R(\alpha,\beta,\gamma)\,\psi_{p,\lambda}. \tag{13.11}$$

In the present notation, the state $\psi_{p\lambda}$ can be equivalently denoted as $|p\,0\,0;\lambda\rangle$. Two different conventions are in vogue for the choice of angles of rotation in R. Jacob and Wick (1959) used $\alpha=\phi,\beta=\theta,\gamma=-\phi$, corresponding to a rotation through an angle θ about the normal to the plane containing $\boldsymbol{p}$ and $\boldsymbol{p}'$. It is found more convenient to adopt the convention of Jacob (1964) and choose $\alpha=\phi,\beta=\theta,\gamma=0$. In this case, the x' and y' axes to be associated with the helicity direction $\hat{\boldsymbol{p}}'$ as z' axis are as indicated in Fig. 13.1. The positive x' direction is along the direction $(\hat{\boldsymbol{p}}\times\hat{\boldsymbol{p}}')\times\hat{\boldsymbol{p}}'$ and the positive y' direction coincides with the unit vector $(\hat{\boldsymbol{p}}\times\hat{\boldsymbol{p}}')$.

The state $|p\,0\,0;\lambda\rangle$ $(=\psi_{p\lambda})$ is a plane wave state with momentum p in the direction of z-axis (chosen coordinate system) and it can be expanded in terms of states $|p\,j\,m;\lambda\rangle$ of definite angular momentum j and projection m. In the chosen coordinate system, $m = \lambda$ for all j

$$|p\,0\,0;\lambda\rangle = \sum_j C_j\,|p\,j\,\lambda;\lambda\rangle, \tag{13.12}$$

where C_j are the coefficients of expansion. Applying a rotation operator $R(\phi,\theta,0)$ on both sides, we obtain

$$|p\,\theta\,\phi,\lambda\rangle = \sum_{jm} C_j\, D^j_{m\lambda}(\phi,\theta,0)\,|p\,j\,m;\lambda\rangle. \tag{13.13}$$

The expansion coefficients C_j are determined by specifying the normalizations of the plane wave states $|p\,\theta\,\phi,\lambda\rangle$ and the angular momentum eigenstates $|p\,j\,m;\lambda\rangle$ and by using the orthogonality relations of the rotation matrices. The plane wave state $|p\,\theta\,\phi,\lambda\rangle$ is normalized such that

$$\langle p'\,\theta'\,\phi';\lambda'\,|\,p\,\theta\,\phi;\lambda\rangle = \delta_{p,p'}\,\delta_2(\theta\phi,\theta'\phi')\,\delta_{\lambda,\lambda'}, \tag{13.14}$$

where $\delta_2(\theta\phi,\theta'\phi')$ stands for

$$\delta_2(\theta\phi,\theta'\phi') = \delta(\cos\theta - \cos\theta')\,\delta(\phi-\phi'). \tag{13.15}$$

The eigenstates of total angular momentum obey the normalization

$$\langle p'\,j'\,m';\lambda'\,|\,p\,j\,m;\lambda\rangle = \delta_{pp'}\,\delta_{jj'}\,\delta_{mm'}\,\delta_{\lambda\lambda'}. \tag{13.16}$$

The orthogonality relations of d-matrices are given by

$$\int_0^\pi d^j_{m\mu}(\beta)\,d^{j'}_{m\mu}(\beta)\,\sin\beta\,d\beta = \frac{2}{2j+1}\delta_{jj'}, \tag{13.17}$$

$$\frac{1}{2}\sum_j (2j+1)d^j_{m\mu}(\beta)\,d^j_{m\mu}(\beta') = \delta(\cos\beta - \cos\beta'). \tag{13.18}$$

Using the normalizations (13.14) and (13.16) of the plane wave states and the angular momentum states and the orthogonality of d-matrices (13.18), we obtain the expansion coefficient C_j.

$$C_j = \sqrt{\frac{2j+1}{4\pi}} \tag{13.19}$$

Thus, we obtain the important result of the expansion of the plane wave state as a sum of angular momentum states for a particle of arbitrary spin s.

$$|p\,\theta\,\phi;\lambda\rangle = \sum_{jm} \sqrt{\frac{2j+1}{4\pi}}\,D^j_{m\lambda}(\theta,\phi,0)|p\,j\,m;\lambda\rangle. \tag{13.20}$$

Since total angular momentum of the particle and its helicity are invariant under rotation, it is possible to obtain the inverse relation which enables us to project states of definite total angular momentum and helicity from the plane wave state.

$$|p\,j\,m\,;\lambda\rangle = \sqrt{\frac{2j+1}{4\pi}}\int D^{j*}_{m\lambda}(\phi,\theta,0)|p\,\theta\,\phi;\lambda\rangle\,d\Omega, \tag{13.21}$$

where

$$d\Omega = \sin\theta\,d\theta\,d\phi. \tag{13.22}$$

Equivalently, the transformation matrix that corresponds to a transition from the angular momentum state to the plane wave state is

$$\langle p\,j\,m;\lambda\,|\,p\,\theta\,\phi;\lambda\rangle = \sqrt{\frac{2j+1}{4\pi}}D^{j}_{m\lambda}(\phi,\theta,0). \tag{13.23}$$

It is easy to verify that the normalizations (13.14) and (13.16) are consistent with the definitions (13.20) and (13.21), using the orthogonality relations of d-matrices. From Eq. (13.20), we find

$$\begin{aligned}\langle p'\,\theta'\,\phi';\lambda'|p\,\theta\,\phi;\lambda\rangle &= \sum_{jm}\sum_{j'm'}\frac{[j'][j]}{4\pi}D^{j'*}_{m'\lambda'}(\phi',\theta',0)D^{j}_{m\lambda}(\phi,\theta,0)\\ &\quad\times\langle p'\,j'\,m';\lambda'\,|\,p\,j\,m;\lambda\rangle\\ &= \delta_{pp'}\sum_{jm}\frac{2j+1}{4\pi}D^{j*}_{m\lambda}(\phi'\,\theta'\,0)D^{j}_{m\lambda}(\phi,\theta,0)\\ &= \delta_{pp'}\,\delta_2(\theta\,\phi,\theta'\,\phi'),\end{aligned} \tag{13.24}$$

using the normalization (13.16) and the orthogonality relation (13.18) of the d-matrices. Similarly, starting with Eq. (13.21) and using the normalization (13.14) and the orthogonality relation (13.17), we obtain

$$\begin{aligned}\langle p'\,j'\,m';\lambda'|p\,j\,m;\lambda\rangle &= \frac{[j'][j]}{4\pi}\int D^{j*}_{m\lambda}(\phi,\theta,0)\,D^{j'}_{m'\lambda'}(\phi',\theta',0)\\ &\quad\times\langle p'\,\theta'\,\phi';\lambda'|p\,\theta\,\phi;\lambda\rangle d\Omega\,d\Omega'\\ &= \delta_{pp'}\,\delta_{jj'}\,\delta_{mm'}.\end{aligned} \tag{13.25}$$

Equation (13.20) is the expansion of the angular function of a plane wave. It may be noted that the angular dependence of the wave function is given by a D-function instead of a spherical harmonic function which occurs in the case of spin-zero particle. For spin-zero particle,

$$\lambda = 0;\quad j\to l;\quad D^{j}_{m0}(\phi,\theta,0)\to\sqrt{\frac{4\pi}{2l+1}}Y^{m*}_{l}(\theta,\phi). \tag{13.26}$$

Hence, for spin-zero particle, Eqs. (13.20), (13.21) and (13.23) reduce to

$$|p\,\theta\,\phi;\lambda=0\rangle \;\rightarrow\; \sum_{lm} Y_l^{m*}(\theta,\phi)\,|lm\rangle = \sum_{lm} Y_l^{m*}(\hat{\boldsymbol{p}})\,Y_l^{m}(\hat{\boldsymbol{r}}), \tag{13.27}$$

$$|p\,j\,m;\lambda\rangle \;\rightarrow\; Y_l^{m}(\hat{\boldsymbol{r}}), \tag{13.28}$$

$$\langle p\,j\,m;\lambda\,|\,p\,\theta\,\phi;\lambda\rangle \;\rightarrow\; Y_l^{m*}(\theta,\phi) = Y_l^{m*}(\hat{\boldsymbol{p}}). \tag{13.29}$$

13.2. Two-Particle Helicity States

In the two-body scattering such as $a + b \rightarrow c + d$, the initial and final states are two-particle states. A non-interacting two-particle plane wave state with helicities λ_1 and λ_2 can be written as a direct product of two one-particle states (Martin and Spearman, 1970; Jacob, 1964).

$$|\boldsymbol{p}_1\,\boldsymbol{p}_2;\lambda_1\,\lambda_2\rangle = |\boldsymbol{p}_1;\lambda_1\rangle \otimes |\boldsymbol{p}_2;\lambda_2\rangle. \tag{13.30}$$

It is advantageous to go to the centre of momentum (c.m) frame and analyse the wave function in terms of centre of mass motion and relative motion in c.m. system.

$$|\boldsymbol{p}_1\,\boldsymbol{p}_2;\lambda_1\,\lambda_2\rangle = |\boldsymbol{P}\rangle \otimes |\boldsymbol{p};\lambda_1\,\lambda_2\rangle, \tag{13.31}$$

where $|\boldsymbol{P}\rangle$ is the state vector denoting the c.m. motion and $|\boldsymbol{p};\lambda_1\,\lambda_2\rangle$, the relative motion of the two-particle system.

In any physical problem, we are concerned only with the wave function denoting the relative motion in c.m. system and our aim is to construct the two-particle helicity states of definite total angular momentum.

