1. Show that for complex scalar fields $\Phi$ and a positive definite hermitian operator $M$,

$$\int D\Phi^* D\Phi \exp\left\{ i \int d^4x d^4y [\Phi^*(x)M(x,y)\Phi(y)] + i \int d^4x [J^*(x)\Phi(x) + \Phi^*(x)J(x)] \right\} = \mathcal{N} \frac{1}{\det M} \exp\left\{ -i \int d^4x d^4y J^*(x)M^{-1}(x,y)J(y) \right\},$$

(1)

for some infinite constant $\mathcal{N}$.

We shall prove the analog of eq. (1) for the integration over $n$ complex variables,

$$\int dz_1^* dz_2^* dz_3^* \cdots dz_n^* dz_n \exp\left\{ i(z_i^* M_{ij}z_j + J_i^* z_i + z_i^* J_i) \right\} = \mathcal{N} \frac{1}{\det M} \exp\left\{ -i J_i^* (M^{-1})_{ij} J_j \right\},$$

(2)

where $M_{ij}$ are the matrix elements of an hermitian $n \times n$ matrix $M$. We can write

$$z_i = x_i + iy_i, \quad M = M_R + iM_I, \quad J = J_R + iJ_I,$$

where $M_R^T = M_R$ and $M_I^T = -M_I$. Then,

$$z_i^* M_{ij} z_j + J_i^* z_i + z_i^* J_i = x_i(M_R)_{ij} x_j + y_i(M_R)_{ij} y_j + y_i(M_I)_{ij} x_j - x_i(M_I)_{ij} y_j + 2 [(J_R)_i x_i + (J_I)_i y_i].$$

(3)

We can introduce real variables $v_k$, where $k = 1, 2, \ldots, 2n$, such that

$$v_k = \begin{cases} x_k, & \text{for } k = 1, 2, \ldots, n, \\ y_{k-n}, & \text{for } k = n+1, n+2, \ldots 2n. \end{cases}$$

Eq. (3) can then be rewritten as

$$z_i^* M_{ij} z_j + J_i^* z_i + z_i^* J_i = \frac{1}{2} v_k \mathcal{M}_{kl} v_l + \mathcal{J}_k v_k,$$

where $\mathcal{M}$ is the real symmetric $2n \times 2n$ matrix,

$$\mathcal{M} \equiv \begin{pmatrix} 2M_R & -2M_I \\ 2M_I & 2M_R \end{pmatrix},$$

written in block form, and

$$\mathcal{J}_k = \begin{cases} 2(J_R)_k, & \text{for } k = 1, 2, \ldots, n, \\ 2(J_I)_{k-n}, & \text{for } k = n+1, n+2, \ldots 2n. \end{cases}$$

In light of the Jacobian,

$$\frac{\partial (z^*, z)}{\partial (x, y)} = \det \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = 2i,$$
we have\(^1\)
\[ dz^* dz = \frac{\partial(z^*, z)}{\partial(x, y)} dxdy = 2i dxdy , \quad (4) \]
and the left hand side of eq. (2) can be written as
\[ (2i)^n \int_{-\infty}^{\infty} dv_1 \int_{-\infty}^{\infty} dv_2 \cdots \int_{-\infty}^{\infty} dv_{2n} \exp\left\{ i\left( \frac{1}{2}v_k \mathcal{M}_{\ell \ell} v_\ell + \mathcal{J}_k v_k \right) \right\} . \]
To evaluate this integral, we first consider
\[ I(\mathcal{M}, \mathcal{J}) \equiv \int_{-\infty}^{\infty} dv_1 \int_{-\infty}^{\infty} dv_2 \cdots \int_{-\infty}^{\infty} dv_{2n} \exp\left\{ -\frac{1}{2}v_k \mathcal{M}_{\ell \ell} v_\ell + \mathcal{J}_k v_k \right\} . \quad (5) \]
Introduce the following change of variables,
\[ v_k = (\mathcal{M}^{-1})_{km} \mathcal{J}_m + w_k . \]
Inserting this into eq. (5) and noting that the corresponding Jacobian is 1, we obtain
\[ I(\mathcal{M}, \mathcal{J}) = \exp\left\{ \frac{1}{2} \mathcal{J}_k (\mathcal{M}^{-1})_{km} \mathcal{J}_m \right\} \int_{-\infty}^{\infty} dw_1 \int_{-\infty}^{\infty} dw_2 \cdots \int_{-\infty}^{\infty} dw_{2n} \exp\left\{ -\frac{1}{2}w_k \mathcal{M}_{\ell \ell} w_\ell \right\} \]
\[ = \frac{(2\pi)^n}{(\det \mathcal{M})^{1/2}} \exp\left\{ \frac{1}{2} \mathcal{J}_k (\mathcal{M}^{-1})_{km} \mathcal{J}_m \right\} , \quad (6) \]
after using eq. (14.7) of Schwartz to evaluate the 2\(n\)-dimensional gaussian integral.
Using the arguments in class to justify the substitutions, \( \mathcal{M} \rightarrow -i\mathcal{M} \) and \( \mathcal{J} \rightarrow i\mathcal{J} \), and noting that \( \det(-i\mathcal{M}) = (-i)^{2n} \det \mathcal{M} \) since \( \mathcal{M} \) is a \( 2n \times 2n \) matrix,
\[ I(-i\mathcal{M}, i\mathcal{J}) = \frac{(2\pi)^n}{(\det \mathcal{M})^{1/2}} \exp\left\{ -\frac{1}{2}i \mathcal{J}_k (\mathcal{M}^{-1})_{km} \mathcal{J}_m \right\} . \]
We need to evaluate \( \det \mathcal{M} \) and \( \mathcal{M}^{-1} \). First, we note that
\[ \mathcal{M} = \begin{pmatrix} 2M_R & -2M_I \\ 2M_I & 2M_R \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i1 & -i1 \end{pmatrix} \begin{pmatrix} M_R - iM_I & O \\ O & M_R + iM_I \end{pmatrix} \begin{pmatrix} 1 & -i1 \\ i1 & i1 \end{pmatrix} , \quad (7) \]
where \( 1 \) is the \( n \times n \) identity matrix and \( O \) is the \( n \times n \) zero matrix. It is straightforward to evaluate
\[ \det \begin{pmatrix} 1 & 1 \\ i1 & -i1 \end{pmatrix} = (-2i)^n , \quad \det \begin{pmatrix} 1 & -i1 \\ i1 & i1 \end{pmatrix} = (2i)^n . \]

\(^1\)Employing the formalism of differential forms, \( dz^* \wedge dz = (dx - idy) \wedge (dx + idy) = 2i dx \wedge dy \), which is equivalent to eq. (4). In deriving this result, we have used the well known properties of differential forms, \( dx \wedge dx = dy \wedge dy = 0 \) and \( dx \wedge dy = -dy \wedge dx \). For an elementary introduction to differential forms, see e.g., James J. Callahan, \textit{Advanced Calculus: A Geometric View} (Springer Science, New York, 2010) Chapter 10.3.
Hence, recalling that $M = M_R + iM_I$, it follows that
\[
\det M = 2^{2n}|\det M|^2.
\]

