1. Show that for complex scalar fields $\Phi$ and a positive definite hermitian operator $M$,

$$\int \mathcal{D}\Phi^*\,\mathcal{D}\Phi \exp\left\{i\int d^4x\,d^4y\big[\Phi^*(x)M(x,y)\Phi(y)\big]+i\int d^4x\,\big[J^*(x)\Phi(x)+\Phi^*(x)J(x)\big]\right\}$$
$$=\mathcal{N}\,\frac{1}{\det M}\exp\left\{-i\int d^4x\,d^4y\,J^*(x)M^{-1}(x,y)J(y)\right\}\,, \qquad (1)$$

for some infinite constant $\mathcal{N}$.

We shall prove the analog of eq. (1) for the integration over $n$ complex variables,

$$\int dz_1^*dz_1dz_2^*dz_2\cdots dz_n^*dz_n\,\exp\big\{i(z_i^*M_{ij}z_j+J_i^*z_i+z_i^*J_i)\big\}=\mathcal{N}\,\frac{1}{\det M}\exp\big\{-iJ_i^*(M^{-1})_{ij}J_j\big\}\,, \qquad (2)$$

where $M_{ij}$ are the matrix elements of an hermitian $n\times n$ matrix $M$. We can write

$$z_i=x_i+iy_i\,,\qquad M=M_R+iM_I\,,\qquad J=J_R+iJ_I\,,$$

where $M_R^{\mathsf{T}}=M_R$ and $M_I^{\mathsf{T}}=-M_I$. Then,

$$z_i^*M_{ij}z_j+J_i^*z_i+z_i^*J_i=x_i(M_R)_{ij}x_j+y_i(M_R)_{ij}y_j+y_i(M_I)_{ij}x_j-x_i(M_I)_{ij}y_j+2\big[(J_R)_ix_i+(J_I)_iy_i\big]\,. \qquad (3)$$

We can introduce real variables $v_k$, where $k=1,2,\ldots,2n$, such that

$$v_k=\begin{cases}x_k\,, & \text{for } k=1,2,\ldots,n\,,\\ y_{k-n}\,, & \text{for } k=n+1,n+2,\ldots 2n\,.\end{cases}$$

Eq. (3) can then be rewritten as

$$z_i^*M_{ij}z_j+J_i^*z_i+z_i^*J_i=\tfrac{1}{2}v_k\mathcal{M}_{k\ell}v_\ell+\mathcal{J}_kv_k\,,$$

where $\mathcal{M}$ is the real symmetric $2n\times 2n$ matrix,

$$\mathcal{M}\equiv\left(\begin{array}{c|c}2M_R & -2M_I\\ \hline 2M_I & 2M_R\end{array}\right)\,,$$

written in block form, and

$$\mathcal{J}_k=\begin{cases}2(J_R)_k\,, & \text{for } k=1,2,\ldots,n\,,\\ 2(J_I)_{k-n}\,, & \text{for } k=n+1,n+2,\ldots 2n\,.\end{cases}$$

In light of the Jacobian,

$$\frac{\partial(z^*,z)}{\partial(x,y)}=\det\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}=2i\,,$$

we have[1]

$$dz^*\,dz = \frac{\partial(z^*,z)}{\partial(x,y)}\,dxdy = 2i\,dxdy\,, \tag{4}$$

and the left hand side of eq. (2) can be written as

$$(2i)^n \int_{-\infty}^{\infty} dv_1 \int_{-\infty}^{\infty} dv_2 \cdots \int_{-\infty}^{\infty} dv_{2n}\ \exp\bigl\{i(\tfrac{1}{2}v_k\mathcal{M}_{k\ell}v_\ell + \mathcal{J}_k v_k)\bigr\}\,.$$

To evaluate this integral, we first consider

$$I(\mathcal{M},\mathcal{J}) \equiv \int_{-\infty}^{\infty} dv_1 \int_{-\infty}^{\infty} dv_2 \cdots \int_{-\infty}^{\infty} dv_{2n}\ \exp\bigl\{-\tfrac{1}{2}v_k\mathcal{M}_{k\ell}v_\ell + \mathcal{J}_k v_k\bigr\}\,. \tag{5}$$