To start with, let us consider the relative motion of the two particles to be along the z-axis, one particle moving along the positive z-axis and the other particle moving with the same momentum p along the negative z-axis. Then

$$|\boldsymbol{p};\lambda_1\,\lambda_2\rangle = |p,\,\theta=0,\,\phi=0;\lambda_1\,\lambda_2\rangle = \psi_{p\lambda_1}\,\chi_{p\lambda_2}, \tag{13.32}$$

where $\psi_{p\lambda_1}$ denotes the one-particle state with momentum p along the positive z-axis and helicity λ_1, and $\chi_{p\lambda 2}$ as defined in Eq. (13.9), denotes the state of the other particle with momentum p along the negative z-axis and helicity λ_2. The resultant helicity λ of the two-particle system is

$$\lambda = \lambda_1 - \lambda_2. \tag{13.33}$$

The two-particle state vectors $|p\,\theta\,\phi;\lambda_1\,\lambda_2\rangle$, representing relative motion along any arbitrary direction can be generated by a suitable rotation $R(\phi,\theta,0)$.

$$|p\,\theta\,\phi;\lambda_1\,\lambda_2\rangle = R(\phi,\theta,0)|p\,0\,0;\lambda_1\,\lambda_2\rangle. \tag{13.34}$$

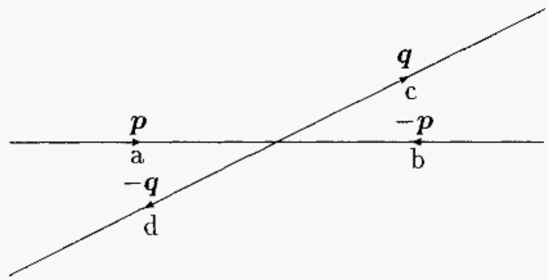

Figure 13.2. The two-body scattering in c.m. system

The plane wave state is a sum over all angular momentum eigenstates and conversely an angular momentum eigenstate can be obtained by angular momentum projection of plane wave state. Using the procedure followed in Sec. 13.1, expressions for two-particle plane wave state and angular momentum eigenfunctions are obtained.

$$|p\,\theta\,\phi\,;\,\lambda_1\,\lambda_2\rangle = \sum_j \sqrt{\frac{2j+1}{4\pi}}\, D^j_{m\lambda}(\phi,\theta,0)\,|p\,j\,m\,;\,\lambda_1\,\lambda_2\rangle, \qquad (13.35)$$

$$|p\,j\,m\,;\,\lambda_1\,\lambda_2\rangle = \sqrt{\frac{2j+1}{4\pi}} \int D^{j\,*}_{m\lambda}(\phi,\theta,0)\,|p\,\theta\,\phi\,;\,\lambda_1\,\lambda_2\rangle d\Omega. \qquad (13.36)$$

The normalizations of the state vectors in the two representations are given by

$$\langle p\,\theta'\,\phi'\,;\,\lambda_1'\,\lambda_2'\,|\,p\,\theta\,\phi\,;\,\lambda_1\,\lambda_2\rangle = \delta_2(\theta\phi,\theta'\phi')\,\delta_{\lambda_1\lambda_1'}\,\delta_{\lambda_2\lambda_2'}, \qquad (13.37)$$

$$\langle p\,j'\,m'\,;\,\lambda_1'\,\lambda_2'\,|\,p\,j\,m\,;\,\lambda_1\,\lambda_2\rangle = \delta_{jj'}\,\delta_{m\,m'}\,\delta_{\lambda_1\lambda_1'}\,\delta_{\lambda_2\lambda_2'}. \qquad (13.38)$$

13.3. Scattering of Particles with Spin

13.3.1. SCATTERING CROSS SECTION

Consider a two-body scattering of particles with spin

$$a + b \to c + d \qquad (13.39)$$

in the c.m. system as described in Fig. 13.2. The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{2\pi}{p}\right)^2 |\langle q\,\theta\,\phi\,;\,\lambda_c\,\lambda_d|\,T(W)\,|p\,0\,0\,;\,\lambda_a\,\lambda_b\rangle|^2, \qquad (13.40)$$

where p denotes the relative momentum of the two particles along the z-axis in the initial state and q denotes the relative momentum of the scattered particles in the final state making an angle ϕ,θ with the incident direction in the c.m. frame. The total energy in the c.m. system is denoted by W and it is conserved in any reaction.

$$W = (p^2 + m_a^2)^{\frac{1}{2}} + (p^2 + m_b^2)^{\frac{1}{2}} = (q^2 + m_c^2)^{\frac{1}{2}} + (q^2 + m_d^2)^{\frac{1}{2}}. \quad (13.41)$$

For evaluating the T-matrix, it is transformed to jm representation.

$$\langle \theta\,\phi; \lambda_c\,\lambda_d \,|\, T(W)\,|0\,0\,; \lambda_a\,\lambda_b\rangle = \sum_{jm}\sum_{j'm'} \langle \theta\,\phi; \lambda_c\,\lambda_d \,|\, j'\,m'; \lambda_c\,\lambda_d\rangle$$
$$\times \langle j'\,m'; \lambda_c\,\lambda_d \,|\, T(W)\,|j\,m\,; \lambda_a\,\lambda_b\rangle \langle j\,m; \lambda_a\,\lambda_b\,|0\,0\,; \lambda_a\,\lambda_b\rangle. \quad (13.42)$$

The rotational invariance implies the conservation of angular momentum and hence j is a good quantum number.

$$\langle j'\,m'; \lambda_c\,\lambda_d \,|\, T(W)\,|j\,m\,; \lambda_a\,\lambda_b\rangle = \delta_{jj'}\,\delta_{mm'}\,\langle \lambda_c\,\lambda_d \,|\, T^j(W)\,|\lambda_a\,\lambda_b\rangle. \quad (13.43)$$

Using Eqs. (13.35), Eq. (13.42) becomes

$$\langle \theta\,\phi; \lambda_c\,\lambda_d \,|\, T(W)\,|0\,0\,; \lambda_a\,\lambda_b\rangle = \sum_{jm} \frac{2j+1}{4\pi} D^{j*}_{m\lambda_f}(\phi,\theta,0)$$
$$\times \langle \lambda_c\,\lambda_d \,|\, T^j(W)\,|\lambda_a\,\lambda_b\rangle D^j_{m\lambda_i}(0,0,0), \quad (13.44)$$

with $\lambda_i = \lambda_a - \lambda_b$ and $\lambda_f = \lambda_c - \lambda_d$. Since

$$D_{m\lambda_i}(0,0,0) = \delta_{m\lambda_i}, \quad (13.45)$$

we obtain

$$\langle \theta\,\phi; \lambda_c\,\lambda_d \,|\, T(W)\,|0\,0\,; \lambda_a\,\lambda_b\rangle = \sum_{j} \frac{2j+1}{4\pi} D^{j*}_{\lambda_i\lambda_f}(\phi,\theta,0)$$
$$\times \langle \lambda_c\,\lambda_d \,|\, T^j(W)\,|\lambda_a\,\lambda_b\rangle. \quad (13.46)$$

Denoting the scattering amplitude in the helicity basis by $f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta,\phi)$, the differential cross section becomes

$$\frac{d\sigma}{d\Omega} = |f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta,\phi)|^2. \quad (13.47)$$

From Eqs. (13.40),(13.46) and (13.47), we find

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta,\phi) = \sum_j (2j+1) D^{j*}_{\lambda_i\lambda_f}(\phi,\theta,0)\, f^j_{\lambda_f\lambda_i}(W), \quad (13.48)$$

with

$$f^j_{\lambda_f \lambda_i}(W) = \frac{1}{2p}\langle \lambda_c \lambda_d \,|\, T^j(W) \,|\lambda_a \lambda_b \rangle. \tag{13.49}$$

For scattering of spinless particles,

$$j \to l; \qquad \lambda_a, \lambda_b, \lambda_c, \lambda_d \to 0; \qquad \lambda_i, \lambda_f \to 0. \tag{13.50}$$

$$D^{j^*}_{\lambda_i \lambda_f}(\phi, \theta, 0) \to D^{l^*}_{00}(\phi, \theta, 0) = \left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}} Y^0_l(\theta, \phi) = P_l(\cos\theta). \tag{13.51}$$

$$f_{\lambda_c \lambda_d; \lambda_a \lambda_b}(\theta, \phi) \to \sum_l (2l+1) P_l(\cos\theta) f_l(W). \tag{13.52}$$

The amplitude $f_l(W)$ $(= T_l(W)/2p)$ is known as the partial wave scattering amplitude for spinless particles. When the particles considered have spin, the total angular momentum j is a good quantum number and for each j, there are several scattering amplitudes which depend on helicity states but the number of independent amplitudes get reduced by invoking parity and time reversal invariance.

Equations (13.47) and (13.48) are general expressions applicable for scattering of particles with arbitrary spin. These formulae are relativistically correct and they are applicable equally well to massless particles and to particles without spin. It is found that the D-functions that occur for particles with spin reduce to Legendre functions for particles without spin.

Let us now explicitly square the scattering amplitude (13.48) and obtain an expression for the differential cross section and total cross section.

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \rho_{\lambda_a \lambda_b} \sum_{jj'} (2j+1)(2j'+1) f^j_{\lambda_f \lambda_i}(W) f^{j'^*}_{\lambda_f \lambda_i}(W) \\ &\quad \times D^{j^*}_{\lambda_i \lambda_f}(\phi, \theta, 0)\, D^{j'}_{\lambda_i \lambda_f}(\phi, \theta, 0), \end{aligned} \tag{13.53}$$

where $\rho_{\lambda_a \lambda_b}$ denotes the density matrix that describes the initial state. Using the symmetry property of the D-functions and using the C.G. series (5.48), we obtain

$$\begin{aligned} &D^{j^*}_{\lambda_i \lambda_f}(\phi, \theta, 0)\, D^{j'}_{\lambda_i \lambda_f}(\phi, \theta, 0) \\ &\quad = (-1)^{\lambda_i - \lambda_f} D^j_{-\lambda_i, -\lambda_f}(\phi, \theta, 0)\, D^{j'}_{\lambda_i \lambda_f}(\phi, \theta, 0) \\ &\quad = (-1)^{\lambda_i - \lambda_f} \sum_l \begin{bmatrix} j & j' & l \\ -\lambda_i & \lambda_i & 0 \end{bmatrix} \begin{bmatrix} j & j' & l \\ -\lambda_f & \lambda_f & 0 \end{bmatrix} D^l_{00}(\phi, \theta, 0). \end{aligned} \tag{13.54}$$

Note that

$$D^l_{00}(\phi,\theta,0) = \sqrt{\frac{4\pi}{2l+1}}Y_l^0(\theta,\phi) = P_l(\cos\theta). \tag{13.55}$$

If the incident and the target particles are not polarized and if the polarization of the final particles are not observed, we need to sum over λ_c and λ_d and average over λ_a and λ_b.

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2s_a+1)(2s_b+1)}\sum_{(\lambda)}\sum_{jj'}(2j+1)(2j'+1)\mathrm{Re}\{f^j_{\lambda_f\lambda_i} f^{j'*}_{\lambda_f\lambda_i}\}$$
$$\times\sum_l(-1)^{\lambda_i-\lambda_f}\begin{bmatrix} j & j' & l \\ -\lambda_i & \lambda_i & 0\end{bmatrix}\begin{bmatrix} j & j' & l \\ -\lambda_f & \lambda_f & 0\end{bmatrix}P_l(\cos\theta), \tag{13.56}$$

where the summation index (λ) stands for helicities λ_a, λ_b, λ_c, λ_d of all incident and scattered particles and Re stands for real part of $\{f^j_{\lambda_f\lambda_i} f^{j'*}_{\lambda_f\lambda_i}\}$ In the above formula, the statistical weight $(2s+1)$ has to be replaced by 2 for a massless particle.

Integrating (13.56) over the solid angle, we obtain the total cross section

$$\sigma = \frac{4\pi}{(2s_a+1)(2s_b+1)}\sum_{(\lambda)}\sum_j(2j+1)\,|f^j_{\lambda_f\lambda_i}|^2, \tag{13.57}$$

using the following relations:

$$\int P_l(\cos\theta)\,d\Omega = 4\pi\,\delta_{l0}, \tag{13.58}$$

$$\begin{bmatrix} j & j' & l \\ -\lambda & \lambda & 0\end{bmatrix}\delta_{l0} = \frac{(-1)^{j+\lambda}}{[j]}\,\delta_{jj'}\,. \tag{13.59}$$

13.3.2. INVARIANCE UNDER PARITY AND TIME REVERSAL

From Eq. (13.56), we find that, for each value of j, there are in total $(2s_a + 1)(2s_b + 1)(2s_c + 1)(2s_d + 1)$ helicity amplitudes. Invariance under parity and time reversal reduces the number of independent amplitudes.

