Removing Infrared Divergences

Summing Soft Photons

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Overview

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- Soft Photons
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Divergences

- Divergences occur when Feynman diagrams (integrals) and cross-sections have terms that tend to infinity in high energy (or short range) or low energy (or long distance) limits.
- Low energy, infrared, divergences occur when there are massless or effectively massless particles.
- High energy, ultraviolet, divergences arise in many situations. From terms that scale as energy and give infinite integrals to examination of theories at small distances.

We're interested in IR divergences at the moment, but as a taste...

- Ultraviolet divergences are removed through renormalization and regularization.
- This can mean the isolation and reworking of the divergent term, instituting a cutoff as to only use the theory at physically reasonable scales, or instituting a regulator.

IR divergences are very interesting from a physical perspective and can be understood because,

"In any practical experiment involving charged particles it is impossible to specify completely the final state of the system. Because individual photons can be emitted with arbitrarily small energies, there will always be a possibility that some photons will escape detection." [5]

Divergences

Infrared Divergences

From a mathematical standpoint we see that infrared divergences appear when a particle has no mass, for instance in the electron self energy we have

$$\int \frac{d^4k}{(2\pi)^4}\gamma^\mu \frac{i(\not{q}-\not{k}+m)}{(q-k)^2-m^2+i\epsilon}\gamma^\mu \frac{1}{k^2} \tag{1}$$

And we can see that in the limit as $k \to 0$ the integral goes to infinity.

IR Regulators

Cutoffs

IR divergences, like UV divergences, must be regularizeable, so the non-physically observable parts must cancel out. (All physical observables must be finite.)

There are a few ways of managing these divergences. One is to institute a cutoff, or limit, such that you only integrate in a regime in which your integral is finite and corresponds to what you physically observe.

IR Regulators

Adding m_γ

Another way to remove divergence is to add a regulator. In this case we can for instance add a photon mass. If we add a fictitious photons mass, for instance in the electron self energy, the singularity disappears,

$$\int \frac{d^4k}{(2\pi)^4}\gamma^\mu \frac{i(\not{q}-\not{k}+m)}{(q-k)^2-m^2+i\epsilon}\gamma^\mu \frac{1}{k^2-m_\gamma^2}. \tag{2}$$

This mass should be removeable though since we do not observe a massive photon. In other words in limits or in consideration of all factors, nothing observable can be left with a dependence on the photon mass.

Summing Soft Photons

The IR divergences can be seen as arising from the incomplete consideration of all the factors in a cross-section. An arbitrary number of soft photons may exist in the final state. So it can be shown that even though $e^+e^- \to \mu^+\mu^-$ is divergent and so is $e^+e^- \to \mu^+\mu^-\gamma$, their sum is IR finite.

The cross section is shown to be

$$\begin{aligned} &\sigma(e^+e^- \to \mu^+\mu^-(+\gamma)) \\ &\qquad \equiv \sigma(e^+e^- \to \mu^+\mu^-) + \sigma(e^+e^- \to \mu^+\mu^-\gamma) \\ &\qquad\qquad = \sigma_0 \left\{ 1 + \frac{3e_R^2}{16\pi^2} \right\}. \end{aligned} \tag{3}$$

Bremsstrahlung

A quick example of how exactly this works is with bremsstrahlung.

In the process of elastic scattering of an electron with a static electric field there are two processes. One is the elastic scattering itself and the other is soft bremsstrahlung. When taken as separate processes in perturbation theory, IR divergences emerge, but when combined they cancel. This makes sense since they really should be treated as one process experimentally.

Let's look at the example of $e^+e^- \to \mu^+\mu^-(+\gamma)$

To leading order $e^+e^- \to \mu^+\mu^-$ only has one s-channel diagram. It's cross section is

$$\sigma_0 = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \, \frac{d\sigma}{d\Omega} = \frac{e_R^4}{12\pi Q^2} \tag{4}$$

where $Q^2 = s = E_{CM}^2$.