Introduce the following change of variables,

$$v_k = (\mathcal{M}^{-1})_{km}\mathcal{J}_m + w_k\,.$$

Inserting this into eq. (5) and noting that the corresponding Jacobian is 1, we obtain

$$I(\mathcal{M},\mathcal{J}) = \exp\bigl\{\tfrac{1}{2}\mathcal{J}_k(\mathcal{M}^{-1})_{km}\mathcal{J}_m\bigr\} \int_{-\infty}^{\infty} dw_1 \int_{-\infty}^{\infty} dw_2 \cdots \int_{-\infty}^{\infty} dw_{2n}\ \exp\bigl\{-\tfrac{1}{2}w_k\mathcal{M}_{k\ell}w_\ell\bigr\}$$

$$= \frac{(2\pi)^n}{(\det\mathcal{M})^{1/2}}\ \exp\bigl\{\tfrac{1}{2}\mathcal{J}_k(\mathcal{M}^{-1})_{km}\mathcal{J}_m\bigr\}\,, \tag{6}$$

after using eq. (14.7) of Schwartz to evaluate the $2n$-dimensional gaussian integral.

Using the arguments in class to justify the substitutions, $\mathcal{M}\to -i\mathcal{M}$ and $\mathcal{J}\to i\mathcal{J}$, and noting that $\det(-i\mathcal{M}) = (-i)^{2n}\det\mathcal{M}$ since $\mathcal{M}$ is a $2n\times 2n$ matrix,

$$I(-i\mathcal{M}, i\mathcal{J}) = \frac{(2\pi i)^n}{(\det\mathcal{M})^{1/2}}\ \exp\bigl\{-\tfrac{1}{2}i\mathcal{J}_k(\mathcal{M}^{-1})_{km}\mathcal{J}_m\bigr\}\,.$$

We need to evaluate $\det\mathcal{M}$ and $\mathcal{M}^{-1}$. First, we note that

$$\mathcal{M} = \left(\begin{array}{c|c} 2M_R & -2M_I \\ \hline 2M_I & 2M_R \end{array}\right) = \left(\begin{array}{c|c} \mathbb{1} & \mathbb{1} \\ \hline i\mathbb{1} & -i\mathbb{1} \end{array}\right) \left(\begin{array}{c|c} M_R - iM_I & \mathbb{O} \\ \hline \mathbb{O} & M_R + iM_I \end{array}\right) \left(\begin{array}{c|c} \mathbb{1} & -i\mathbb{1} \\ \hline \mathbb{1} & i\mathbb{1} \end{array}\right), \tag{7}$$

where $\mathbb{1}$ is the $n\times n$ identity matrix and $\mathbb{O}$ is the $n\times n$ zero matrix. It is straightforward to evaluate

$$\det\left(\begin{array}{c|c} \mathbb{1} & \mathbb{1} \\ \hline i\mathbb{1} & -i\mathbb{1} \end{array}\right) = (-2i)^n\,, \qquad \det\left(\begin{array}{c|c} \mathbb{1} & -i\mathbb{1} \\ \hline \mathbb{1} & i\mathbb{1} \end{array}\right) = (2i)^n\,.$$

Hence, recalling that $M = M_R + iM_I$, it follows that $\det\mathcal{M} = 2^{2n}|\det M|^2$.

[1]Employing the formalism of differential forms, $dz^*\wedge dz = (dx - idy)\wedge(dx + idy) = 2i\,dx\wedge dy$, which is equivalent to eq. (4). In deriving this result, we have used the well known properties of differential forms, $dx\wedge dx = dy\wedge dy = 0$ and $dx\wedge dy = -dy\wedge dx$. For an elementary introduction to differential forms, see e.g., James J. Callahan, *Advanced Calculus: A Geometric View* (Springer Science, New York, 2010) Chapter 10.3.