The helicity defined by $\boldsymbol{J}\cdot\hat{\boldsymbol{p}}$ changes sign under space inversion. A state with helicity λ is transformed into a state with helicity $-\lambda$. If $\mathcal{P}$ is the parity operator,

$$\mathcal{P}\,|jm;\lambda_a\,\lambda_b\rangle = \eta_a\eta_b(-1)^{j-s_a-s_b}|jm;-\lambda_a,-\lambda_b\rangle, \tag{13.60}$$

where η_a, η_b denote the intrinsic parities of the two particles with spin s_a and s_b. $\mathcal{P}$ is a unitary operator and invariance of the S-matrix under parity implies that $\mathcal{P}^\dagger S \mathcal{P} = \mathcal{S}$. Since $S = 1 + iT$, it follows that $\mathcal{P}^\dagger T \mathcal{P} = T$.

$$\begin{aligned}\langle \lambda_c, \lambda_d | T^j(W) | \lambda_a, \lambda_b \rangle &= \langle \lambda_c, \lambda_d | \mathcal{P}^\dagger T^j(W) \mathcal{P} | \lambda_a, \lambda_b \rangle \\ &= \eta_a \eta_b \eta_c \eta_d (-1)^{s_c + s_d - s_a - s_b} \\ &\quad \times \langle -\lambda_c, -\lambda_d | T^j(W) | -\lambda_a, -\lambda_b \rangle. \end{aligned} \tag{13.61}$$

Under time reversal, both $\boldsymbol{J}$ and $\boldsymbol{p}$ change sign and hence the helicity does not change. By applying the time reversal operator T to the state $|j\, m : \lambda_a \lambda_b\rangle$, we obtain a new state with the same angular momentum and helicities but with an opposite eigenvalue of J_z. With the phase conventions of Jacob and Wick (1959),

$$\mathcal{T} |j\, m; \lambda_a\, \lambda_b\rangle = (-1)^{j-m} |j\, -m; \lambda_a\, \lambda_b\rangle. \tag{13.62}$$

The operator T is antiunitary and hence the invariance under time reversal implies $\mathcal{T}^\dagger S^\dagger \mathcal{T} = S$.

$$\begin{aligned}\langle \lambda_c\, \lambda_d | T^j(W) | \lambda_a\, \lambda_b \rangle &= \langle jm; \lambda_c\, \lambda_d | T^j(W) | jm; \lambda_a\, \lambda_b \rangle \\ &= \langle jm; \lambda_c\, \lambda_d | \mathcal{T}^\dagger T^{j\dagger}(W) \mathcal{T} | jm; \lambda_a\, \lambda_b \rangle \\ &= (-1)^{j-m} \langle jm; \lambda_c\, \lambda_d | \mathcal{T}^\dagger T^{j\dagger}(W) | j\, -m; \lambda_a \lambda_b \rangle \\ &= (-1)^{j-m} \langle j\, -m; \lambda_a\, \lambda_b | T^j(W) \mathcal{T} | j\, m; \lambda_c \lambda_d \rangle \\ &= \langle j\, -m; \lambda_a\, \lambda_b | T^j(W) | j\, -m; \lambda_c \lambda_d \rangle \\ &= \langle \lambda_a \lambda_b | T^j(W) | \lambda_c \lambda_d \rangle. \end{aligned} \tag{13.63}$$

This yields the familiar result that under time reversal invariance, the transition $a + b \to c + d$ is equal to the inverse transition $c + d \to a + b$.

For identical particles, we have a further relation.

$$\langle \lambda_c\, \lambda_d | T^j(W) | \lambda_a\, \lambda_b \rangle = \langle \lambda_d\, \lambda_c | T^j(W) | \lambda_b\, \lambda_a \rangle. \tag{13.64}$$

13.3.3. POLARIZATION STUDIES

Since the polarizations of the particles are considered separately, formulas giving polarizations take a simple form in the Helicity Formalism. The longitudinal polarization can obviously be introduced by giving different weights to the positive and negative helicity amplitudes in Eq. (13.56). However, it is the angular distribution of the transverse polarization that is more informative.

Transverse polarization is usually defined by means of the expectation value of a transverse component of the spin. The definition of transverse components of spin is somewhat arbitrary in the relativistic case and for a massless particle, the transverse component cannot be defined at all. So, in what follows, we consider only the transverse polarization of a particle with finite mass, for which one can go to the rest frame by Lorentz transformation. The helicity remains unchanged in Lorentz transformation and so also the density matrix in helicity basis. Using the known non-relativistic form for spin matrices, we obtain after simplification that (the reader is referred to solved problem 13.1 for derivation)

$$\begin{aligned} \langle s_y \rangle &= \mathrm{Tr}\,(s_y\,\rho) \\ &= \sum_{\lambda} \{(s+\lambda)(s-\lambda+1)\}^{1/2}\,\mathrm{Im}(\rho_{\lambda-1,\lambda}), \end{aligned} \qquad (13.65)$$

where Im(...) denotes the imaginary part of the quantity within the bracket. Using the algebraic form of C.G. coefficient,

$$\begin{aligned} \begin{bmatrix} s & 1 & s \\ \lambda & -1 & \lambda-1 \end{bmatrix} &= \left\{ \frac{(s+\lambda)(s-\lambda+1)}{2s(1+s)} \right\}^{1/2} \\ &= -\begin{bmatrix} s & 1 & s \\ \lambda-1 & 1 & \lambda \end{bmatrix}, \end{aligned} \qquad (13.66)$$

Equation (13.65) can be rewritten as

$$\langle s_y \rangle = -\{2s(1+s)\}^{1/2} \sum_{\lambda} \begin{bmatrix} s & 1 & s \\ \lambda-1 & 1 & \lambda \end{bmatrix} \mathrm{Im}(\rho_{\lambda-1,\lambda}). \qquad (13.67)$$

We shall consider two specific cases. 1. The incident particle a is transversely polarized with the polarization $\langle s_{ay} \rangle$. What is the "polarized cross section" i.e., the part of the cross section $d\sigma/d\Omega$ which is proportional to $\langle s_{ay} \rangle$? 2. The incident and target particles are unpolarized. What is the transverse polarization $\langle s_{cy'} \rangle$ of the outgoing particle c in the reaction?

Case 1

If the incident particle a has transverse polarization $\langle s_{ay} \rangle$, then its spin density matrix can be written as (the reader is referred to solved problem 13.2 for derivation)

$$\rho_a = \frac{1}{2s_a+1}\left[1 + \frac{3}{s_a(s_a+1)}\langle s_{ay}\rangle s_{ay} + \cdots\right]. \qquad (13.68)$$

If we restrict our consideration to vector polarization and neglect higher order tensor contributions, the density matrix for the initial system is

$$\rho_i = \rho_a \rho_b = \frac{1}{(2s_a+1)(2s_b+1)}\left[1 + \frac{3}{s_a(s_a+1)}\langle s_{ay}\rangle\, s_{ay}\right]. \qquad (13.69)$$

The cross section depends on the density matrix for the final state which is evaluated if the scattering amplitude f and the density matrix of the initial state ρ_i are known.

$$\begin{aligned}\operatorname{Tr}\rho_f &= \operatorname{Tr}(f\rho_i f^\dagger) = \operatorname{Tr}(\rho_i f^\dagger f) \\ &= \sum_{\lambda_i\lambda_i'}(\rho_i)_{\lambda_i'\lambda_i}(f^\dagger f)_{\lambda_i\lambda_i'},\end{aligned} \tag{13.70}$$

where $(f^\dagger f)$ can be considered as the density matrix ρ_f^0 corresponding to the final state when the incident particles are unpolarized. Using Eq. (13.69) for the density matrix for the initial system, the polarized cross section $\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{p}}$ that is propotional to $\langle s_{ay}\rangle$ is obtained from (13.70).

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{p}} &= \frac{3}{(2s_a+1)(2s_b+1)s_a(s_a+1)}\langle s_{ay}\rangle \\ &\quad\times\sum_{\lambda_i}(s_{ay})_{\lambda_i-1,\lambda_i}\operatorname{Im}(f^\dagger f)_{\lambda_i,\lambda_i-1}.\end{aligned} \tag{13.71}$$

Expanding $(f^\dagger f)_{\lambda_i,\lambda_i-1}$ as $\sum_{\lambda_f}(f^\dagger)_{\lambda_i,\lambda_f}(f)_{\lambda_f,\lambda_i-1}$ and substituting the expansion (13.48) for the scattering amplitude f, we obtain

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{p}} &= \frac{3}{(2s_a+1)(2s_b+1)s_a(s_a+1)}\langle s_{ay}\rangle\sum_{jj'}\sum_{\lambda_f\lambda_i}(2j+1)(2j'+1) \\ &\quad\times\operatorname{Im}\left\{D^j_{\lambda_i,\lambda_f}(\phi,\theta,0)\,D^{j'*}_{\lambda_i-1,\lambda_f}(\phi,\theta,0)\,f^{j*}_{\lambda_f,\lambda_i}\,f^{j'}_{\lambda_f,\lambda_i-1}\right\} \\ &\quad\times(s_{ay})_{\lambda_i-1,\lambda_i}.\end{aligned} \tag{13.72}$$

Equation (13.72) can be simplified by coupling the two D-matrices by using C.G. series (5.48).

$$\begin{aligned}&D^j_{\lambda_i,\lambda_f}(\phi,\theta,0)\,D^{j'*}_{\lambda_i-1,\lambda_f}(\phi,\theta,0) \\ &\quad= (-1)^{\lambda_i-1-\lambda_f}D^j_{\lambda_i,\lambda_f}(\phi,\theta,0)\,D^{j'}_{1-\lambda_i,-\lambda_f}(\phi,\theta,0) \\ &\quad= (-1)^{\lambda_i-1-\lambda_f}\sum_L\begin{bmatrix} j & j' & L \\ \lambda_i & 1-\lambda_i & 1\end{bmatrix}\begin{bmatrix} j & j' & L \\ \lambda_f & -\lambda_f & 0\end{bmatrix}D^L_{10},\end{aligned} \tag{13.73}$$

with

$$\begin{aligned}D^L_{10}(\phi,\theta,0) &= \mathrm{e}^{-i\phi}\,d^L_{10}(\theta) \\ &= \mathrm{e}^{-i\phi}\left[-\{L(L+1)\}^{-\frac{1}{2}}\sin\theta\,P_L'(\cos\theta)\right].\end{aligned} \tag{13.74}$$

Using Eq. (13.66), we obtain the matrix element of s_{ay}.

$$\begin{aligned}(s_{ay})_{\lambda_i-1,\lambda_i} &= \{(s_a+\lambda_i)(s_a-\lambda_i+1)\}^{1/2} \\ &= -\{2s_a(1+s_a)\}^{1/2}\begin{bmatrix} s_a & 1 & s_a \\ \lambda_i-1 & 1 & \lambda_i\end{bmatrix}. \end{aligned} \tag{13.75}$$

Substituting Eqs. (13.73) - (13.75) into Eq. (13.72), we obtain the polarized cross section arising from the transverse polarization $\langle s_{ay}\rangle$ of particle *a*.