Now we need to calculate the vertex correction. This is a lengthy derivation I am not going to go through in detail. It involves regulating a UV divergence as well. The adding and regularization with m_γ is covered in chapter 19 of Schwartz. The result of the loop correction to order e_R^6 is

$$\sigma_V = \frac{e_R^2}{8\pi^2}\sigma_0 \left\{ -\ln^2 \frac{m_\gamma^2}{Q^2} - 3\ln\frac{m_\gamma^2}{Q^2} - \frac{7}{2} + \frac{\pi^2}{3} \right\} \tag{5}$$

where σ_V is the virtual cross section correction. The squared logarithm is characteristic of IR divergences and is called the "Sudakov double logarithm."

$e^+e^- \to \mu^+\mu^-(+\gamma)$

Real Emission

Again skipping the details of the derivation, the real emission diagrams give,

$$\sigma_R = \frac{e_R^2}{8\pi^2}\sigma_0 \left\{ \ln^2 \frac{m_\gamma^2}{Q^2} + 3\ln\frac{m_\gamma^2}{Q^2} - \frac{\pi^2}{3} + 5 \right\} \tag{6}$$

combining this with the virtual correction

$$\sigma_R + \sigma_V = \frac{3e_R^2}{16\pi^2}\sigma_0 \tag{7}$$

and thus the total cross section is

$$\sigma_{tot} = \sigma_0 \left\{ 1 + \frac{3e_R^2}{16\pi^2} \right\} \tag{8}$$

and we see that amazingly the divergences cancelled!

Jets

The idea of having an arbitrary number of soft photons (or collinear photons) in the final state leads to the idea of jets. Very briefly, there is a chance that an observable final state photon jet is produced. This contribution is dependent on the angular and energy resolution of the experiment.

Schwartz notes that “In physical cross sections, an experimental resolution parameter acts as an IR regulator.” [1] So we have no need for a photon mass at all here, but the calculations are easier when done with it rather than using the parameters of an experiment.

Jets

For example in the case we looked at we know the total cross section is made of two parts

$$\sigma_{tot} = \sigma_{2\to 2} + \sigma_{2\to 3}. \tag{9}$$

But we have limitations on the energy and the angle measurable by the experiment, so the resolution of the energy and angle become the regulators. When $E_\gamma < E_{res}$ and $\theta_{\gamma\mu} < \theta_{res}$ the rate looks like a $\mu^+\mu^-$ pair is produced. But when $E_\gamma > E_{res}$ and $\theta_{\gamma\mu} > \theta_{res}$ then the rates $\sigma_{2\to 2}$ and $\sigma_{2\to 3}$ are dependent on the resolution. The two-body jet final state is thus

$$\sigma_{2\to 2} = \sigma_0 \left(1 - \frac{e_R^2}{8\pi^2}\left\{\ln\frac{1}{\theta_{res}}\left[\ln\left(\frac{Q}{2E_{res}} - 1\right) - \frac{3}{4} + 3\frac{E_{res}}{Q}\right] + \cdots\right\}\right) \tag{10}$$

Going Further

A few things beyond the scope of this discussion:

- Other loops
- Vacuum polarization correction - scale dependent effective charge
- Initial state radiation
- Summing higher order soft photons
- Bloch-Nordsieck theorem
- Kinoshita-Lee-Nauenberg theorem

Summary

- IR divergences arise from massless or effectively massless particles
- These divergences can be removed by adding a fictitious mass, a cutoff, or by summing over all diagrams including soft final state radiation.
- Looking at the relatively simple $e^+e^- \to \mu^+\mu^-(+\gamma)$ interaction we see that the logarithmic photon mass terms cancel exactly.
- Collinear or soft photons give rise to jets, which will depend on the angular and energetic resolution of the detector.

Applications

For further study

From the KLN theorem comes the idea that any physical observable must be "infrared safe." Infrared safety pertains to the Jets we saw as well as a number of other topics:

- Decay widths
- Protons in QCD
- Final state energy cuts
- Lamb Shift

References

[1] Schwartz, Matthew D., *Quantum Theory and the Standard Model.* New York: Cambridge University Press, 2014. Print.

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[5] D. Yennie, S. Frautschi, and H. Suura, *Ann, Phys*. **13**, 379 (1961).

[6] Schwartz, Matthew. *Lecture 19: Infrared Divergences*. Lecture for QFT 1, Harvard, 2009. (http://isites.harvard.edu/fs/docs/icb.topic792163.files/19-infrared.pdf).