Next, to compute $M^{-1}$, we again use eq. (7) to obtain
\[
M^{-1} = \frac{1}{4} \begin{pmatrix}
1 & 1 \\
i1 & -i1
\end{pmatrix}
\begin{pmatrix}
M^*-1 & O \\
O & M^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & -i1 \\
i1 & i1
\end{pmatrix}.
\]

Thus, using matrix notation,
\[
\mathcal{J}_k(M^{-1})_{km} \mathcal{J}_m = (J_R J_I) \begin{pmatrix}
1 & 1 \\
i1 & -i1
\end{pmatrix}
\begin{pmatrix}
M^*-1 & O \\
O & M^{-1}
\end{pmatrix}
\begin{pmatrix}
J^* \\
J
\end{pmatrix} = J^* M^{-1} J
\]
\[
= 2J^* M^{-1} J,
\]
where we have used the fact that $M$ is hermitian (which implies that $M^{-1}$ is hermitian) to conclude that $JM^{-1} J^* \equiv J^* (M^{-1})_{ij} J_j = J^*_i (M^*-1)_{ji} J_j = J_j (M^*-1)_{ji} J^*_i \equiv JM^*-1 J^*$. Hence,
\[
I(-iM, i\mathcal{J}) = \frac{(\pi i)^n}{\det M} \exp\{-iJ^*_i (M^{-1})_{ij} J_j\}.
\]

Note that since $M$ is positive definite, $\det M$ is positive and we can dispense with the absolute value signs.

We conclude that
\[
\int dz_1^* dz_2^* dz_3^* \cdots dz_n^* dz_n \exp\{i(z_1^* M_{ij} z_j + J^*_i z_i + z_i^* J_i)\}
\]
\[
= (2i)^n I(-iM, i\mathcal{J}) = \frac{(-2\pi)^n}{\det M} \exp\{-iJ^*_i (M^{-1})_{ij} J_j\}.
\]
That is, we have confirmed eq. (2), with $\mathcal{N} = (-2\pi)^n$. In the limit of $n \to \infty$, the constant $\mathcal{N}$ is infinite. The generalization to eq. (1) is now straightforward.

2. (a) Derive the result:
\[
\int d^4z \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z) \delta \Phi(y)} = -\delta^4(x - y),
\]
and interpret diagrammatically. Here, \( W[J] \) is the generating functional for the connected Green functions and \( \Gamma[\Phi] \) is the generating functional for the one particle irreducible (1PI) Green functions.

We begin with the definition of the effective action,

\[
\Gamma[\Phi] = W[J] - \int d^4 x \, J(x) \Phi(x),
\]

(9)

where \( \Phi(x) \) is the classical field [which was denoted in class by \( \Phi_c(x) \)]. It follows that

\[
\frac{\delta W[J]}{\delta J(x)} = \Phi(x), \quad \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} = -J(x).
\]

(10)

Taking a second functional derivative yields,

\[
\frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} = \frac{\delta \Phi(x)}{\delta \Phi(y)}, \quad \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(x) \delta \Phi(y)} = -\frac{\delta J(x)}{\delta \Phi(y)}.
\]

Hence, it follows that

\[
\int d^4 z \, \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z) \delta \Phi(y)} = -\int d^4 z \, \frac{\delta \Phi(x)}{\delta \Phi(z)} \frac{\delta J(z)}{\delta \Phi(y)} = -\frac{\delta \Phi(x)}{\delta \Phi(y)} = -\delta^4(x - y),
\]

(11)

where we have used the chain rule for functional derivatives at the second step above.

Recall that the two-point 1PI Green function, \( \Gamma^{(2)}(x_1, x_2) \), and the two-point connected Green function \( G^{(2)}_c(x_1, x_2) \), are defined as

\[
\Gamma^{(2)}(x_1, x_2) = \left. \left( \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(x) \delta \Phi(y)} \right) \right|_{\Phi = 0}, \quad G^{(2)}_c(x_1, x_2) = -i \left( \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} \right) \bigg|_{J = 0},
\]

We shall assume that the quantum field \( \phi(x) \) has no vacuum expectation value,\(^2\) in which case

\[
G^{(1)}_c(x) \equiv \langle \Omega | \phi(x) | \Omega \rangle = \frac{\delta W[J]}{\delta J(x)} \bigg|_{J = 0} = \Phi(x) \bigg|_{J = 0} = 0.
\]

(12)

That is, setting \( J = 0 \) implies that \( \Phi = 0 \) and vice versa.

Using the above results, eq. (11) implies that

\[
\int d^4 z \, \Gamma^{(2)}(x, z) G^{(2)}_c(z, y) = i \delta^4(x - z),
\]

(13)

In momentum space, eq. (13) yields

\[
\Gamma^{(2)}(p, -p) G^{(2)}_c(p, -p) = i.
\]

Since \( G^{(2)}_c(p, -p) \) is the momentum space propagator, it follows that \( i \Gamma^{(2)}(p, -p) \) is the negative of the inverse propagator in momentum space.

---

\(^2\)If \( \langle \Omega | \phi(x) | \Omega \rangle = v \neq 0 \), then one can redefine the quantum field by redefining \( \phi \rightarrow \phi + v \).
(b) By taking one further functional derivative, show that $\Gamma$ generates the amputated connected three-point function.

We shall take a functional derivative of eq. (11). On the right hand side of eq. (11), we have

$$\frac{\delta}{\delta J(w)} \delta^4(x - y) = 0,$$

since $\delta^4(x - y)$ is the analog of the Kronecker delta, $\delta_{ij}$, for an infinite dimensional function space. On the left hand side of eq. (11),

$$\frac{\delta}{\delta J(w)} \left[ \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z) \delta \Phi(y)} \right] = \frac{\delta^3 W[J]}{\delta J(w) \delta J(x) \delta J(z)} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z) \delta \Phi(y)} + \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z) \delta \Phi(y)}.$$

In the second term on the right hand side above, we use the chain rule,

$$\frac{\delta}{\delta J(w)} \frac{\delta}{\delta \phi(v)} = \int d^4u \frac{\delta^2 W[J]}{\delta J(w) \delta J(v)} \frac{\delta}{\delta \phi(v)},$$

after using the definition of the classical field $\phi(v)$ given in eq. (10). Hence, eq. (11) yields

$$\int d^4z \frac{\delta^3 W[J]}{\delta J(w) \delta J(x) \delta J(z)} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z) \delta \Phi(y)} + \int d^4z d^4v \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta^3 \Gamma[\Phi]}{\delta \Phi(z) \delta \Phi(y)} = 0. \quad (14)$$

We now multiply eq. (14) by

$$\frac{\delta^2 W[J]}{\delta J(y) \delta J(u)},$$

and integrate over $d^4y$. Using the result of eq. (11), we obtain

$$\int d^4z d^4y \frac{\delta^3 W[J]}{\delta J(w) \delta J(x) \delta J(z)} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z) \delta \Phi(y)} \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(z) \delta \Phi(y)} = - \int d^4z \frac{\delta^3 W[J]}{\delta J(w) \delta J(x) \delta J(z)} \delta^4(z - u)$$

$$= - \frac{\delta^3 W[J]}{\delta J(w) \delta J(x) \delta J(u)}.\quad (15)$$

Applying this result to eq. (14) yields

$$\frac{\delta^3 W[J]}{\delta J(w) \delta J(x) \delta J(u)} = \int d^4v d^4y d^4z \frac{\delta^3 \Gamma[\Phi]}{\delta \Phi(v) \delta \Phi(z) \delta \Phi(y)} \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta^2 W[J]}{\delta J(y) \delta J(u)} \frac{\delta^2 W[J]}{\delta J(w) \delta J(v)} \frac{\delta^2 W[J]}{\delta J(y) \delta J(u)}. \quad (15)$$