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{P}} &= \frac{3}{(2s_a+1)(2s_b+1)}\sqrt{\frac{2}{s_a(1+s_a)}}\langle s_{ay}\rangle \\ &\times \sum_{jj'}\sum_{L}\sum_{\lambda_i\lambda_f}(2j+1)(2j'+1)\mathrm{Im}\left\{f^{j\,*}_{\lambda_f,\lambda_i}\, f^{j'}_{\lambda_f,\lambda_i-1}\,\mathrm{e}^{-i\phi}\right\} \\ &\times(-1)^{\lambda_i-\lambda_f}\begin{bmatrix} j & j' & L \\ \lambda_i & 1-\lambda_i & 1\end{bmatrix}\begin{bmatrix} j & j' & L \\ \lambda_f & -\lambda_f & 0\end{bmatrix} \\ &\times\begin{bmatrix} s_a & 1 & s_a \\ \lambda_i & -1 & \lambda_i-1\end{bmatrix}\{L(L+1)\}^{-\frac{1}{2}}\sin\theta\, P_L'(\cos\theta). \end{aligned} \tag{13.76}$$

Case 2

Let us now consider the transverse polarization of one final particle, say *c,* when the initial particles are not polarized and when the polarization of the other final particle *d* is not observed. The polarization of particle *c* of spin s_c normal to the production plane is

$$\langle s_{cy'}\rangle = \frac{\mathrm{Tr}\,(s_{cy'}\rho_f)}{\mathrm{Tr}\,\rho_f}, \tag{13.77}$$

where $\mathrm{Tr}\rho_f$ is just the differential cross section $d\sigma/d\Omega$. So,

$$\langle s_{cy'}\rangle\frac{d\sigma}{d\Omega} = \mathrm{Tr}\,(s_{cy'}\rho_f). \tag{13.78}$$

Using Eqs. (13.65) and (13.67), we obtain

$$\begin{aligned}\mathrm{Tr}\,(s_{cy'}\rho_f) &= \sum_{\lambda_c}\{(s_c+\lambda_c)(s_c-\lambda_c+1)\}^{\frac{1}{2}}\,\mathrm{Im}\,(\rho_f)_{\lambda_c-1,\lambda_c} \\ &= -\{2s_c(1+s_c)\}^{1/2}\sum_{\lambda_c}\begin{bmatrix} s_c & 1 & s_c \\ \lambda_c-1 & 1 & \lambda_c\end{bmatrix} \\ &\quad\times\mathrm{Im}\,(\rho_f)_{\lambda_c-1,\lambda_c}. \end{aligned} \tag{13.79}$$

Since the particles in the initial state are not polarized, the elements of the spin density matrix of the final state is given by

$$(\rho_f)_{\lambda'_f,\lambda_f} = \frac{1}{(2s_a+1)(2s_b+1)} f_{\lambda'_f,\lambda_i} f^*_{\lambda_f,\lambda_i} \tag{13.80}$$

where, for brevity, single helicity quantum number is used to denote a two-particle helicity state as shown below.

$$\lambda_i = \{\lambda_a, \lambda_b\}; \;\; \lambda_f = \{\lambda_c, \lambda_d\}; \;\; \lambda'_f = \{\lambda_c - 1, \lambda_d\}. \tag{13.81}$$

Substituting Eq. (13.48) for the helicity amplitudes $f_{\lambda'_f,\lambda_i}, f^*_{\lambda_f,\lambda_i}$, we obtain

$$\begin{aligned}(\rho_f)_{\lambda'_f,\lambda_f} &= \sum_{jj'} \frac{(2j+1)(2j'+1)}{(2s_a+1)(2s_b+1)} D^{j*}_{\lambda_i\lambda'_f}(\phi,\theta,0)\, D^{j'}_{\lambda_i\lambda_f}(\phi,\theta,0) \\ &\quad \times f^j_{\lambda'_f,\lambda_i} f^{j'*}_{\lambda_f,\lambda_i}. \end{aligned} \tag{13.82}$$

Coupling the two rotation matrices using C.G. series (5.48) and using Eq. (13.79), we obtain

$$\begin{aligned}\langle s_{cy'}\rangle \frac{d\sigma}{d\Omega} &= \frac{1}{(2s_a+1)(2s_b+1)} \left[-\{2s_c(1+s_c)\}^{\frac{1}{2}}\right] \begin{bmatrix} s_c & 1 & s_c \\ \lambda_c - 1 & 1 & \lambda_c \end{bmatrix} \\ &\quad \times \sum_{(\lambda)} \sum_{jj'L} (2j+1)(2j'+1)(-1)^{\lambda_i - \lambda'_f} \begin{bmatrix} j & j' & L \\ -\lambda_i & \lambda_i & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} j & j' & L \\ -\lambda'_f & \lambda_f & 1 \end{bmatrix} D^L_{0,1}(\phi,\theta,0)\, f^j_{\lambda'_f\lambda_i} f^{j'*}_{\lambda_f\lambda_i}. \end{aligned} \tag{13.83}$$

Using the analytical expression for the rotation matrix,

$$D^L_{01}(\phi,\theta,0) = d^L_{01}(\theta) = \{L(L+1)\}^{-\frac{1}{2}} \sin\theta\, P'_L(\cos\theta), \tag{13.84}$$

we finally obtain

$$\begin{aligned}\langle s_{cy'}\rangle \frac{d\sigma}{d\Omega} &= \frac{1}{(2s_a+1)(2s_b+1)} \left[-\{2s_c(1+s_c)\}^{\frac{1}{2}}\right] \begin{bmatrix} s_c & 1 & s_c \\ \lambda_c - 1 & 1 & \lambda_c \end{bmatrix} \\ &\quad \times \sum_{(\lambda)} \sum_{jj'L} (2j+1)(2j'+1)(-1)^{\lambda_i - \lambda'_f} \begin{bmatrix} j & j' & L \\ -\lambda_i & \lambda_i & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} j & j' & L \\ -\lambda'_f & \lambda_f & 1 \end{bmatrix} \frac{\sin\theta\, P'_L(\cos\theta)}{\sqrt{L(L+1)}} \,\mathrm{Im}\left\{ f^j_{\lambda'_f\lambda_i} f^{j'*}_{\lambda_f\lambda_i} \right\}. \end{aligned} \tag{13.85}$$

A similar formula may be obtained for $\langle s_{cx'}\rangle$ and may be shown to vanish, as one expects, if the scattering matrix satisfies the symmetry condition for parity conservation discussed in Sec. 13.3.2.

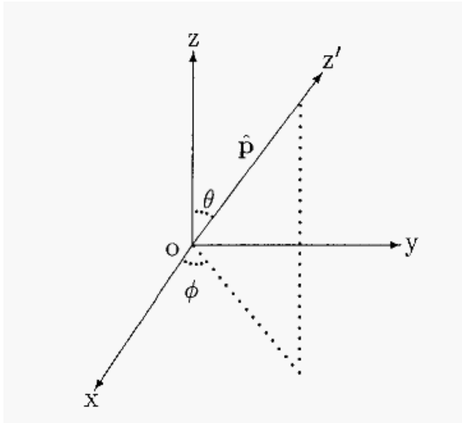

Figure 13.3. The unprimed coordinate system is the rest frame of γ and the primed coordinate system is the helicity frame for the decay products α and β.

13.4. Two-Body Decay

Let us now investigate the two-body decay of an unstable resonance or, more generally, of a system of definite angular momentum and parity (Lee and Yang, 1958; Byers and Fenster, 1963; Jackson, 1965). The observables are the intensity and polarization of the angular distributions of the decay products. There are two main objectives. 1. One is to obtain information on the mechanism of production of a resonance. In this case, it is better to work in terms of the density matrix elements themselves since they give direct information on the population of the angular momentum substates. 2. The other is to determine the spin and parity of the resonance by studying various moments of angular distributions. For this, it is often convenient to express the density matrix in terms of multipole parameters.

To be specific, we choose the rest frame of γ with a fixed z axis (quantization axis) to describe its two-body decay into α and β (vide Fig. 13.3). If $\boldsymbol{p}$ and $-\boldsymbol{p}$ are the momenta of α and β in this frame, then the state vector of the two particles containing the angular and helicity information is denoted by $|p\,\theta\,\phi;\lambda_\alpha\lambda_\beta\rangle$ which can be expanded in terms of angular momentum eigenstates.

$$|p\,\theta\,\phi;\lambda_\alpha\lambda_\beta\rangle = \sum_{jm}\sqrt{\frac{2j+1}{4\pi}}\,D^j_{m\lambda}(\phi,\theta,0)\,|jm;\lambda_\alpha\lambda_\beta\rangle, \tag{13.86}$$

with

$$\lambda = \lambda_\alpha - \lambda_\beta. \tag{13.87}$$

The amplitude for the decay $\psi \to \alpha + \beta$ from a definite state $|jm\rangle$ of γ is given by (suppressing the label p hereafter)

$$\begin{aligned} f_{\lambda m}(\lambda_\alpha \lambda_\beta) &= \langle \theta\,\phi : \lambda_\alpha \lambda_\beta | \, H \, | jm \rangle \\ &= \sqrt{\frac{2j+1}{4\pi}}\, D^{j*}_{m\lambda}(\phi, \theta, 0) \langle jm; \lambda_\alpha \lambda_\beta | \, H \, | jm \rangle. \end{aligned} \quad (13.88)$$

Since the interaction Hamiltonian H is a scalar under rotation, its matrix element depends on λ_α and λ_β but not on m. So, let us denote the matrix element by $H(\lambda_\alpha, \lambda_\beta)$.

If the resonant state γ is denoted by the density matrix ρ_i, then the density matrix ρ_f corresponding to the final state is given by

$$\begin{aligned} (\rho_f)_{\lambda_f, \lambda_{f'}} &= \sum_{mm'} f_{\lambda_f m}\, \rho_{mm'}\, f^*_{\lambda_{f'} m'} \\ &= \sum_{m,m'} \frac{2j+1}{4\pi} D^{j*}_{m\lambda_f}(\phi, \theta, 0)\, D^{j}_{m'\lambda_{f'}}(\phi, \theta, 0) \langle jm; \lambda_\alpha \lambda_\beta | \, H \, | jm \rangle \\ &\quad \times \langle jm | \, \rho_i \, | jm' \rangle \, \langle jm; \lambda'_\alpha \lambda'_\beta | \, H \, | jm' \rangle^* \end{aligned} \quad (13.89)$$

with $\lambda_{f'} = \lambda'_\alpha - \lambda'_\beta$ and $\lambda_f = \lambda_\alpha - \lambda_\beta$.

