DUE: THURSDAY, FEBRUARY 11, 2016

1. Consider a field theory of a real pseudoscalar field $\phi(x)$ coupled to the electron field $\psi(x)$. The interaction Lagrangian is:

$$\mathscr{L}_{\rm int} = -i\lambda\,\overline{\psi}(x)\,\gamma_5\,\psi(x)\phi(x)\,,$$

where λ is a real coupling constant (called the Yukawa coupling).

(a) Using functional techniques, derive the Feynman rule for the interaction vertex of this theory.

(b) Calculate the $\mathcal{O}(\lambda^2)$ contribution of the pseudoscalar to the anomalous magnetic moment of the electron.

2. Consider the function of a *real* parameter z

$$F(z) \equiv \int_0^1 dx\, \ln\bigl[1 - zx(1-x) - i\epsilon\bigr]\,,$$

which appears in the computation of the one-loop correction to the 4-point Green function in scalar field theory.

(a) Evaluate Im $F(z)$. For what values of z does $\operatorname{Im} F$ vanish?

HINT: First, determine the imaginary part of the integrand. Note that $\ln\bigl[1 - zx(1-x) - i\epsilon\bigr]$ should be interpreted as the principal value of the complex-valued logarithm, with the branch cut along the negative real axis. Since ϵ is a positive infinitesimal, the sign of the imaginary part is uniquely determined. Then, carry out the integration, noting that the imaginary part of the integrand may vanish over part (and in some cases all) of the integration range.

(b) Consider the 1PI 4-point Green function, $\Gamma^{(4)}$, in a field theory of a real scalar field with mass m and an interaction Lagrangian density given by $\mathscr{L}_I = -\lambda\phi^4/4!$. Using the Feynman rules for this theory, write down an integral expression for the full $\mathcal{O}(\lambda^2)$ contribution to $\Gamma^{(4)}$. From the integral expression, evaluate $\operatorname{Im}\Gamma^{(4)}$ up to order λ^2 by making use of the *cutting rules* given in Section 24.1.2 [pp. 456–459] of Schwartz.

(c) An explicit one-loop computation of $\Gamma^{(4)}$ yields

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = -\lambda - \frac{\lambda^2}{32\pi^2}\left[F\left(\frac{s}{m^2}\right) + F\left(\frac{t}{m^2}\right) + F\left(\frac{u}{m^2}\right) + G(m^2)\right]\,, \tag{1}$$

where $s \equiv (p_1+p_2)^2$, $t \equiv (p_1-p_3)^2$, $u \equiv (p_1-p_4)^2$ are Lorentz-invariant kinematic variables, the function F is defined in part (a), and the function G is a real function.[1] Using eq. (1) and the results of part (a), compute $\operatorname{Im}\Gamma^{(4)}$ and check that your calculation in part (b) is correct.

[1] In fact, the function G is infinite, but this infinity can be removed by renormalization. Since we are only interested here in $\operatorname{Im}\Gamma^{(4)}$, we can safely ignore any details associated with the renormalization procedure.

3. The Lagrangian of QED is given by:

$$\mathscr{L}_{\rm QED} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ + eA\!\!\!/)\psi - m\overline{\psi}\psi - \frac{1}{2a}(\partial_\mu A^\mu)^2 \,.$$

(a) Compute the tree-level photon propagator (in momentum space).

(b) Show that this Lagrangian is not invariant under the infinitesimal gauge transformation:

$$\begin{aligned} \delta\psi &= ie\Lambda(x)\psi(x)\,, \\ \delta A_\mu &= \partial_\mu\Lambda(x)\,, \end{aligned}$$

where $\Lambda(x)$ is an arbitrary real function of x that vanishes (sufficiently fast) as $|\vec{\boldsymbol{x}}| \to \infty$.