Recall the definition of the connected $n$-point Green function,

$$G_c^{(n)}(x_1, x_2, \ldots, x_n) = i^{1-n} \frac{\delta^n W[J]}{\delta J(x_1) \delta J(x_2) \ldots \delta J(x_n)} \bigg|_{J=0}, \quad (16)$$
and the \( n \)-point 1PI Green function,
\[
\Gamma^{(n)}(x_1, x_2, \ldots, x_n) = \frac{\delta^n \Gamma[\phi]}{\delta \phi(x_1) \delta \phi(x_2) \cdots \delta \phi(x_n)} \bigg|_{\phi=0}.
\]
(17)

In light of eq. (12), we can set \( J = \phi = 0 \) in eq. (15). Using eqs. (16) and (17), it then follows that
\[
G^{(3)}_c(w, x, u) = i \int d^4v d^4y d^4z \Gamma^{(3)}(v, z, y) G^{(2)}_c(v, w) G^{(2)}_c(z, x) G^{(2)}_c(y, u).
\]
(18)

To invert this equation, we make use of the inverse propagator, which satisfies
\[
\int d^4z G^{(2)}_c(x, z) G^{-1}_{c}(z, y) = \delta^4(x - y).
\]

Then, we can rewrite eq. (18) as
\[
i \Gamma^{(3)}(v, z, y) = \int d^4w d^4x d^4u G^{(2)}_c(w, v) G^{-1}_{c}(x, z) G^{(2)}_c(y, u) G^{(3)}_c(w, x, u).
\]

The effect of the factors of \( G^{(2)}_c^{-1} \) is to remove the explicit propagators that appear on the external legs of the three-point Green function. That is, \( i \Gamma^{(3)} \) is obtained from \( G^{(3)}_c \) by amputating the full propagators on the three external legs.

3. Consider the quantum field theory of a real scalar field governed by the Lagrangian,
\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4.
\]

(a) Evaluate the generating functional for the connected Green functions, \( W[J] \), perturbatively, keeping all terms up to and including terms of \( \mathcal{O}(\lambda) \).

The generating functional for the connected Green functions, \( W[J] \), is determined by
\[
Z[J] = \exp \{ iW[J] \},
\]
(19)

where
\[
Z[J] = \frac{\int \mathcal{D}\phi \exp \left\{ i \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 + J\phi \right] \right\}}{\int \mathcal{D}\phi \exp \left\{ i \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right] \right\}}.
\]
(20)

Expanding the numerator of eq. (20) to \( \mathcal{O}(\lambda) \),
\[
\int \mathcal{D}\phi \exp \left\{ i \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + J\phi \right] \right\} \left[ 1 - i \int d^4y \frac{\lambda}{4!} \phi^4(y) \right] &= \left[ 1 - i \frac{\lambda}{4!} \int d^4y \left( \frac{1}{i} \frac{\delta}{\delta J(y)} \right)^4 \right] \int \mathcal{D}\phi \exp \left\{ i \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + J\phi \right] \right\},
\]
since each \( i^{-1} \delta/\delta J(x) \) operator brings down a factor of \( \phi(x) \).
Next we use the result obtained in class,

\[
Z_0[J] = \frac{\int \mathcal{D}\phi \exp \left\{ i \int d^4x \left[ \frac{1}{2} (\partial_{\mu}\phi)^2 - \frac{1}{2} m^2 \phi^2 + J\phi \right] \right\}}{\int \mathcal{D}\phi \exp \left\{ i \int d^4x \left[ \frac{1}{2} (\partial_{\mu}\phi)^2 - \frac{1}{2} m^2 \phi^2 \right] \right\}}
\]

\[
= \exp \left\{ -\frac{i}{2} \int d^4x_1 d^4x_2 J(x_1) \Delta_F(x_1 - x_2) J(x_2) \right\},
\]

where \( i \Delta_F \) is the free-field propagator. It then follows that

\[
Z[J] = \mathcal{N} \left[ 1 - \frac{i\lambda}{4!} \int d^4y \left( \frac{1}{i \delta J(y)} \right)^4 \right] \exp \left\{ -\frac{i}{2} \int d^4x_1 d^4x_2 J(x_1) \Delta_F(x_1 - x_2) J(x_2) \right\}, \tag{21}
\]

where \( \mathcal{N} \) is the \( J \)-independent constant,

\[
\mathcal{N} \equiv \frac{\int \mathcal{D}\phi \exp \left\{ i \int d^4x \left[ \frac{1}{2} (\partial_{\mu}\phi)^2 - \frac{1}{2} m^2 \phi^2 \right] \right\}}{\int \mathcal{D}\phi \exp \left\{ i \int d^4x \left[ \frac{1}{2} (\partial_{\mu}\phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right] \right\}}.
\tag{22}
\]

There is no need to evaluate \( \mathcal{N} \) using eq. (22), since it can be determined at the end of our computation using \( Z[0] = 1 \).

To evaluate eq. (21), we first compute

\[
\frac{1}{i \delta J(y)} \exp \left\{ -\frac{i}{2} \int d^4x_1 d^4x_2 J(x) \Delta_F(x_1 - x_2) J(x_2) \right\}
\]

\[
= -\int d^4x \Delta_F(y - x) J(x) \exp \left\{ -\frac{i}{2} \int d^4x_1 d^4x_2 J(x_1) \Delta_F(x_1 - x_2) J(x_2) \right\},
\]

where we have used the fact that \( \Delta_F(x - y) = \Delta_F(y - x) \). Taking a second functional derivative yields

\[
\left( \frac{1}{i \delta J(y)} \right)^2 \exp \left\{ -\frac{i}{2} \int d^4x_1 d^4x_2 J(x) \Delta_F(x_1 - x_2) J(x_2) \right\}
\]

\[
= \left\{ i\Delta_F(0) + \left[ \int d^4x \Delta_F(y - x) J(x) \right]^2 \right\} \exp \left\{ -\frac{i}{2} \int d^4x_1 d^4x_2 J(x) \Delta_F(x_1 - x_2) J(x_2) \right\}.
\]