The angular distribution $I(\theta,\phi)$ of the decay particles is obtained by taking the trace of ρ_f.

$$\begin{aligned} I(\theta, \phi) &= \operatorname{Tr} \rho_f \\ &= \sum_{\lambda_\alpha \lambda_\beta} \sum_{mm'} \frac{2j+1}{4\pi} D^{j*}_{m\lambda}(\phi, \theta, 0) D^{j}_{m'\lambda}(\phi, \theta, 0) \\ &\quad \times H(\lambda_\alpha, \lambda_\beta)\, H^*(\lambda_\alpha, \lambda_\beta) (\rho_i)_{mm'}, \end{aligned} \quad (13.90)$$

with the notation

$$\lambda = \lambda_\alpha - \lambda_\beta; \quad H(\lambda_\alpha, \lambda_\beta) = \langle jm; \lambda_\alpha\, \lambda_\beta | \, H \, | j\, m \rangle. \quad (13.91)$$

Separating the terms that depend on m and m', we get

$$\begin{aligned} &\sum_{mm'} D^{j*}_{m\lambda}(\phi, \theta, 0)\, D^{j}_{m'\lambda}(\phi, \theta, 0) (\rho_i)_{mm'} \\ &\qquad = \sum_{mm'} (-1)^{m-\lambda}\, D^{j}_{-m,-\lambda}(\phi, \theta, 0)\, D^{j}_{m'\lambda}(\phi, \theta, 0)\, (\rho_i)_{mm'} \\ &\qquad = \sum_{mm'} (-1)^{m-\lambda}\, e^{i(m-m')\phi}\, d^{j}_{-m,-\lambda}(\theta)\, d^{j}_{m'\lambda}(\theta)\, (\rho_i)_{mm'}. \end{aligned} \quad (13.92)$$

The rotation matrices $d^j(\theta)$ are known and hence the angular distribution can be obtained in terms of the density matrix of the initial system. The normalized angular distribution is given by

$$\frac{I(\theta,\phi)}{\int I(\theta,\phi)\,d\Omega}. \tag{13.93}$$

It is easy to show that

$$\int I(\theta,\phi)\,d\Omega = \sum_{\lambda_\alpha,\lambda_\beta} |H(\lambda_\alpha,\lambda_\beta)|^2, \tag{13.94}$$

since

$$\int D^{j*}_{m\lambda}(\phi,\theta,0)\,D^{j}_{m'\lambda}(\phi,\theta,0)\,d\Omega = \frac{4\pi}{2j+1}\,\delta_{mm'}, \tag{13.95}$$

and

$$\sum_{mm'}(\rho_i)_{mm'}\,\delta_{mm'} = \mathrm{Tr}\,\rho_i = 1. \tag{13.96}$$

Let us now illustrate the above discussion by considering the decay of a spin-1 system into two spin-zero particles. For this, there is only one helicity matrix element $H(0,0)$ since $\lambda_\alpha = \lambda_\beta = 0$. Since $j = 1$ and $\lambda = 0$, the required d^j matrix elements are

$$d^1_{10} = -\frac{1}{\sqrt{2}}\sin\theta; \quad d^1_{00} = \cos\theta; \quad d^1_{-1,0} = \frac{1}{\sqrt{2}}\sin\theta. \tag{13.97}$$

Substituting these values of d^1 matrix elements, the normalized angular distribution of the decay particle is obtained in terms of the spin density matrix of the parent system.

$$\begin{aligned}\frac{I(\theta,\phi)}{\int I(\theta,\phi)\,d\Omega} &= \frac{3}{4\pi}\Big[\cos^2\theta\,\rho_{0,0} + \tfrac{1}{2}\sin^2\theta\,(\rho_{1,1}+\rho_{-1,-1}) \\ &\quad - \sin^2\theta\,\mathrm{Re}(e^{2i\phi}\,\rho_{1,-1}) \\ &\quad -\sqrt{\tfrac{1}{2}}\,\sin 2\theta\,\mathrm{Re}(e^{i\phi}\,\rho_{1,0} - e^{-i\phi}\,\rho_{-1,0})\Big]. \end{aligned} \tag{13.98}$$

As discussed in Sec. 11.2, the density matrix can be expanded in terms of spherical tensor parameters which are also known as multipole parameters.

Using Eq. (11.23), the elements of the density matrix can be written as

$$\begin{aligned}(\rho_i)_{mm'} &= \frac{1}{2j+1}\sum_{k=0}^{2j}\sum_{m_k=-k}^{+k}\langle T_k^{m_k}\rangle^* \langle jm|\, T_k^{m_k}\,|jm'\rangle \\ &= \frac{1}{2j+1}\sum_{k=0}^{2j}\sum_{m_k=-k}^{+k}\langle T_k^{m_k}\rangle^* \left[\begin{array}{ccc} j & k & j \\ m' & m_k & m\end{array}\right]\langle j||\, T_k^{m_k}\,||j\rangle \\ &= \frac{1}{2j+1}\sum_{k=0}^{2j}\sum_{m_k=-k}^{+k}\langle T_k^{m_k}\rangle^* \left[\begin{array}{ccc} j & k & j \\ m' & m_k & m\end{array}\right][k]. \qquad (13.99)\end{aligned}$$

The product of two rotation matrices that occur in Eq. (13.90) can be simplified using the formula (5.48), familiarly known as the C.G. series.

$$\begin{aligned}&D^{j^*}_{m\lambda}(\phi,\theta,0)\, D^{j}_{m'\lambda}(\phi,\theta,0) \\ &\quad = (-1)^{m-\lambda}\, D^{j}_{-m,-\lambda}(\phi,\theta,0)\, D^{j}_{m'\lambda}(\phi,\theta,0) \\ &\quad = (-1)^{m-\lambda}\sum_L \left[\begin{array}{ccc} j & j & L \\ -m & m' & M\end{array}\right]\left[\begin{array}{ccc} j & j & L \\ -\lambda & \lambda & 0\end{array}\right] D^L_{M0}(\phi,\theta,0). \qquad (13.100)\end{aligned}$$

The resulting rotation matrix $D^L_{M0}(\phi,\theta,0)$ can have only integer values for L and it can be expressed as a spherical harmonic using Eq. (5.76).

$$D^L_{M0}(\phi,\theta,0) = \sqrt{\frac{4\pi}{2L+1}}\, Y_L^{M^*}(\theta,\phi). \qquad (13.101)$$

Substituting Eqs. (13.99 - 13.101) into Eq. (13.90), we obtain

$$\begin{aligned}I(\theta,\phi) &= \sum_{\lambda_\alpha\lambda_\beta}\sum_{mm'}\sum_L \{4\pi(2L+1)\}^{-1/2}(-1)^{m-\lambda} \\ &\quad\times \left[\begin{array}{ccc} j & j & L \\ -m & m' & M\end{array}\right]\left[\begin{array}{ccc} j & j & L \\ -\lambda & \lambda & 0\end{array}\right] Y_L^{M^*}(\theta,\phi)\,|H(\lambda_\alpha,\lambda_\beta)|^2 \\ &\quad\times \sum_{k\, m_k}\left[\begin{array}{ccc} j & k & j \\ m' & m_k & m\end{array}\right][k]\,\langle T_k^{m_k}\rangle^*. \qquad (13.102)\end{aligned}$$

Equation (13.102) is simplified by performing first the summation over m and then replacing the summation over m' by M.

$$\begin{aligned}&\sum_{mm'}(-1)^{m-\lambda}\left[\begin{array}{ccc} j & j & L \\ -m & m' & M\end{array}\right]\left[\begin{array}{ccc} j & k & j \\ m' & m_k & m\end{array}\right] \\ &\qquad = \sum_M (-1)^M(-1)^{j-k+\lambda}\,\frac{[j]}{[k]}\,\delta_{Lk}\,\delta_{M,-m_k}. \qquad (13.103)\end{aligned}$$

Also

$$\sum_L (-1)^M Y_L^{M*} \langle T_L^{-M} \rangle^* = \sum_M Y_L^{M*} \langle T_L^M \rangle. \tag{13.104}$$

Substituting these results in Eq. (13.102) and replacing k and m_k by L and $-M$ because of the delta functions, we finally obtain

$$\begin{aligned} I(\theta,\phi) &= \sum_{\lambda_\alpha \lambda_\beta} \sum_L (-1)^{j-\lambda} \frac{[j]}{\sqrt{4\pi}[L]} \begin{bmatrix} j & j & L \\ \lambda & -\lambda & 0 \end{bmatrix} \\ &\quad \times |H(\lambda_\alpha, \lambda_\beta)|^2 \sum_M \langle T_L^M \rangle Y_L^{M*}(\theta,\phi). \end{aligned} \tag{13.105}$$

Integrating over the solid angle and using the following identities

$$\int Y_L^{M*}(\theta,\phi)\, d\Omega = \sqrt{4\pi}\, \delta_{L0}\, \delta_{M0}, \tag{13.106}$$

$$\begin{bmatrix} j & j & 0 \\ \lambda & -\lambda & 0 \end{bmatrix} = \frac{(-1)^j}{[j]}, \tag{13.107}$$

$$\langle T_0^0 \rangle = 1, \tag{13.108}$$

we retrieve the result (13.94).

$$\int I(\theta,\phi)\, d\Omega = \sum_{\lambda_\alpha, \lambda_\beta} |H(\lambda_\alpha, \lambda_\beta)|^2.$$

By inspection of Eq. (13.105), it is seen that the statistical tensors $\langle T_L^M \rangle$ are related to the spherical harmonic moments of $I(\theta,\phi)$.

$$\begin{aligned} \int I(\theta,\phi)\, Y_L^M(\theta,\phi)\, d\Omega &= \sum_{\lambda_\alpha \lambda_\beta} (-1)^{j-\lambda} \frac{[j]}{\sqrt{4\pi}[L]} \begin{bmatrix} j & j & L \\ \lambda & -\lambda & 0 \end{bmatrix} \\ &\quad \times |H(\lambda_\alpha, \lambda_\beta)|^2 \langle T_L^M \rangle. \end{aligned} \tag{13.109}$$

Case 1: Decay into two spinless particles

In the case of decay into two spinless particles,

$$\lambda_\alpha = \lambda_\beta = \lambda = 0.$$

Equation (13.109) now reduces to

$$\int I(\theta,\phi)\, Y_L^M(\theta,\phi)\, d\Omega = \sum_{\lambda_\alpha\lambda_\beta} (-1)^j \frac{[j]}{\sqrt{4\pi}[L]} \begin{bmatrix} j & j & L \\ 0 & 0 & 0 \end{bmatrix} \times |H(0,0)|^2 \langle T_L^M \rangle. \quad (13.110)$$

Here j is an integer and L should be even because of the parity C.G. coefficient. Since

$$\int I(\theta,\phi)\, d\Omega = |H(0,0)|^2, \quad (13.111)$$

it follows that the normalized spherical harmonic moments of angular distribution is

$$\frac{\int I(\theta,\phi)\, Y_L^M(\theta,\phi)\, d\Omega}{\int I(\theta,\phi)\, d\Omega} = \sum_{\lambda_\alpha\lambda_\beta} (-1)^j \frac{[j]}{\sqrt{4\pi}[L]} \begin{bmatrix} j & j & L \\ 0 & 0 & 0 \end{bmatrix} \langle T_L^M \rangle. \quad (13.112)$$

Case 2: Decay into a spin-$\frac{1}{2}$ and a spin-zero particle

From parity considerations, the two amplitudes $H(\frac{1}{2},0)$ and $H(-\frac{1}{2},0)$ are related.

$$H(-\tfrac{1}{2},0) = \epsilon\, H(\tfrac{1}{2},0), \quad (13.113)$$

where $\epsilon = \pm 1$. From (13.60), it follows that

$$\epsilon = \eta_\alpha\, \eta_\beta\, \eta_\gamma\, (-1)^{j-\frac{1}{2}}. \quad (13.114)$$

If parity is conserved in the decay, then $\epsilon = \pm 1$ corresponding to the orbital angular momentum $(l = j \mp \frac{1}{2})$ of the ab system. The conservation of parity requires that the product of intrinsic parities $\eta_\alpha\eta_\beta\eta_\gamma = (-1)^l$. Thus determines the intrinsic parity of the γ resonance. However

$$|H(-\tfrac{1}{2},0)|^2 = |H(\tfrac{1}{2},0)|^2. \quad (13.115)$$

Consequently,

$$\int I(\theta,\phi)\, d\Omega = |H(\tfrac{1}{2},0)|^2 + |H(-\tfrac{1}{2},0)|^2 = 2\,|H(\tfrac{1}{2},0)|^2, \quad (13.116)$$

and the normalized angular distribution is given by

$$\frac{\int I(\theta,\phi)\,Y_L^M(\theta,\phi)\,d\Omega}{\int I(\theta,\phi)\,d\Omega} = (-1)^{j-\frac{1}{2}}\frac{[j]}{\sqrt{4\pi}[L]}\begin{bmatrix} j & j & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \times\langle T_L^M\rangle\left\{\frac{1+(-1)^L}{2}\right\}. \tag{13.117}$$

The C.G. coefficient ensures that the spherical harmonic moments with L > $2j$ vanish, and so the observation of a statistically significant non-vanishing average value of Y_L^M means that the spin of the γ resonance is at least $\frac{1}{2}L$.

