DUE: TUESDAY, MARCH 1, 2016

1. In class, I defined the matrix-valued covariant derivative operator in the adjoint representation, $\mathscr{D}_\mu$, by

$$\mathscr{D}_\mu V_\nu \equiv (D_\mu V_\nu)_a T^a = \partial_\mu V_\nu + ig[A_\mu\,,\,V_\nu]\,,$$

where $V_\nu \equiv V_\nu^a T^a$ is a matrix-valued adjoint field and $(D_\mu)_{ab} \equiv \delta_{ab}\partial_\mu + g f_{cab} A^c_\mu$ is the covariant derivative acting on a field in the adjoint representation. The commutation relations satisfied by the generators of the Lie group G are given by $[T_a\,,\,T_b] = i f_{abc} T_c$, and the indices a, b and c take on d_G possible values, where d_G is the dimension of G.

(a) Prove that for any pair of matrix-valued adjoint fields V and W,

$$[\mathscr{D}_\mu\,,\,V]\,W = (\mathscr{D}_\mu V)W\,,$$

where $[\quad,\quad]$ is the usual matrix commutator. This means that $\mathscr{D}_\mu V = [\mathscr{D}_\mu\,,\,V]$ holds as an operator equation.

(b) Prove that for any matrix-valued adjoint field V,

$$[\mathscr{D}_\mu\,,\,\mathscr{D}_\nu]V = ig[F_{\mu\nu}\,,\,V]\,,$$

where $F_{\mu\nu} \equiv F^a_{\mu\nu} T^a$ is the matrix-value field strength tensor of the non-abelian gauge theory.

(c) Starting from the non-abelian Maxwell equation,

$$\mathscr{D}_\mu\, F^{\mu\nu} = j^\nu\,,$$

prove that the current j^ν is covariantly conserved. That is,

$$\mathscr{D}_\mu j^\mu = 0\,.$$

2. (a) Compute the differential cross section at $\mathcal{O}(\alpha_s^2)$ for $q\bar{q} \to t\bar{t}$ (where $q \neq t$ is any light quark and t is the top quark), in terms of the center-of-mass energy $\sqrt{s}$ and the squared four-momentum transfer t. Integrate your result over t to obtain the total cross section as a function of the squared center-of-mass energy s. In your calculation, average over initial colors and spins and sum over final colors and spins. You may assume that the initial quark and anti-quark are massless, but do *not* neglect the mass of the top-quark.

(b) Compute the differential cross section at $\mathcal{O}(\alpha_s^2)$ for $gg \to t\bar{t}$, where g is a gluon, in terms of the squared center-of-mass energy $\sqrt{s}$ and the squared four-momentum transfer t. Integrate your result over t to obtain the total cross section as a function of s. In your calculation, average over initial colors and spins and sum over final colors and spins.

3. Consider the following model Lagrangian density for a theory with two real scalar fields ϕ_1 and ϕ_2 and a Dirac fermion field ψ,

$$\mathscr{L} = \tfrac{1}{2}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right] + \tfrac{1}{2}\mu^2\left(\phi_1^2 + \phi_2^2\right) - \tfrac{1}{4}\lambda\left(\phi_1^2 + \phi_2^2\right)^2 + i\overline{\psi}\partial\!\!\!/\psi - g\overline{\psi}(\phi_1 + i\gamma_5\phi_2)\psi\,,$$

where the parameters μ^2 and λ are assumed to be positive.

(a) Show that this theory possesses the following global symmetry,

$$\begin{aligned}
\phi_1 &\to \phi_1\cos\alpha - \phi_2\sin\alpha \\
\phi_2 &\to \phi_1\sin\alpha + \phi_2\cos\alpha \\
\psi &\to \exp\left\{-\tfrac{1}{2}i\alpha\gamma_5\right\}\psi\,.
\end{aligned}$$

Show that the solution to the classical field equations with the minimum energy breaks this symmetry spontaneously.

(b) Without loss of generality, one can assume that ϕ_1 possesses a non-zero vacuum expectation value, $\langle\phi_1\rangle = v$, in the ground state, whereas $\langle\phi_2\rangle = 0$. Define new scalar fields $\sigma(x)$ and $\pi(x)$ such that,

$$\big(\phi_1(x)\,,\,\phi_2(x)\big) = \big(v + \sigma(x)\,,\,\pi(x)\big)\,.$$

Write out the Lagrangian in terms of the new scalar fields $\sigma(x)$ and $\pi(x)$, and show that the fermion acquires a mass. Evaluate the fermion mass in terms of g and v.

(c) What are the masses of the physical scalar bosons of this model?

4. Consider the abelian Higgs model (i.e., scalar electrodynamics where the U(1) gauge symmetry is spontaneously broken).

(a) Suppose you wish to do calculations in the R_ξ–gauge. Derive the Faddeev-Popov Lagrangian and the corresponding Feynman rules for the ghost propagator and vertices.

(b) In the abelian Higgs model, m_H and m_V are the masses of the Higgs boson (H) and the massive vector boson (V), respectively. Assuming that $m_H > 2m_V$, compute the tree-level rate for the decay $H \to VV$.

(c) The Equivalence Theorem states that the S-matrix amplitude involving external longitudinally polarized gauge bosons may be evaluated in the R_ξ gauge by substituting the corresponding Goldstone bosons as external particles. This equality holds up to corrections of order m_V/E_V, where E_V is the vector boson energy. Verify this theorem by applying it to the Higgs boson decay of part (b).

HINT: Work in the rest frame of the Higgs boson, and assume that $m_H \gg m_V$. Show that the longitudinal polarization vector of V can be approximated by $\epsilon_L^\mu(p) \simeq p^\mu/m_V$ in the parameter regime where $E_V \gg m_V$. The class handout on the polarization sum for massless spin-one particles (and its extension to the case of massive spin-one particles) may be of use here.