Taking a third functional derivative yields

\[
\left( \frac{1}{i \delta J(y)} \right)^3 \exp \left\{ -\frac{i}{2} \int d^4x_1 d^4x_2 J(x) \Delta_F(x_1 - x_2) J(x_2) \right\}
\]

\[
= \left\{ -3i\Delta_F(0) \int d^4x \Delta_F(y - x) J(x) - \left[ \int d^4x \Delta_F(y - x) J(x) \right]^3 \right\}
\]

\[
\times \exp \left\{ -\frac{i}{2} \int d^4x_1 d^4x_2 J(x) \Delta_F(x_1 - x_2) J(x_2) \right\}.
\]
Finally, taking a fourth functional derivative yields

\[
\left( \frac{1}{i} \frac{\delta}{\delta J(y)} \right)^4 \exp \left\{ -i \frac{1}{2} \int d^4x_1 d^4x_2 J(x) \Delta_F(x_1 - x_2) J(x_2) \right\} \\
= \left\{ -3[\Delta_F(0)]^2 + 6i\Delta_F(0) \left[ \int d^4x \Delta_F(y - x) J(x) \right]^2 + \left[ \int d^4x \Delta_F(y - x) J(x) \right]^4 \right\} \\
\times \exp \left\{ -i \frac{1}{2} \int d^4x_1 d^4x_2 J(x) \Delta_F(x_1 - x_2) J(x_2) \right\}.
\]

The end result is

\[
Z[J] = \mathcal{N} \left\{ 1 + \frac{i\lambda}{8} \left[ \int d^4y [\Delta_F(0)]^2 - 2i\Delta_F(0) \int d^4y d^4x_1 d^4x_2 \Delta_F(y - x_1) \Delta_F(y - x_2) J(x_1) J(x_2) \right. \right.
\]

\[
- \frac{1}{3} \int d^4y d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Delta_F(y - x_1) \cdots \Delta_F(y - x_4) J(x_1) \cdots J(x_4) \left. \right\} \right\}
\]

\[
\times \exp \left\{ -i \frac{1}{2} \int d^4x_1 d^4x_2 J(x) \Delta_F(x_1 - x_2) J(x_2) \right\}.
\]

Using \(Z[0] = 1\), it follows that to \(\mathcal{O}(\lambda)\),

\[
\mathcal{N} = 1 - \frac{i\lambda}{8} \int d^4y [\Delta_F(0)]^2.
\]

Thus,

\[
Z[J] = \left\{ 1 - \frac{i\lambda}{4!} \left[ 6i\Delta_F(0) \int d^4y d^4x_1 d^4x_2 \Delta_F(y - x_1) \Delta_F(y - x_2) J(x_1) J(x_2) \right. \right.
\]

\[
+ \int d^4y d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Delta_F(y - x_1) \cdots \Delta_F(y - x_4) J(x_1) \cdots J(x_4) \left. \right\} \right\}
\]

\[
\times \exp \left\{ -i \frac{1}{2} \int d^4x_1 d^4x_2 J(x) \Delta_F(x_1 - x_2) J(x_2) \right\}.
\]

Since we are only keeping terms of \(\mathcal{O}(\lambda)\), we can also rewrite \(Z[J]\) in the following form:

\[
Z[J] = \exp \left\{ -i \frac{1}{2} \int d^4x_1 d^4x_2 J(x) \Delta_F(x_1 - x_2) J(x_2) \right. \]

\[
- \frac{i\lambda}{4!} \left[ 6i\Delta_F(0) \int d^4y d^4x_1 d^4x_2 \Delta_F(y - x_1) \Delta_F(y - x_2) J(x_1) J(x_2) \right. \right.
\]

\[
+ \int d^4y d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Delta_F(y - x_1) \cdots \Delta_F(y - x_4) J(x_1) \cdots J(x_4) \left. \right\} \right\}. 
\]
Hence, using eq. (19) it follows that

\[
W[J] = -\frac{1}{2} \int d^4x_1 d^4x_2 J(x) \Delta_F(x_1 - x_2) J(x_2) \\
- \frac{i\lambda}{4} \Delta_F(0) \int d^4y d^4x_1 d^4x_2 \Delta_F(y - x_1) \Delta_F(y - x_2) J(x_1) J(x_2) \\
- \frac{\lambda}{4!} \int d^4y d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Delta_F(y - x_1) \cdots \Delta_F(y - x_4) J(x_1) \cdots J(x_4).
\]

(b) Using the result of part (a), compute the four-point connected Green function, take the appropriate Fourier transform, and verify the momentum space Feynman rule for the four-point scalar interaction obtained in class,

\[
- \frac{i\lambda}{4} \Delta_F(0) \int d^4y d^4x_1 d^4x_2 \Delta_F(y - x_1) \Delta_F(y - x_2) J(x_1) J(x_2) \\
- \frac{\lambda}{4!} \int d^4y d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Delta_F(y - x_1) \cdots \Delta_F(y - x_4) J(x_1) \cdots J(x_4).
\]

Using eqs. (16) and (23), it immediately follows that

\[
G_c^{(4)}(x_1, x_2, x_3, x_4) = -i\lambda \int d^4y \Delta_F(y - x_1) \Delta_F(y - x_2) \Delta_F(y - x_3) \Delta_F(y - x_4).
\]

In particular, note that the coefficient of 1/4! is canceled due to the fact that there are 4! ways in taking the functional derivatives in eq. (16). Converting to momentum space,

\[
G_c^{(4)}(p_1, p_2, p_3, p_4)(2\pi)^4\delta^4(p_1 + p_2 + p_3 + p_4)
= \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i(p_1 x_1 + \cdots + p_4 x_4)} G_c^{(4)}(x_1, x_2, x_3, x_4)
= -i\lambda \int d^4y d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i(p_1 x_1 + \cdots + p_4 x_4)} \Delta_F(y - x_1) \cdots \Delta_F(y - x_4)
= -i\lambda \int d^4y d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i(p_1 x_1 + \cdots + p_4 x_4)} e^{ip_1(x_1 - y)} \Delta_F(y - x_1) \cdots e^{ip_4(x_4 - y)} \Delta_F(y - x_4).
\]

We can now perform the integration over \(x_1, \ldots, x_4\) using the expression for the free-field propagator in momentum space,

\[
\frac{1}{p^2 - m^2 + i\epsilon} = \int d^4x e^{-ipx} \Delta_F(x),
\]

where \(m\) is the mass of the scalar field. Employing the integral representation of the momentum
conserving delta function,
\[
\int d^4 ye^{i(y(p_1 + \cdots + p_4))} = (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4),
\]

the end result is
\[
G_c^{(4)}(p_1, p_2, p_3, p_4) = -i\lambda \frac{i}{p_1^2 - m^2 + i\epsilon} \cdots \frac{i}{p_4^2 - m^2 + i\epsilon}.
\]

If we now amputate the four external propagators, we arrive at the Feynman rule for the four-point scalar interaction shown in eq. (24).

(c) Evaluate the classical field \(\phi_c(x)\) and the generating functional for the 1PI Green functions, \(\Gamma[\phi_c]\), perturbatively, keeping all terms up to and including terms of \(O(\lambda)\). Then, repeat part (b) for the four-point 1PI Green function.