The distribution of the longitudinal polarization of the spin-$\frac{1}{2}$ particle that comes from the decay is

$$P_l(\theta,\phi) = \frac{I_{\frac{1}{2}}(\theta,\phi) - I_{-\frac{1}{2}}(\theta,\phi)}{I_{\frac{1}{2}}(\theta,\phi) + I_{-\frac{1}{2}}(\theta,\phi)}. \tag{13.118}$$

The denominator $I_{\frac{1}{2}}(\theta,\phi) + I_{-\frac{1}{2}}(\theta,\phi)$ is just equal to $I(\theta,\phi)$. Hence

$$P_l(\theta,\phi)\,I(\theta,\phi) = I_{\frac{1}{2}}(\theta,\phi) - I_{-\frac{1}{2}}(\theta,\phi). \tag{13.119}$$

Using Eq. (13.105), we obtain the helicity distributions.

$$\begin{aligned} P_l(\theta,\phi)\,I(\theta,\phi) &= \sum_{LM}\frac{[j]}{\sqrt{4\pi}[L]}\langle T_L^M\rangle Y_L^{M*}(\theta,\phi)\left\{(-1)^{j-\frac{1}{2}}\begin{bmatrix} j & j & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}\right. \\ &\quad \left. -(-1)^{j+\frac{1}{2}}\begin{bmatrix} j & j & L \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}\right\}|H(\tfrac{1}{2},0)|^2 \\ &= \sum_{LM}\frac{[j]}{\sqrt{4\pi}[L]}\langle T_L^M\rangle Y_L^{M*}(\theta,\phi)(-1)^{j-\frac{1}{2}}\begin{bmatrix} j & j & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \\ &\quad \times\{1-(-1)^L\}\,|H(\tfrac{1}{2},0)|^2. \end{aligned} \tag{13.120}$$

After normalization, the longitudinal polarization of the angular distribution is

$$\begin{aligned} &\frac{\int P_l(\theta,\phi)\,I(\theta,\phi)\,Y_L^M(\theta,\phi)\,d\Omega}{\int I(\theta,\phi)\,d\Omega} \\ &\quad = \frac{[j]}{\sqrt{4\pi}[L]}(-1)^{j-\frac{1}{2}}\begin{bmatrix} j & j & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}\langle T_L^M\rangle\left\{\frac{1-(-1)^L}{2}\right\}. \end{aligned} \tag{13.121}$$

It is observed that the longitudinal polarization yields information about odd L multipole parameters while the particle distribution gives information about even L multipole parameters. These studies do not throw any light on the parity of the resonant (parent) state. Only the study of the transverse polarization of the decay products gives valuable information on the parity of the parent state.

Since we are considering two-body decay, of which one particle has spin-$\frac{1}{2}$ and the other spin zero, we need to consider only the transverse polarization of the spin-$\frac{1}{2}$ particle. The transverse polarization is the expectation value of σ_x or σ_y operator. Let us illustrate the method by calculating the x component of polarization.

$$P_x = \langle \sigma_x \rangle = \frac{\mathrm{Tr}\,(\sigma_x \rho_f)}{\mathrm{Tr}\,\rho_f}. \tag{13.122}$$

Equivalently,

$$P_x\, I(\theta, \phi) = \mathrm{Tr}\,(\sigma_x \rho_f). \tag{13.123}$$

To evaluate Tr $(\sigma_x \rho_f)$ we proceed in steps. First let us show that Tr $(\sigma_x \rho_f)$ is just the real part of the spin density matrix element $(\rho_f)_{\frac{1}{2},-\frac{1}{2}}$.

$$\begin{aligned}
\mathrm{Tr}\,(\sigma_x \rho_f) &= \tfrac{1}{2}\mathrm{Tr}\,\{(\sigma_+ + \sigma_-)\,\rho_f\} \\
&= \tfrac{1}{2}\left\{\sum_{\lambda\lambda'}(\sigma_+)_{\lambda\lambda'}(\rho_f)_{\lambda'\lambda} + \sum_{\lambda\lambda'}(\sigma_-)_{\lambda\lambda'}(\rho_f)_{\lambda'\lambda}\right\} \\
&= \tfrac{1}{2}\left\{\sum_{\lambda\lambda'}\delta_{\lambda',\lambda-1}(\rho_f)_{\lambda'\lambda} + \sum_{\lambda\lambda'}\delta_{\lambda'-1,\lambda}(\rho_f)_{\lambda'\lambda}\right\} \\
&= \tfrac{1}{2}\left\{\sum_{\lambda}(\rho_f)_{\lambda-1,\lambda} + \sum_{\lambda'}(\rho_f)_{\lambda',\lambda'-1}\right\} \\
&= \tfrac{1}{2}\sum_{\lambda}\{(\rho_f)_{\lambda-1,\lambda} + (\rho_f)_{\lambda,\lambda-1}\} \\
&= \mathrm{Re}\sum_{\lambda}(\rho_f)_{\lambda,\lambda-1}.
\end{aligned} \tag{13.124}$$

The last step is obtained by invoking the Hermitian property of the density matrix. For the spin-$\frac{1}{2}$ particle, the helicity can assume only two values $+\frac{1}{2}$ and $-\frac{1}{2}$ and hence λ in the above expression can take only one value $\frac{1}{2}$. Hence we obtain a simple result that

$$\mathrm{Tr}\,(\sigma_x \rho_f) = \mathrm{Re}\,(\rho_f)_{\frac{1}{2},-\frac{1}{2}}. \tag{13.125}$$

Using Eq. (13.69), we obtain (suppressing for the present the Euler angles of rotation (ϕ,θ,0) in the rotation matrix)

$$\begin{aligned}\mathrm{Tr}\,(\sigma_x\rho_f) &= \mathrm{Re}\sum_{mm'}\frac{2j+1}{4\pi}D^{j^*}_{m,\frac{1}{2}}\,D^{j}_{m',-\frac{1}{2}}H(\tfrac{1}{2},0)\,H(-\tfrac{1}{2},0)^*(\rho_i)_{mm'}\\ &= \mathrm{Re}\sum_{mm'}\frac{2j+1}{4\pi}(-1)^{m-\frac{1}{2}}D^{j}_{-m,-\frac{1}{2}}D^{j}_{m',-\frac{1}{2}}\,\epsilon\,|H(\tfrac{1}{2},0)|^2\,(\rho_i)_{mm'}\\ &= \mathrm{Re}\sum_{mm'}\frac{2j+1}{4\pi}(-1)^{m-\frac{1}{2}}\sum_{L}\left[\begin{array}{ccc} j & j & L\\ -m & m' & M\end{array}\right]\\ &\quad\times\left[\begin{array}{ccc} j & j & L\\ -\frac{1}{2} & -\frac{1}{2} & -1\end{array}\right]D^{L}_{M,-1}\,\epsilon\,|H(\tfrac{1}{2},0)|^2\,(\rho_i)_{mm'}.\end{aligned}\qquad(13.126)$$

The above result is obtained using the C.G. series for the coupling of the rotation matrices and the relation between the helicity amplitudes, viz., $H(-\frac{1}{2},0)=\epsilon\,H(\frac{1}{2},0)$ Expressing the density matrix of the initial resonant state in terms of the multipole parameters as given in Eq. (13.99),

$$(\rho_i)_{mm'}=\frac{1}{2j+1}\sum_{km_k}\langle T_k^{m_k}\rangle^*\left[\begin{array}{ccc} j & k & j\\ m' & m_k & m\end{array}\right][k],$$

it will be convenient to separate the terms that depend upon *m* and *m'* and perform the summation over *m* and replace the summation over *m'* by *M.*

$$\begin{aligned}&\sum_{mm'}(-1)^{m-\frac{1}{2}}\left[\begin{array}{ccc} j & j & L\\ -m & m' & M\end{array}\right]\left[\begin{array}{ccc} j & k & j\\ m' & m_k & m\end{array}\right]\\ &\qquad=(-1)^{2j-k}(-1)^{j-\frac{1}{2}}\frac{[j]}{[k]}\sum_{M}(-1)^{M}\,\delta_{L,k}\,\delta_{M,-m_k}.\end{aligned}\qquad(13.127)$$

Substituting the above result, we get after simplification

$$\begin{aligned}P_x\,I(\theta,\phi)=\mathrm{Tr}\,(\sigma_x\,\rho_f) &= \mathrm{Re}\sum_{LM}\frac{[j]}{4\pi}(-1)^{j-\frac{1}{2}}(-1)^{M}\left[\begin{array}{ccc} j & j & L\\ \frac{1}{2} & \frac{1}{2} & 1\end{array}\right]\\ &\quad\times D^{L}_{M,-1}(\phi,\theta,0)\,\langle T_L^{-M}\rangle^*\,\epsilon\,|H(\tfrac{1}{2},0)|^2.\end{aligned}\qquad(13.128)$$

In a similar way, one can calculate the transverse polarization P_y.

$$P_y\,I(\theta,\phi)=\mathrm{Tr}\,(\sigma_y\,\rho_f)=\frac{1}{2i}\mathrm{Tr}\,\{(\sigma_+-\sigma_-)\,\rho_f\}.\qquad(13.129)$$

Following the same procedure as before, we can show that

$$\begin{aligned} P_y I(\theta, \phi) &= -\mathrm{Im}\,(\rho_f)_{\frac{1}{2},-\frac{1}{2}} \\ &= -\mathrm{Im} \sum_{LM} \frac{[j]}{4\pi} (-1)^{j-\frac{1}{2}} (-1)^M \begin{bmatrix} j & j & L \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \\ &\quad \times D^L_{M,-1}(\phi, \theta, 0) \, \langle T_L^{-M} \rangle^* \, \epsilon \, |H(\tfrac{1}{2}, 0)|^2. \end{aligned} \tag{13.130}$$

Thus the study of the transverse polarization, which depends on ϵ, will yield the parity of the resonant state.

