DUE: THURSDAY, MARCH 17, 2016

1. Consider the spontaneous breaking of a gauge group G down to U(1). The unbroken generator $Q = c_a T^a$ is some real linear combination of the generators of G.

(a) Prove that $x_b \equiv c_b/g_b$ is an (unnormalized) eigenvector of the vector boson squared-mass matrix, M^2_{ab}, with zero eigenvalue.

(b) Suppose that A_μ is the massless gauge field that corresponds to the generator Q. Show that the covariant derivative can be expressed in the following form:

$$D_\mu = \partial_\mu + ieQA_\mu + \ldots\,, \tag{1}$$

where we have omitted terms in eq. (1) corresponding to all the other gauge bosons and

$$e = \left[\sum_a \left(\frac{c_a}{g_a}\right)^2\right]^{-1/2}. \tag{2}$$

HINT: The vector boson mass matrix is diagonalized by an orthogonal transformation $\mathcal{O}M^2\mathcal{O}^T$ as shown in class. The rows of the matrix $\mathcal{O}$ are constructed from the *orthonormal* eigenvectors of M^2.

(c) Evaluate Q in the adjoint representation (that is, $Q = c_a T^a$, where the $(T^a)_{bc} = -if_{abc}$ are the generators of the gauge group in the adjoint representation). Show that $Q_{bc}x_c = 0$, where x_c is defined in part (a). What is the physical interpretation of this result?

(d) Prove that the commutator $[Q, M^2] = 0$, where Q is the unbroken U(1) generator in the adjoint representation and M^2 is the gauge boson squared-mass matrix. Conclude that one can always choose the eigenstates of the gauge boson squared-mass matrix to be states of definite unbroken U(1)-charge.

2. In class, we examined in detail the structure of a spontaneously broken SU(2)×U(1)$_Y$ gauge theory, in which the symmetry breaking was due to the vacuum expectation value of a complex $Y = 1$, SU(2) doublet of scalar fields. In this problem, a different representation of scalar fields will be employed.

(a) Consider an SU(2)×U(1)$_Y$ gauge theory with a $Y = 0$, SU(2) triplet of *real* scalar fields, Φ. The scalar potential is given by

$$V(\Phi) = -\tfrac{1}{2}m^2\Phi^{\mathsf{T}}\Phi + \lambda(\Phi^{\mathsf{T}}\Phi)^2\,,$$

where m^2 and λ are real parameters. After spontaneous symmetry breaking, the electrically neutral ($Q = 0$) member of the scalar triplet acquires a vacuum expectation value (where $Q = T_3 + Y/2$). Identify the subgroup that remains unbroken. Compute the vector boson masses and the physical Higgs scalar masses in this model. Deduce the Feynman rules for the three-point interactions among the Higgs and vector bosons.

HINT: Since the triplet of scalar fields corresponds to the adjoint representation of SU(2), the corresponding SU(2) generators that act on the triplet of scalar fields can be chosen to be $(T^a)_{bc} = -i\epsilon_{abc}$. The hypercharge operator annihilates the $Y = 0$ fields. Define $L^a = ig_a T^a$, and follow the methods outlined in class.

(b) Consider an SU(2)×U(1)$_Y$ gauge theory with a $Y = 2$, SU(2) triplet of *complex* scalar fields (again denoted by Φ). The scalar potential is given by

$$V(\Phi) = -m^2\Phi^\dagger\Phi + \lambda_1(\Phi^\dagger\Phi)^2 - \lambda_2\sum_a(\Phi^\dagger\mathcal{T}^a\Phi)(\Phi^\dagger\mathcal{T}^a\Phi)\,,$$

where $m^2 > 0$ and $\lambda_1 > \lambda_2 > 0$. The $\mathcal{T}^a$ are hermitian generators in the 3-dimensional representation of SU(2) in a basis where $\mathcal{T}^3$ is diagonal.[1]

Again, assume that the electrically neutral ($Q = 0$) member of the scalar triplet acquires a vacuum expectation value (where $Q = T_3 + Y/2$). After symmetry breaking, identify the subgroup that remains unbroken. Compute the vector boson masses and the physical Higgs scalar masses in this model.

HINT: In order to use the methods of part (a), one can rewrite the complex scalar fields in terms of their real and imaginary parts. In this case, the real antisymmetric generators iT^a are 6×6 matrices,

$$iT^a = \begin{pmatrix} -\mathrm{Im}\,\mathcal{T}^a & -\mathrm{Re}\,\mathcal{T}^a \\ \mathrm{Re}\,\mathcal{T}^a & -\mathrm{Im}\,\mathcal{T}^a \end{pmatrix},$$

which have been expressed in terms of the 3×3 hermitian generators $\mathcal{T}^a$.

(c) If both doublet and triplet Higgs fields exist in nature, what does this exercise imply about the parameters of the Higgs Lagrangian?

3. In the Standard Model, the Higgs boson H couples to two gluons via a one-loop triangle diagram containing top quarks in the loop.[2]

(a) Compute the amplitude for the decay of the Higgs boson to two gluons ($H \to gg$), as a function of m_t, m_H, G_F (the Fermi constant) and $\alpha_s \equiv g_s^2/(4\pi)$, using perturbation theory in the one loop approximation. Simplify your answer by invoking the kinematics of the problem, *i.e.* the conservation of four-momentum and the on-shell conditions for the external particles.

HINT: Two diagrams contribute to $H \to gg$, which differ due to the interchange of the two outgoing gluons. In obtaining the decay amplitude, you should make use of the unitary gauge Feynman rules of the Standard Model.

(b) Denote the amplitude for $H \to gg$ by $\mathcal{M}_{\mu\nu}$, where μ and ν are the Lorentz indices of the two gluons. Gauge invariance implies that $k_1^\mu\mathcal{M}_{\mu\nu} = k_2^\nu\mathcal{M}_{\mu\nu} = 0$, where k_1 and k_2 are

[1] The $\mathcal{T}^a$ are given by the standard spin-1 matrices defined in the $|j\,m\rangle$ basis in quantum mechanics.

[2] In this problem, you should work in the approximation where all quarks are massless, with the exception of the top quark, in which case only triangle diagrams with top quarks in the loop contribute.

the respective gluon momenta.[3] Check that your amplitude obtained in part (a) respect this requirement.

(c) Work out all integrals explicitly and evaluate the imaginary part of $\mathcal{M}_{\mu\nu}$. For what range of m_t/m_H is the amplitude purely real? Check your result for the imaginary part by using Cutkosky's rules [cf. problem 2 of Problem Set 2].

HINT: You may find the following integral useful:

$$\int_0^1 \frac{dy}{y} \log\bigl[1 - 4Ay(1-y)\bigr] = -2(\sin^{-1}\sqrt{A})^2$$

for $0 \leq A \leq 1$. For values of A outside this region, you may analytically continue the above result. The imaginary part of this integral is easily computed once the $i\epsilon$ factor is restored in the argument of the logarithm.

(d) Evaluate $\mathcal{M}_{\mu\nu}$ in the limit of $m_t \to \infty$.

(e) The dominant decay of the Higgs boson is into a pair of bottom quarks, $H \to b\bar{b}$. Evaluate the ratio of decay rates:

$$\frac{\Gamma(H \to gg)}{\Gamma(H \to b\bar{b})}$$

in the limit where $m_t \gg m_H$. In obtaining the decay rates into $b\bar{b}$ and gg respectively, you should sum the squared-amplitude over the final state spins and colors, and then evaluate the results numerically.

[3] In this computation, no three gluon vertex appears since the gluon does not couple directly to the Higgs boson. Consequently, the Ward identities of QED also apply here.