The effective action is given by eq. (9), where the classical field is defined by eq. (10). Using eq. (23), it follows that
\[
\Phi(x) = -\int d^4x_1 \Delta_F(x - x_1)J(x_1) - \frac{1}{2}i\lambda \Delta_F(0) \int d^4y d^4x_1 \Delta_F(y - x)\Delta_F(y - x_1)J(x_1)
\]
\[
-\frac{\lambda}{6} \int d^4y d^4x_1 d^4x_2 d^4x_3 \Delta_F(y - x)\Delta_F(y - x_1)\Delta_F(y - x_2)\Delta_F(y - x_3)J(x_1)J(x_2)J(x_3).
\]

(25)

We must invert this equation and solve for \(J(x)\). This can be done using an iterative process. Operate on eq. (25) with the operator \(\Box_x + m^2 - i\epsilon\). Using
\[
(\Box_x + m^2 - i\epsilon)\Delta_F(x - y) = -\delta^4(x - y),
\]

it follows that
\[
(\Box_x + m^2 - i\epsilon)\Phi(x) = J(x) + \frac{1}{2}i\lambda \Delta_F(0) \int d^4x_1 \Delta_F(x - x_1)J(x_1)
\]
\[
+\frac{\lambda}{6} \int d^4x_1 d^4x_2 d^4x_3 \Delta_F(x - x_1)\Delta_F(x - x_2)\Delta_F(x - x_3)J(x_1)J(x_2)J(x_3).
\]

(27)

At \(O(\lambda^0)\), we have \(J(x) = (\Box_x + m^2 - i\epsilon)\Phi(x)\). Thus, in the \(O(\lambda)\) term in eq. (27), we can replace \(J(x_k)\) with \((\Box_{x_k} + m^2 - i\epsilon)\Phi(x_k)\), for \(k = 1, 2, 3\). We can then move the operators \((\Box_{x_k} + m^2 - i\epsilon)\) so that they operate on the \(\Delta_F(x - x_k)\) by two successive integrations by parts. Using eq. (26), we produce three delta functions, after which the integrals over \(x_1, x_2\) and \(x_3\) are trivially done. The end result is
\[
(\Box_x + m^2 - i\epsilon)\Phi(x) = J(x) - \frac{1}{2}i\lambda \Delta_F(0)\Phi(x) - \frac{1}{6}i\lambda [\Phi(x)]^3.
\]
Hence, to $\mathcal{O}(\lambda)$,

$$J(x) = (\Box_x + m^2 - i\epsilon)\Phi(x) + \frac{1}{2}i\lambda\Delta_F(0)\Phi(x) + \frac{1}{6}\lambda[\Phi(x)]^3. \tag{28}$$

We can use the same procedure to rewrite $W[J]$ in terms of the classical field $\Phi(x)$. We simply insert eq. (28) into eq. (23), and keep only terms up to $\mathcal{O}(\lambda)$. This yields

$$W[J] = \frac{1}{2} \int d^4x \Phi(x) \left\{ (\Box_x + m^2)\Phi(x) + \frac{1}{2}i\lambda\Delta_F(0)\Phi(x) + \frac{1}{6}[\Phi(x)]^3 \right\} - \frac{1}{4}i\lambda\Delta_F(0) \int d^4x [\Phi(x)]^3 - \lambda \frac{4!}{4!} \int d^4x [\Phi(x)]^4,$$

after taking the $\epsilon \to 0$ limit. Using eq. (9) to obtain the effective action, we note that

$$\int d^4x J(x)\Phi(x) = \int d^4x \Phi(x) \left\{ (\Box_x + m^2)\Phi(x) + \frac{1}{2}i\lambda\Delta_F(0)\Phi(x) + \frac{1}{6}[\Phi(x)]^3 \right\},$$

where we have again used eq. (28) and have kept only terms up to $\mathcal{O}(\lambda)$. Hence, we end up with

$$\Gamma[\Phi] = -\frac{1}{2} \int d^4x \Phi(x) \left( \Box_x + m^2 \right)\Phi(x) - \frac{1}{4}i\lambda\Delta_F(0) \int d^4x [\Phi(x)]^3 - \frac{\lambda}{4!} \int d^4x [\Phi(x)]^4. \tag{29}$$

Finally, we make use of eq. (17) to compute the 1PI four-point function,

$$\Gamma^{(4)}(x_1, \ldots, x_4) = \left. \frac{\delta^n\Gamma[\Phi]}{\delta\Phi(x_1)\cdots\delta\Phi(x_4)} \right|_{\Phi=0}.$$

Using eq. (29),

$$\Gamma^{(4)}(x_1, \ldots, x_4) = -\lambda \int d^4x \delta^4(x-x_1)\delta^4(x-x_2)\delta^4(x-x_3)\delta^4(x-x_4).$$

In momentum space,

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4)(2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4)$$

$$= \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i(p_1x_1+\cdots+p_4x_4)} \Gamma^{(4)}(x_1, x_2, x_3, x_4)$$

$$= -\lambda \int d^4x e^{ix(p_1+\cdots+p_4)}$$

$$= -(2\pi)^4 \lambda \delta^4(p_1 + p_2 + p_3 + p_4).$$

That is,

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = -\lambda.$$

The Feynman rule for the four-point scalar interaction corresponds to $i\Gamma^{(4)}(p_1, p_2, p_3, p_4)$. 

11
4. Consider a scalar field theory defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi(x) \partial_\mu \phi(x) - V(\phi(x)),$$

and the corresponding equation of motion,

$$\Box \phi(x) + V'(\phi) = 0,$$

where $\Box \equiv \partial^\mu \partial_\mu$ and $V' \equiv dV/d\phi$.

(a) Starting from eq. (14.122) on p. 276 of Schwartz, derive the equation of motion for the Green function $\langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle$,

$$\Box \langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle = -\langle \Omega | T \{ V'(\phi(x)) \phi(y) \} | \Omega \rangle - i\delta^4(x-y).$$

Consider a scalar field theory where the Lagrangian density is given by,

$$\mathcal{L}[\phi] = -\frac{1}{2} \phi(x) \Box \phi(x) + \mathcal{L}_{\text{int}}[\phi].$$

In the case of $\mathcal{L}_{\text{int}} = -V(\phi)$, this Lagrangian density differs from eq. (30) by a total divergence, which can be neglected. That is, the corresponding actions,

$$S[\phi] = \int d^4x \mathcal{L}[\phi],$$

are the same. Then, eq. (14.122) on p. 276 of Schwartz states that

$$-i \Box \frac{\delta Z[J]}{\delta J(x)} = \left\{ \frac{\delta}{\delta J(x)} + J(x) \right\} Z[J].$$

Take a functional derivative of eq. (33) with respect to $J(y)$,

$$-i \Box \frac{\delta^2 Z[J]}{\delta J(x) \delta J(y)} = \left\{ \frac{\delta}{\delta J(x)} + J(x) \right\} \frac{\delta Z[J]}{\delta J(y)} + \delta^4(x-y) Z[J],$$

after using the product rule for differentiating and

$$\frac{\delta J(x)}{\delta J(y)} = \delta^4(x-y).$$

We now make use of the definition of the generating functional,

$$Z[J] = \frac{\int \mathcal{D}\phi \exp \left\{ iS[\phi] + i \int d^4x \ J(x) \phi(x) \right\}}{\int \mathcal{D}\phi \exp \{ iS[\phi] \}},$$

More accurately, one should employ functional derivatives in eq. (33) rather than partial derivatives.
where the action $S[\phi]$ is defined in eq. (32). It follows that

$$
\left(\frac{1}{i}\right)^2 \frac{\delta^2 Z[J]}{\delta J(x) \delta J(y)} = \frac{\int \mathcal{D}\phi \phi(x) \phi(y) \exp \left\{ iS[\phi] + i \int d^4 x \ J(x) \phi(x) \right\}}{\int \mathcal{D}\phi \exp \left\{ iS[\phi] \right\}},
$$