Hitherto, we have considered only the parity conserving two-body decay. For parity non-conserving weak decay such as the decay of hyperons, only small modifications are necessary. The interaction Hamiltonian, in this case, is a sum of two terms, one scalar H_e and the other pseudoscalar H_o.

$$H = H_e + H_o. \tag{13.131}$$

Under parity operation,

$$\mathcal{P}^{-1} H_e \mathcal{P} = H_e; \quad \mathcal{P}^{-1} H_o \mathcal{P} = -H_o. \tag{13.132}$$

This means that the matrix element is a sum of two terms,

$$H_e(\lambda_\alpha, \lambda_\beta) + H_o(\lambda_\alpha, \lambda_\beta), \tag{13.133}$$

where H_e and H_o obeys the following relations:

$$H_e(-\lambda_\alpha, -\lambda_\beta) = \eta_\alpha \eta_\beta \eta_\gamma (-1)^{j-s_\alpha-s_\beta}; \tag{13.134}$$

$$H_o(-\lambda_\alpha, -\lambda_\beta) = -\eta_\alpha \eta_\beta \eta_\gamma (-1)^{j-s_\alpha-s_\beta}. \tag{13.135}$$

To be specific, let us consider a weak decay of a hyperon into a baryon of spin-$\frac{1}{2}$ and a meson of spin zero. The various distributions involve the following combinations of $H_e(\frac{1}{2}, 0)$ and $H_o(\frac{1}{2}, 0)$:

$$\begin{aligned} a &= \frac{2\,\mathrm{Re}(H_e(\frac{1}{2},0)\, H_o^*(\frac{1}{2},0))}{|H_e(\frac{1}{2},0)|^2 + |H_o(\frac{1}{2},0)|^2}, \\ b &= \frac{2\,\mathrm{Im}\,(H_e(\frac{1}{2},0)\, H_o^*(\frac{1}{2},0))}{|H_e(\frac{1}{2},0)|^2 + |H_o(\frac{1}{2},0)|^2}, \\ c &= \frac{|H_e(\frac{1}{2},0)|^2 - |H_o(\frac{1}{2},0)|^2}{|H_e(\frac{1}{2},0)|^2 + |H_o(\frac{1}{2},0)|^2}. \end{aligned} \tag{13.136}$$

It is easy to observe that $a^2 + b^2 + c^2 = 1$. The various changes that occur in our earlier study of parity conserving two-body decay can easily be determined and we only quote the final results for normalized angular distribution and the longitudinal polarization. Equations (13.117) and (13.121) get modified to yield

$$\frac{\int I(\theta,\phi)\, Y_L^M(\theta,\phi)\, d\Omega}{\int I(\theta,\phi)\, d\Omega} = (-1)^{j-\frac{1}{2}} \frac{[j]}{\sqrt{4\pi}[L]} \begin{bmatrix} j & j & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \langle T_L^M \rangle \times \left\{ \frac{1+(-1)^L}{2} + \frac{a(1-(-1)^L)}{2} \right\}, \qquad (13.137)$$

$$\frac{\int P_l(\theta,\phi)\, I(\theta,\phi)\, Y_L^M(\theta,\phi)\, d\Omega}{\int I(\theta,\phi)\, d\Omega} = (-1)^{j-\frac{1}{2}} \frac{[j]}{\sqrt{4\pi}[L]} \begin{bmatrix} j & j & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \langle T_L^M \rangle \times \left\{ \frac{1-(-1)^L}{2} + \frac{a(1+(-1)^L)}{2} \right\}. \qquad (13.138)$$

The reader may note the interchange of the roles played by even and odd L in the above equations.

For the relativistic treatment of angular momentum states for three-body system and for the three-body decay, the reader is referred to Wick (1962) and Berman and Jacob (1965).

13.5. Muon Capture

We shall now apply the helicity formalism to discuss the capture of muon by spin-zero target nucleus,

$$\mu^- + A(j_i = 0) \rightarrow B(j_f \geq 1) + \nu_\mu, \qquad (13.139)$$

and investigate the asymmetry in the angular distribution of the recoil nucleus B and its polarization.

The usual source of muon is from π decay and it is polarized in the direction of its flight. When it is incident on a target, it is slowed down and caught in Bohr orbits. It cascades down to lower orbits emitting X-rays known as muonic X-rays and ultimately reaches the 1s orbit before it is captured by the nucleus through weak interaction. It is observed that depolarization takes place during the process of slowing down and cascading, but yet there is a residual polarization of order 15 to 20% in the 1s orbit at the time of capture by spin zero nucleus.

The muon polarization which coincides with the direction of incident muon is assumed as the z-axis of the rest frame of the initial system as shown in Fig. 13.4. This corresponds in the final state, to the centre of momentum system, with the recoil momentum $\boldsymbol{p} = -\nu$, making an angle

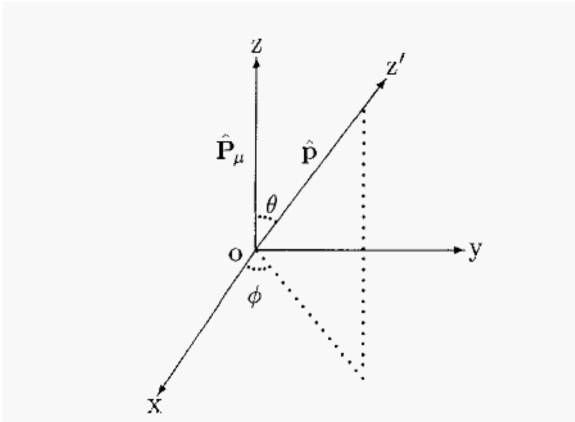

Figure 13.4. The muon polarization is along the z-axis of the rest frame of the muon-nucleon system and the momentum of the recoiling nucleus is along the z'-axis of the rotating frame which is otherwise called the helicity frame.

θ,ϕ with the z-axis. For describing this process, we have two frames of reference, one is the fixed frame of reference with z-axis in the direction of muon polarization and the other, the rotating frame of reference with z'-axis coinciding with the direction of recoiling nucleus. The latter frame of reference is obtained from the former by rotation through Euler angles (ϕ,θ,0)

Since the target nucleus is of zero spin, the total angular momentum of the initial system (μ- + A) is $\frac{1}{2}$ and is described by the state vector $|\frac{1}{2}\, m\rangle$. The final state is the recoiling nucleus B with spin j_f and helicity 1f, and the muon neutrino ν_μ with spin-$\frac{1}{2}$ and helicity $-\frac{1}{2}$. Expanding the final state in terms of definite angular momentum following Eq. (13.35),

$$|\theta, \phi; \lambda_f, -\tfrac{1}{2}\rangle = \sum_{jM} \sqrt{\frac{2j+1}{4\pi}} D^j_{M,\lambda}(\phi, \theta, 0)|jM; \lambda_f, -\tfrac{1}{2}\rangle, \qquad (13.140)$$

the transition amplitude can be obtained in the helicity basis.

$$\begin{aligned} f_{\lambda m} &= \langle \theta\phi; \lambda_f, -\tfrac{1}{2}\,|H|\,\tfrac{1}{2}m\rangle \\ &= \sum_{jM} \sqrt{\frac{2j+1}{4\pi}} D^{j\,*}_{M\lambda}(\phi, \theta, 0)\,\langle jM; \lambda_f, -\tfrac{1}{2}\,|H|\,\tfrac{1}{2}m\rangle. \end{aligned} \qquad (13.141)$$

Since H is a scalar under rotation, $j = \frac{1}{2}$ and $M = m$, there can be only two partial wave helicity amplitudes $\langle \frac{1}{2}M; \lambda_f, -\frac{1}{2}|H|\frac{1}{2}m\rangle$ corresponding to the total angular momentum $\frac{1}{2}$. These partial wave helicity amplitudes will hereafter be represented by H_λ where $\lambda = \lambda_f + \frac{1}{2} = \pm\frac{1}{2}$. Thus,

$$f_{\lambda m} = \sqrt{\frac{1}{2\pi}}\, D^{\frac{1}{2}\,*}_{m\lambda}(\phi,\theta,0)\, H_\lambda. \tag{13.142}$$

The elements of the density matrix for the final system is given by

$$(\rho_f)_{\lambda,\lambda'} = \sum_m f_{\lambda m}\,(\rho_i)_{mm'}\, f^*_{\lambda' m'}, \tag{13.143}$$

where ρ_i denotes the density matrix for the initial system which is taken to be in the diagonal form in the rest frame.

$$\rho_i = \tfrac{1}{2}(1 + \boldsymbol{\sigma}\cdot\boldsymbol{P}_\mu) = \tfrac{1}{2}(1 + 2\,\boldsymbol{s}\cdot\boldsymbol{P}_\mu), \tag{13.144}$$

where σ denotes the Pauli spin operator, $\boldsymbol{s}$ the spin of the muon and $\boldsymbol{P}_\mu$ the polarization of the muon which is in the z direction. Substituting the eigenvalue of s_z in the density matrix of the initial state,

$$(\rho_f)_{\lambda,\lambda'} = \tfrac{1}{2}\sum_m f_{\lambda m}\,(1 + 2mP_\mu)\, f^*_{\lambda' m}. \tag{13.145}$$

Using Eq. (13.142) and the explicit form of $D^{\frac{1}{2}}$ rotation matrices, we obtain the following results:

$$\begin{aligned}
\sum_m f_{\lambda m}\, f^*_{\lambda' m} &= \frac{1}{2\pi}\sum_m D^{\frac{1}{2}\,*}_{m\lambda}(\phi,\theta,0)\, D^{\frac{1}{2}}_{m\lambda'}(\phi,\theta,0)\, H_\lambda\, H^*_{\lambda'} \\
&= \frac{1}{2\pi}\sum_m d^{\frac{1}{2}\,\dagger}_{\lambda m}(\theta)\, d^{\frac{1}{2}}_{m\lambda'}(\theta)\, H_\lambda\, H^*_{\lambda'} \\
&= \frac{1}{2\pi}\,|H_\lambda|^2\,\delta_{\lambda,\lambda'}. &\quad (13.146) \\
\sum_m f_{\lambda m}\, m\, f^*_{\lambda' m} &= \frac{1}{2\pi}\sum_m D^{\frac{1}{2}\,*}_{m\lambda}(\phi,\theta,0)\, m\, D^{\frac{1}{2}}_{m\lambda'}(\phi,\theta,0)\, H_\lambda\, H^*_{\lambda'} \\
&= \frac{1}{4\pi}\Big\{2\,\lambda\,|H_\lambda|^2\cos\theta\,\delta_{\lambda,\lambda'} - H_{\frac{1}{2}}\, H^*_{-\frac{1}{2}}\sin\theta\,\delta_{\lambda-\lambda',1} \\
&\qquad - H_{-\frac{1}{2}}\, H^*_{\frac{1}{2}}\sin\theta\,\delta_{\lambda-\lambda',-1}\Big\}. &\quad (13.147)
\end{aligned}$$

Consolidating the above results, we obtain the angular distribution of the recoil nucleus $I(\theta,\phi)$.

$$I(\theta,\phi) = \mathrm{Tr}\,\rho_f = \frac{1}{4\pi}\sum_\lambda |H_\lambda|^2 + \frac{1}{2\pi}\sum_\lambda \lambda|H_\lambda|^2\, P_\mu\cos\theta. \tag{13.148}$$

Writing it in a more compact form,

$$I(\theta, \phi) = \frac{\Gamma}{4\pi}\,\Lambda(\theta), \tag{13.149}$$

with

$$\Gamma = \sum_{\lambda} |H_\lambda|^2, \quad \Lambda(\theta) = 1 + \alpha\, P_\mu \cos\theta, \tag{13.150}$$

we find the asymmetry coefficient of the recoil angular distribution to be

$$\alpha = 2\,\frac{\sum_\lambda \lambda |H_\lambda|^2}{\sum_\lambda |H_\lambda|^2}. \tag{13.151}$$

The quantity Γ represents the capture rate.