(c) Consider the modified Lagrangian:

$$\mathscr{L} = \mathscr{L}_{\rm QED} + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi\,,$$

where ϕ is a free scalar field. Show that the action is invariant under the generalized (infinitesimal) gauge transformation:

$$\begin{aligned} \delta\psi &= ie\epsilon\,\phi(x)\psi(x)\,, \\ \delta A_\mu &= \epsilon\,\partial_\mu\phi(x)\,, \\ \delta\phi &= -\frac{\epsilon}{a}\,\partial_\mu A^\mu\,, \end{aligned}$$

where ϵ is an infinitesimal parameter. This has a name: it is called the BRST-transformation. The action is therefore said to be BRST-invariant.

4. Consider the Lagrangian for a non-abelian gauge theory, with gauge field A^a_μ and gauge field strength tensor $F^a_{\mu\nu}$,

$$\mathscr{L}_{\rm YM} = \tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}\,,$$

which is invariant under the gauge transformation:

$$\delta A^a_\mu(x) = \epsilon D^{ab}_\mu\omega_b(x)\,, \tag{2}$$

where ϵ is infinitesimal, D_μ is the covariant derivative, and $\omega(x)$ is an arbitrary function of x.

(a) In order to be able to define a propagator for the gauge field, we must add a gauge-fixing term:

$$\mathscr{L}_{\rm GF} = -\frac{1}{2a}(\partial^\mu A^a_\mu)^2\,. \tag{3}$$

Show that under the gauge transformation of eq. (2), the gauge invariance is broken due to an extra term generated:

$$\delta\mathscr{L}_{\rm GF} = -\frac{\epsilon}{a}(\partial^\mu A^a_\mu)(\partial^\nu D^{ab}_\nu\omega_b)\,.$$

(b) Attempt to restore the symmetry by adding a new field $\eta(x)$ and a new term to the Lagrangian:

$$\mathscr{L}_{\rm G} = -\eta_a(\partial^\mu D_\mu^{ab}\omega_b)\,,$$

and by postulating the transformation law:

$$\delta\eta_a = -\frac{\epsilon}{a}(\partial_\mu A_a^\mu)\,. \tag{4}$$

Show that this does not quite work because D_μ is field dependent and:

$$\delta(\mathscr{L}_{\rm GF} + \mathscr{L}_{\rm G}) \neq 0\,.$$

(c) Save the day by promoting ω to a field and postulating the transformation law:

$$\delta\omega_a = \tfrac{1}{2}\epsilon g f_{abc}\omega_b\omega_c\,, \tag{5}$$

where g is the Yang-Mills coupling constant and the f_{abc} are the structure constants of the gauge group. Summation over repeated indices is implied. Note that since the f_{abc} are totally antisymmetric under interchange of a, b and c, the only way to have $\delta\omega \neq 0$ is to require that ω is an anticommuting field. This immediately implies that η is an anticommuting field and ϵ is an anticommuting infinitesimal constant. With this in mind, show that $\mathscr{L}_{\rm YM} + \mathscr{L}_{\rm GF} + \mathscr{L}_{\rm G}$ is invariant under the transformation laws given by eqs. (2)–(5). This enlarged gauge invariance is called BRST invariance (and δ is called an infinitesimal BRST transformation).

(d) Define δ^2 to mean the application of δ with anti-commuting parameter ϵ_1 followed by δ with anti-commuting parameter ϵ_2. Show that when δ^2 is applied to A_μ^a and η_a, the result is zero in each case. However $\delta^2\,\eta_a = 0$ only if the Lagrange field equations for the ghost fields ω_a are satisfied.

(e) Suppose that the gauge fixing term is chosen to be

$$\mathscr{L}_{\rm GF} = B_a\partial_\mu A_a^\mu + \frac{a}{2}B_aB_a\,. \tag{6}$$

Note that the new field B_a has no kinetic energy term; it is thus an auxiliary field. Show that if one solves for B_a using the Lagrange field equations, one regains the usual gauge fixing term given by eq. (3).

(f) Using the new gauge fixing term given in eq. (6), we now modify the BRST transformation law of η and define:

$$\begin{aligned}\delta\,\eta_a &= \epsilon B_a\,,\\ \delta\,B_a &= 0\,.\end{aligned}$$

Show that the full Lagrangian is still invariant under the BRST transformation. Furthermore, verify that $\delta^2 = 0$ when applied to *all* fields of the theory, independently of the field equations.