(37)

and

$$
\mathcal{L}'_{\text{int}} \left[ -i \frac{\delta}{\delta J(y)} \right] Z[J] = \frac{\int \mathcal{D}\phi \mathcal{L}'_{\text{int}}(\phi(x)) \phi(y) \exp \left\{ iS[\phi] + i \int d^4 x \ J(x) \phi(x) \right\}}{\int \mathcal{D}\phi \exp \left\{ iS[\phi] \right\}}.
$$

(38)

Taking another functional derivative with respect to $J(y)$ then yields,

$$
\mathcal{L}'_{\text{int}} \left[ -i \frac{\delta}{\delta J(y)} \right] \mathcal{L}'_{\text{int}} \left[ \frac{1}{i} \frac{\delta Z[J]}{\delta J(x)} \right] = \frac{\int \mathcal{D}\phi \mathcal{L}'_{\text{int}}(\phi(x)) \phi(y) \exp \left\{ iS[\phi] + i \int d^4 x \ J(x) \phi(x) \right\}}{\int \mathcal{D}\phi \exp \left\{ iS[\phi] \right\}}
$$

(39)

Employing eqs. (37) and (38) in eq. (34) and then setting $J = 0$ at the end of the computation, we end up with

$$
\Box_x \frac{\int \mathcal{D}\phi \phi(x) \phi(y) \exp \left\{ iS[\phi] \right\}}{\int \mathcal{D}\phi \exp \left\{ iS[\phi] \right\}} = \frac{\int \mathcal{D}\phi \mathcal{L}'_{\text{int}}(\phi(x)) \phi(y) \exp \left\{ iS[\phi] \right\}}{\int \mathcal{D}\phi \exp \left\{ iS[\phi] \right\}} - i \delta^4(x - y),
$$

(40)

where we have used $Z[0] = 1$.

The $n$-point Green functions are given by

$$
\langle \Omega | T \left[ \phi(x_1) \phi(x_2) \cdots \phi(x_n) \right] | \Omega \rangle = i^{-n} \left. \frac{\delta^n Z[J]}{\delta J(x_1) \delta J(x_2) \cdots \delta J(x_n)} \right|_{J=0}.
$$

Using eq. (36), it follows that

$$
\langle \Omega | T \left[ \phi(x_1) \phi(x_2) \cdots \phi(x_n) \right] | \Omega \rangle = \frac{\int \mathcal{D}\phi \phi(x_1) \phi(x_2) \cdots \phi(x_n) \exp \left\{ iS[\phi] \right\}}{\int \mathcal{D}\phi \exp \left\{ iS[\phi] \right\}}.
$$

(40)

Since $\mathcal{L}_{\text{int}} = -V(\phi)$, we see that for a potential that is polynomial in $\phi$ (or more generally, by expanding $V(\phi)$ as a functional Taylor series in $\phi$), eq. (39) is equivalent to,

$$
\Box_x \langle \Omega | T \left\{ \phi(x) \phi(y) \right\} | \Omega \rangle = -\langle \Omega | T \left\{ V'(\phi(x)) \phi(y) \right\} | \Omega \rangle - i \delta^4(x - y).
$$

That is, eq. (31) is proven.
(b) Derive eq. (31) by the following technique. Start from the path integral definition of the generating functional,

\[ Z[J] = \mathcal{N} \int \mathcal{D}\phi \exp \left\{ i \int d^4 x \left[ \mathcal{L} + J(x)\phi(x) \right] \right\}, \tag{41} \]

where \( \mathcal{N} \) is chosen such that \( Z[0] = 1 \). Perform a change of variables in the path integral, \( \phi(x) \to \phi(x) + \varepsilon(x) \), where \( \varepsilon(x) \) is an arbitrary infinitesimal function of \( x \). Noting that a change of variables\(^4\) does not change the value of of \( Z[J] \), show that to first order in \( \varepsilon(x) \),

\[ \int \mathcal{D}\phi \exp \left\{ i \int d^4 x \left[ \mathcal{L} + J(x)\phi(x) \right] \right\} \int d^4 x \varepsilon(x) \left[ -\Box \Phi - V'(\phi) + J(x) \right] = 0. \tag{42} \]

Since \( \varepsilon(x) \) is arbitrary, we may choose \( \varepsilon(x) = \varepsilon \delta^4(x-y) \), where \( \varepsilon \) is an infinitesimal constant. With this choice for \( \varepsilon(x) \), show that by taking the functional derivative of the eq. (42) with respect to \( J(x) \) and then setting \( J = 0 \), one can derive eq. (31).

The Jacobian corresponding to the change of field variables, \( \phi(x) \to \phi(x) + \varepsilon(x) \) is unity. Applying this change of variables to eq. (41) yields

\[ Z[J] = \mathcal{N} \int \mathcal{D}\phi \exp \left\{ i \int d^4 x \left[ \mathcal{L} + J(x)\phi(x) \right] \right\} \exp \left\{ i \int d^4 x \left[ \partial^\mu \phi \partial_\mu \varepsilon - \varepsilon(x)V'(\phi) + \varepsilon(x)J(x) \right] \right\}, \]

where we have used \( V(\phi + \varepsilon) = V(\phi) + \varepsilon V'(\phi) + \mathcal{O}(\varepsilon^2) \), and we have dropped all terms of \( \mathcal{O}(\varepsilon^2) \). We can further expand the second exponential above, keeping only those terms up to of \( \mathcal{O}(\varepsilon) \). Subtracting the resulting expression from eq. (41) yields

\[ i\mathcal{N} \int \mathcal{D}\phi \exp \left\{ i \int d^4 x \left[ \mathcal{L} + J(x)\phi(x) \right] \right\} \int d^4 x \varepsilon(x) \left[ -\Box \Phi - V'(\phi) + J(x) \right] = 0, \]

after an integration by parts. Since this expression is valid for any infinitesimal function \( \varepsilon(x) \), we may choose \( \varepsilon(x) = \varepsilon \delta^4(x-y) \). We can then carry out the second integration above to obtain,

\[ \mathcal{N} \int \mathcal{D}\phi \exp \left\{ i \int d^4 x \left[ \mathcal{L} + J(x)\phi(x) \right] \right\} \left[ -\Box \Phi - V'(\phi) + J(y) \right] = 0. \tag{43} \]

We now take the functional derivative of eq. (43) with respect to \( J(x) \) and employ eq. (35). Setting \( J = 0 \) at the end of the calculation, we end up with

\[ -i\mathcal{N} \int \mathcal{D}\phi \partial_y \phi(y) \exp \left\{ i \int d^4 x \mathcal{L} \right\} + \mathcal{N} \int \mathcal{D}\phi \delta^4(x-y) \exp \left\{ i \int d^4 x \mathcal{L} \right\} = 0. \]