The longitudinal polarization of the recoil nucleus is

$$\begin{aligned} P_L &= \frac{\mathrm{Tr}(\boldsymbol{J}\cdot\boldsymbol{p})\,\rho_f}{\mathrm{Tr}\,\rho_f} \\ &= \frac{\sum_{\lambda,\lambda'} (\boldsymbol{J}\cdot\boldsymbol{p})_{\lambda,\lambda'}\,(\rho_f)_{\lambda',\lambda}}{\sum_\lambda (\rho_f)_{\lambda,\lambda}}. \end{aligned} \tag{13.152}$$

Since

$$(\boldsymbol{J}\cdot\boldsymbol{p})_{\lambda,\lambda'} = \lambda_f\,\delta_{\lambda,\lambda'} = (\lambda - \tfrac{1}{2})\,\delta_{\lambda,\lambda'}, \tag{13.153}$$

the longitudinal polarization becomes

$$P_L = \frac{\sum_\lambda (\lambda - \frac{1}{2})\,(\rho_f)_{\lambda,\lambda}}{\sum_\lambda (\rho_f)_{\lambda,\lambda}}. \tag{13.154}$$

In the absence of muon polarization ($P_\mu = 0$),

$$P_L^0 = \frac{\sum_\lambda (\lambda - \frac{1}{2})|H_\lambda|^2}{\sum_\lambda |H_\lambda|^2} = \frac{\alpha}{2} - \frac{1}{2}. \tag{13.155}$$

Thus we arrive at a well known relation for the observables in muon capture.

$$\alpha - 2\,P_L^0 = 1. \tag{13.156}$$

Since the muon capture process is completely described by two helicity amplitudes $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$, all the observables in muon capture can be expressed in terms of these amplitudes and their relative phase. Hence it follows that there cannot be more than three independent observables in muon capture. For further details of helicity formalism as applied to muon capture, the reader is referred to Bernabeu (1975) and Subramanian et al. (1976, 1979).

Review Questions

13.1 (a) Write down the non-interacting two-particle wave function in terms of the plane wave helicity basis and the angular momentum basis and obtain the transformation from one basis to the other.
(b) Discuss the advantages of using the helicity formalism for the study of two-particle scattering and obtain expressions for the angular distributions and polarization of the scattered particles.

13.2 (a) Consider the two-body decay of a resonant state and deduce an expression for the angular distribution of the decay products in terms of the decay products in terms of the statistical parameters $\langle T_k^{\mu} \rangle$ defining the initial system. Also find the spherical harmonic moments of the angular distribution.
(b) Apply the above consideration to the decay of a resonant state into (i) two spinless particles and (ii) one spin-$\frac{1}{2}$ and the other spin-zero particle.

13.3 Discuss how is it possible to determine the spin and parity of a resonant state by observing the angular distributions and polarization of the decay products. Restrict your considerations to the decay into two particles.

13.4 Consider muon capture by a spin-zero target nucleus and show that the asymmetry in the angular distribution of the final nucleus with respect to the polarization vector of the initial muon is related to longitudinal polarization of the final nucleus by a simple relation $\alpha - 2P_L^0 = 1$, where α denotes the asymmetry coefficient and P_L^0 denotes the longitudinal polarization of the final nucleus for muon polarization zero.

Problems

13.1 If a particle with spin *j* has transverse polarization, show that

$$\begin{aligned} \langle J_x \rangle &= \{(j+\lambda)(j-\lambda+1)\}^{\frac{1}{2}} \,\mathrm{Re}\, \rho_{\lambda-1,\lambda}, \\ \langle J_y \rangle &= \{(j+\lambda)(j-\lambda+1)\}^{\frac{1}{2}} \,\mathrm{Im}\, \rho_{\lambda-1,\lambda}. \end{aligned}$$

13.2 A particle with spin *s* is transversally polarized. If the transverse polarization is denoted by $\langle s_y \rangle$, then show that its spin density matrix is given by

$$\rho = \frac{1}{2s+1}\left(1 + \sqrt{\frac{3}{s(s+1)}} \langle s_y \rangle \, s_y\right).$$

Show that the density matrix reduces to the familiar formula

$$\rho = \tfrac{1}{2}(1 + \boldsymbol{\sigma} \cdot \boldsymbol{P})$$

for the spin-$\frac{1}{2}$ particle with vector polarization ***P***.

13.3 Discuss the pion-nucleon and nucleon-nucleon scattering using the helicity formalism and enumerate the number of independent scattering amplitudes in each case.

13.4 Discuss the following decays

$$\begin{aligned} &(a)\ Y^*(1385\,\mathrm{MeV}) && \to && \Lambda\,\pi, \\ &(b)\ \Xi^*(1530\,\mathrm{MeV}) && \to && \Xi\,\pi, \end{aligned}$$

and explain how you will determine the spin and parity of the parent systems. (These are parity conserving decays through strong interaction. The spin of the hyperons Λ and Ξ is $\frac{1}{2}$ and the spin of π is zero.)

Solutions to Selected Problems

13.1 The transverse polarization of a particle with spin j is the expectation value of the operators J_x and J_y.

$$\begin{aligned} \langle J_x \rangle &= \mathrm{Tr}\,(J_x\,\rho) = \frac{1}{2}\mathrm{Tr}\,\{(J_+ + J_-)\,\rho\} \\ &= \frac{1}{2}\left\{\sum_{\lambda,\lambda'}(J_+)_{\lambda,\lambda'}(\rho)_{\lambda',\lambda} + \sum_{\lambda,\lambda'}(J_-)_{\lambda',\lambda}\,(\rho)_{\lambda,\lambda'}\right\} \\ &= \frac{1}{2}\left\{\sum_{\lambda,\lambda'}\{(j-\lambda')(j+\lambda'+1)\}^{\frac{1}{2}}\,\delta_{\lambda,\lambda'+1}(\rho)_{\lambda',\lambda}\right. \\ &\qquad \left. + \sum_{\lambda,\lambda'}\{(j+\lambda)(j-\lambda+1)\}^{\frac{1}{2}}\,\delta_{\lambda',\lambda-1}\,(\rho)_{\lambda,\lambda'}\right\} \\ &= \frac{1}{2}\sum_{\lambda}\{(j+\lambda)(j-\lambda+1)\}^{\frac{1}{2}}\,\{(\rho)_{\lambda-1,\lambda} + (\rho)_{\lambda,\lambda-1}\}\,. \end{aligned}$$

Since ρ is a Hermitian matrix, it follows that

$$\langle J_x \rangle = \sum_{\lambda}\{(j+\lambda)(j-\lambda)+1\}^{\frac{1}{2}}\mathrm{Re}\,(\rho)_{\lambda-1,\lambda}.$$

Since $J_y = \frac{1}{2i}(J_+ - J_-)$, it can be shown in a similar manner that

$$\langle J_y \rangle = \sum_{\lambda}\{(j+\lambda)(j-\lambda+1)\}^{\frac{1}{2}}\mathrm{Im}\,(\rho)_{\lambda-1,\lambda}.$$

13.2 Retaining only the first order term and neglecting higher order tensor orientations, the density matrix can be written as

$$\rho = \frac{1}{2s+1}\left(1 + \langle T_1^\mu \rangle T_1^\mu\right),$$

where the tensor operator T_1^μ is normalized such that

$$\mathrm{Tr}\,(T_1^{\mu\dagger} T_1^{\mu'}) = (2s+1)\,\delta_{\mu,\mu'}.$$

The normalized T_1^μ operator is

$$T_1^\mu = \sqrt{\frac{3}{s(s+1)}}\; s_1^\mu .$$

Substituting it in the expression for ρ, we get

$$\rho = \frac{1}{2s+1}\left(1 + \frac{3}{s(s+1)}\langle s_1^\mu \rangle\, s_1^\mu\right).$$

For spin-$\frac{1}{2}$ particle, the density matrix reduces to

$$\begin{aligned}\rho &= \tfrac{1}{2}(1 + 4s^\mu \langle s^\mu \rangle) \\ &= \tfrac{1}{2}(1 + \sigma^\mu \langle \sigma^\mu \rangle) \\ &= \tfrac{1}{2}(1 + \boldsymbol{\sigma}\cdot\boldsymbol{P})\end{aligned}$$

13.3 For each partial wave scattering amplitude, the number of helicity amplitudes is $(2s_a + 1)(2s_b + 1)(2s_c + 1)(2s_d + 1)$. But by the application of invariance and symmetry principles, the number of independent amplitudes is considerably reduced.
For pion-nucleon scattering, the number of helicity amplitudes is 4, since the pion spin is zero and the nucleon spin is $\frac{1}{2}$. Explicitly, the amplitudes are

$$\begin{array}{llll}\text{(i)} & \langle 0, \frac{1}{2}|T|0, \frac{1}{2}\rangle, & \text{(ii)} & \langle 0, \frac{1}{2}|T|0, -\frac{1}{2}\rangle, \\ \text{(iii)} & \langle 0, -\frac{1}{2}|T|0, \frac{1}{2}\rangle, & \text{(iv)} & \langle 0, -\frac{1}{2}|T|0, -\frac{1}{2}\rangle.\end{array}$$

By application of parity conservation, the helicity amplitudes (i) and (iv) are equal and (ii) and (iii) are equal. The application of time reversal invariance implies that amplitudes (ii) and (iii) are equal and so it does not give any new relation. Hence the number of independent amplitudes required for describing the pion-nucleon scattering is only two.

For describing the nucleon-nucleon scattering, the total number of helicity amplitudes required is 16, since the nucleon has spin-$\frac{1}{2}$. The parity invariance reduces the number of independent helicity amplitudes from 16 to 8 and the time reversal invariance reduces further the number of independent helicity amplitudes from 8 to 6. By invoking the relation for the identical particles, the number is further reduced to 5. The five independent partial wave helicity amplitudes are given below in a matrix form.

	$++$	$+-$	$-+$	$--$
$++$	f_1^J	f_5^J	f_5^J	f_2^J
$+-$	f_5^J	f_3^J	f_4^J	f_5^J
$-+$	f_5^J	f_4^J	f_3^J	f_5^J
$--$	f_2^J	f_5^J	f_5^J	f_1^J

The rows and columns denote the helicity states of the final and initial systems, using for brevity + for $+\frac{1}{2}$ and - for $-\frac{1}{2}$ helicity states. For instance, in the table, f_1^J denotes the helicity amplitude

$$f_1^j = \frac{1}{2p}\langle +\tfrac{1}{2}, +\tfrac{1}{2}| T^j | +\tfrac{1}{2}, +\tfrac{1}{2}\rangle = \frac{1}{2p}\langle -\tfrac{1}{2}, -\tfrac{1}{2}| T^j | -\tfrac{1}{2}, -\tfrac{1}{2}\rangle.$$