We can pull \( \Box y \) outside of the path integral (since it does not by itself depend on the field configurations that one is integrating over). Thus,

\[ \Box_y \mathcal{N} \int \mathcal{D}\phi \phi(x)\phi(y) \exp \left\{ i \int d^4 x \mathcal{L} \right\} = -\mathcal{N} \int \mathcal{D}\phi \phi(x)V'(\phi(y)) \exp \left\{ i \int d^4 x \mathcal{L} \right\} \]

\[ -i\delta^4(x-y)\mathcal{N} \int \mathcal{D}\phi \exp \left\{ i \int d^4 x \mathcal{L} \right\}. \tag{44} \]

\(^4\)Just as in the case of ordinary integration, a change of functional integration variables does not change the value of the functional integral.
The constant $N$ is determined from the condition $Z[0] = 1$. That is,

$$N^{-1} = \int \mathcal{D}\phi \exp \left\{ i \int d^4x \mathcal{L} \right\}.$$ 

Thus, eq. (44) can be rewritten as

$$\Box_y \int \mathcal{D}\phi \phi(x) \phi(y) \exp \left\{ i \int d^4x \mathcal{L} \right\} = \int \mathcal{D}\phi \phi(x) V'(\phi(y)) \exp \left\{ i \int d^4x \mathcal{L} \right\} - i\delta^4(x-y).$$

In light of eq. (40), we see that eq. (45) is equivalent to

$$\Box_y \langle \Omega | T\{ \phi(x) \phi(y) \} | \Omega \rangle = -\langle \Omega | T\{ \phi(x) V'(\phi(y)) \} | \Omega \rangle - i\delta^4(x-y).$$

We now redefine the the variables $x$ and $y$ by interchanging $x \leftrightarrow y$ in eq. (46). Because the ordering of the fields that appear inside a time ordered product is irrelevant (since it is the time ordering prescription that dictates the order of the fields in a time-ordered product), and using the fact that $\delta^4(x-y)$ is an even function of its argument, we obtain eq. (31) as expected.

5. Consider a field theory of a real pseudoscalar field coupled to a fermion field. The interaction Lagrangian is:

$$\mathcal{L}_{\text{int}} = -i\lambda \bar{\psi}(x) \gamma_5 \psi(x) \phi(x),$$

where $\lambda$ is a real coupling constant (called the Yukawa coupling). Using functional techniques, derive the Feynman rule for the interaction vertex of this theory.

In class, we derived expressions for the generating functional for a free scalar field theory

$$Z_0[J] = \exp \left\{ -\frac{1}{2} i \int d^4x d^4y J(x) \Delta_F(x-y) J(y) \right\},$$

and the generating functional for a free Dirac fermion field theory,

$$Z_0[\bar{\zeta}, \zeta] = \exp \left\{ -i \int d^4x d^4y \bar{\zeta}(x) S_F(x-y) \zeta(y) \right\},$$

where $\Delta_F$ and $S_F$ are the free-field propagators of the scalar and Dirac fermion fields, respectively. $J$ is a commuting source and $\bar{\zeta}$ and $\zeta$ are anticommuting sources. For a free field theory consisting of both a scalar and a Dirac fermion field, the generating functionals given above can be combined,

$$Z_0[J, \bar{\zeta}, \zeta] = N_0 \int \mathcal{D}\phi \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left\{ i \left[ S_0 + \int d^4x \left[ J(x) \Phi(x) + \bar{\zeta}(x) \psi(x) + \bar{\psi}(x) \zeta(x) \right] \right] \right\}$$

$$= \exp \left\{ -\frac{1}{2} i \int d^4x d^4y J(x) \Delta_F(x-y) J(y) \right\} \exp \left\{ -i \int d^4z d^4w \bar{\zeta}(z) S_F(z-w) \zeta(w) \right\},$$

(48)
where $S_0$ is the action of the free field theory, and $N_0$ is a normalization constant chosen such that $Z[0,0,0] = 1$.

For the interacting theory,

$$Z[J, \zeta, \zeta] = \mathcal{N} \int D\phi D\bar{\psi} D\psi \exp \left\{ i \left[ S_0 + S_{\text{int}} + \int d^4x \left[ J(x)\Phi(x) + \zeta(x)\psi(x) + \bar{\psi}(x)\zeta(x) \right] \right] \right\},$$

where

$$S_{\text{int}} \equiv \int d^4x \mathcal{L}_{\text{int}},$$

and $\mathcal{N}$ is a normalization constant chosen such that $Z[0,0,0] = 1$. In analogy with eq. (21), we can rewrite eq. (49) as

$$Z[J, \zeta, \zeta] = \mathcal{N} \exp \left\{ iS_{\text{int}} \left( \frac{1}{i} \frac{\delta}{\delta J(x)} , i \frac{\delta}{\delta \zeta(x)} \right) \right\} Z_0[J, \zeta, \zeta].$$

Note the appearance of $i\delta/\delta \zeta$ in eq. (50). The reason for the factor of $i$ instead of $1/i$ is due to the anticommutative properties of $\zeta$ and $\zeta$. In particular, note that

$$\frac{\delta}{\delta \zeta(x)} \int d^4y \left[ \zeta(y)\psi(y) + \bar{\psi}(y)\zeta(y) \right] = -\bar{\psi}(x),$$

$$\frac{\delta}{\delta \zeta(x)} \int d^4y \left[ \zeta(y)\psi(y) + \bar{\psi}(y)\zeta(y) \right] = \psi(x),$$

after using the delta functions obtained via

$$\frac{\delta \zeta(y)}{\delta \zeta(x)} = \delta^4(y - x), \quad \frac{\delta \zeta(y)}{\delta \bar{\zeta}(x)} = \delta^4(y - x),$$

to integrate over $y$. The minus sign on the right hand side of eq. (51), which arises when we move $\delta/\delta \zeta(x)$ past $\bar{\psi}(y)$, is properly compensated for by employing $i\delta/\delta \zeta(x)$ in eq. (50).

It is convenient to write the interaction Lagrangian with the spinor indices made explicit,

$$\mathcal{L}_{\text{int}} = -i\lambda \bar{\psi}_\alpha(x) (\gamma_5)_{\alpha\beta} \psi_\beta(x)\phi(x),$$

where repeated indices are summed over. Then, to first order in perturbation theory,

$$\exp \left\{ iS_{\text{int}} \left( \frac{1}{i} \frac{\delta}{\delta J(x)} , i \frac{\delta}{\delta \zeta_\alpha(x)} \right) \right\} = 1 + \lambda(\gamma_5)_{\alpha\beta} \int d^4x \frac{1}{i} \frac{\delta}{\delta J(x)} \frac{\delta}{\delta \zeta_\alpha(x)} \frac{1}{i} \frac{\delta}{\delta \bar{\zeta}_\beta(x)} .$$

It follows that to $\mathcal{O}(\lambda)$,

$$Z[J, \zeta, \zeta] = \left\{ 1 + \lambda(\gamma_5)_{\alpha\beta} \int d^4x \frac{1}{i} \frac{\delta}{\delta J(x)} \frac{\delta}{\delta \zeta_\alpha(x)} \frac{1}{i} \frac{\delta}{\delta \bar{\zeta}_\beta(x)} \right\} Z_0[J, \zeta, \zeta].$$
The three-point Green function is given by

\[ G^{(3)}(y, z, w)_{\rho \sigma} = \langle \Omega | T \left[ \begin{array}{c} \phi(y) \psi_\rho(z) \overline{\psi}_\sigma(w) \end{array} \right] | \Omega \rangle = \frac{1}{i} \frac{\delta}{\delta J(y)} \frac{1}{i} \frac{\delta}{\delta \zeta_\rho(z)} \frac{\delta}{\delta \zeta_\sigma(w)} Z[J, \zeta, \zeta] \bigg|_{J=\zeta=\zeta=0}. \]

(54)

There is no \( O(\lambda^0) \) contribution to \( G^{(3)} \), since a factor of \( J \) arises when one takes a functional derivative with respect to \( J \). Thus, the end result vanishes when taking \( J = 0 \). Thus, at \( O(\lambda) \),

\[ G^{(3)}(y, z, w)_{\rho \sigma} = -\lambda(\gamma_5)_{\alpha \beta} \frac{\delta}{\delta J(y)} \frac{1}{i} \frac{\delta}{\delta \zeta_\rho(z)} \frac{\delta}{\delta \zeta_\sigma(w)} \left\{ \int d^4x \frac{\delta}{\delta J(x)} \frac{\delta}{\delta \xi_\alpha(x)} \frac{\delta}{\delta \xi_\beta(x)} Z_0[J, \zeta, \zeta] \right\} \bigg|_{J=\zeta=\zeta=0} \]

Using eqs. (47) and (48),

\[ G^{(3)}(y, z, w)_{\rho \sigma} = \lambda(\gamma_5)_{\alpha \beta} \frac{\delta}{\delta \zeta_\rho(z)} \frac{1}{i} \frac{\delta}{\delta \zeta_\sigma(w)} \left\{ \int d^4x i\Delta_F(y-x) \frac{\delta}{\delta \xi_\alpha(x)} \frac{\delta}{\delta \xi_\beta(x)} Z_0[\zeta, \zeta] \right\} \bigg|_{\zeta=\zeta=0} \]

\[ = \lambda(\gamma_5)_{\alpha \beta} \frac{\delta}{\delta \zeta_\rho(z)} \frac{1}{i} \frac{\delta}{\delta \zeta_\sigma(w)} \left\{ \int d^4x i\Delta_F(y-x) \left[ -iS_F(0)_{\beta \alpha} Z_0[\zeta, \zeta] \right. \right. \]

\[ \left. \left. - \int d^4x_1 d^4x_2 S_F(x-x_2)_{\beta \tau} \zeta_\tau(x_2) S_F(x_1-x)_{\gamma \alpha} \zeta_\gamma(x_1) Z_0[\zeta, \zeta] \right] \right\} \bigg|_{\zeta=\zeta=0} \]

\[ = \lambda(\gamma_5)_{\alpha \beta} \int d^4x i\Delta_F(y-x) \left[ iS_F(x-w)_{\beta \sigma} iS_F(z-x)_{\rho \alpha} - iS_F(0)_{\beta \alpha} iS_F(z-w)_{\rho \sigma} \right], \]

(55)

after using the delta functions obtained via

\[ \frac{\delta \zeta_\tau(x_2)}{\delta \zeta_\sigma(w)} = \delta_{\tau \sigma} \delta^4(x_2 - w), \quad \frac{\delta \zeta_\gamma(x_1)}{\delta \zeta_\rho(z)} = \delta_{\gamma \rho} \delta^4(x_1 - z), \]

to integrate over \( x_1 \) and \( x_2 \), and noting that \( Z[0,0] = 1 \). Diagrammatically, the two terms represented by eq. (55) are:

The second diagram above is disconnected. Thus, the connected three-point Green function is given by

\[ G^{(3)}_c(y, z, w)_{\rho \sigma} = \lambda(\gamma_5)_{\alpha \beta} \int d^4x i\Delta_F(y-x) iS_F(x-w)_{\beta \sigma} iS_F(z-x)_{\rho \alpha}. \]

The order of the functional derivatives in eq. (54) is determined by the order of the fermion fields inside the time-ordered product [cf. eqs. (51) and (52)]. Different orderings can yield a different overall sign since the fermion fields anticommute.
We can easily read off the Feynman rules in coordinate space:

\[ y \rightarrow x, \quad i\Delta_F(x - y) \]

\[ y, \beta \rightarrow x, \alpha, \quad iS_F(x - y)_{\alpha\beta} \]

\[ \lambda(\gamma_5)_{\alpha\beta} \int d^4x \]

Note that the order of the spinor indices corresponds to traversing the Feynman diagram in the direction opposite to the direction of the fermion line arrows.

The Feynman rules in momentum space are obtained following the same procedure given in problem 3(b). We define

\[ G^{(3)}_c(p_1, p_2, p_3)_{\rho\sigma} = \int d^4x_1 d^4x_2 d^4x_3 e^{ip_1 x_1 + p_2 x_2 + p_3 x_3} G^{(4)}_c(x_1, x_2, x_3) \]

\[ = \lambda(\gamma_5)_{\alpha\beta} \int d^4x_1 d^4x_2 d^4x_3 e^{ip_1 x_1 + p_2 x_2 + p_3 x_3} i\Delta_F(x_1 - x) iS_F(x_2 - x)_{\beta\sigma} \]

\[ = -i\lambda \int d^4x_1 d^4x_2 d^4x_3 e^{ip_1 x_1 + p_2 x_2 + p_3 x_3} e^{ip_1 (x_1 - x)} \Delta_F(x - x_1) \]

\[ \times e^{ip_2 (x_2 - x)} iS_F(x_2 - x)_{\rho\alpha} e^{ip_3 (x_3 - x)} iS_F(x - x_3)_{\beta\sigma} . \]

We can now perform the integration over \( x_1, x_2 \) and \( x_3 \) using the expression for the free-field propagators in momentum space,

\[ \frac{1}{p^2 - m_s^2 + i\epsilon} = \int d^4x e^{-ipx} \Delta_F(x), \quad \frac{1}{p^2 - m_f^2 + i\epsilon} = \int d^4x e^{-ipx} S_F(x)_{\alpha\beta}, \]

where \( m_s \) and \( m_f \) are the masses of the pseudoscalar and fermion, respectively. Finally, using the integral representation of the momentum conserving delta function, the end result is

\[ G^{(3)}_c(p_1, p_2, p_3)_{\rho\sigma} = \frac{i}{p_1^2 - m_s^2 + i\epsilon} \frac{i(p_2 + m_f)_{\rho\alpha}}{p_2^2 - m_f^2 + i\epsilon} \frac{i(p_3 + m_f)_{\beta\sigma}}{p_3^2 - m_f^2 + i\epsilon} \lambda(\gamma_5)_{\alpha\beta} . \]

If we now amputate the three external propagators, we arrive at the momentum space Feynman rule for the pseudoscalar–fermion Yukawa interaction:

\[ \lambda(\gamma_5)_{\alpha\beta} \]

The momentum-space Feynman rule is simply obtained by removing the fields from \( i\mathcal{L}_{\text{int}} \) given in eq. (53). As previously noted, the order of the spinor indices corresponds to traversing the Feynman diagram in the direction opposite to the direction of the fermion line arrows.