Next, to compute $\mathcal{M}^{-1}$, we again use eq. (7) to obtain

$$\mathcal{M}^{-1} = \frac{1}{4}\left(\begin{array}{c:c} \mathbb{1} & \mathbb{1} \\ \hdashline i\mathbb{1} & -i\mathbb{1} \end{array}\right)\left(\begin{array}{c:c} M^{*\,-1} & \mathbb{O} \\ \hdashline \mathbb{O} & M^{-1} \end{array}\right)\left(\begin{array}{c:c} \mathbb{1} & -i\mathbb{1} \\ \hdashline \mathbb{1} & i\mathbb{1} \end{array}\right).$$

Thus, using matrix notation,

$$\begin{aligned} \mathcal{J}_k(\mathcal{M}^{-1})_{km}\mathcal{J}_m &= (J_R \quad J_I)\left(\begin{array}{c:c} \mathbb{1} & \mathbb{1} \\ \hdashline i\mathbb{1} & -i\mathbb{1} \end{array}\right)\left(\begin{array}{c:c} M^{*\,-1} & \mathbb{O} \\ \hdashline \mathbb{O} & M^{-1} \end{array}\right)\left(\begin{array}{c:c} \mathbb{1} & -i\mathbb{1} \\ \hdashline \mathbb{1} & i\mathbb{1} \end{array}\right)\begin{pmatrix} J_R \\ J_I \end{pmatrix} \\ &= (J \quad J^*)\left(\begin{array}{c:c} M^{*\,-1} & \mathbb{O} \\ \hdashline \mathbb{O} & M^{-1} \end{array}\right)\begin{pmatrix} J^* \\ J \end{pmatrix} \\ &= JM^{*\,-1}J^* + J^*M^{-1}J \\ &= 2J^*M^{-1}J\,, \end{aligned}$$

where we have used the fact that $M$ is hermitian (which implies that $M^{-1}$ is hermitian) to conclude that $JM^{-1}J^* \equiv J_i^*(M^{-1})_{ij}J_j = J_i^*(M^{*\,-1})_{ji}J_j = J_j(M^{*\,-1})_{ji}J_i^* \equiv JM^{*\,-1}J^*$. Hence,

$$I(-i\mathcal{M}, i\mathcal{J}) = \frac{(\pi i)^n}{\det M}\exp\bigl\{-iJ_i^*(M^{-1})_{ij}J_j\bigr\}\,.$$

Note that since $M$ is positive definite, $\det M$ is positive and we can dispense with the absolute value signs.

We conclude that

$$\begin{aligned} &\int dz_1^* dz_1 dz_2^* dz_2 \cdots dz_n^* dz_n \, \exp\bigl\{i(z_i^* M_{ij} z_j + J_i^* z_i + z_i^* J_i)\bigr\} \\ &\qquad\qquad = (2i)^n I(-i\mathcal{M}, i\mathcal{J}) = \frac{(-2\pi)^n}{\det M}\exp\bigl\{-iJ_i^*(M^{-1})_{ij}J_j\bigr\}\,. \qquad (8) \end{aligned}$$

That is, we have confirmed eq. (2), with $\mathcal{N} = (-2\pi)^n$. In the limit of $n \to \infty$, the constant $\mathcal{N}$ is infinite. The generalization to eq. (1) is now straightforward.

2. (a) Derive the result:

$$\int d^4z \, \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)} = -\delta^4(x-y)\,,$$

and interpret diagrammatically. Here, $W[J]$ is the generating functional for the connected Green functions and $\Gamma[\Phi]$ is the generating functional for the one particle irreducible (1PI) Green functions.

We begin with the definition of the effective action,

$$\Gamma[\Phi] = W[J] - \int d^4x\, J(x)\Phi(x)\,, \tag{9}$$

where $\Phi(x)$ is the classical field [which was denoted in class by $\Phi_c(x)$]. It follows that

$$\frac{\delta W[J]}{\delta J(x)} = \Phi(x)\,, \qquad\qquad \frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = -J(x)\,. \tag{10}$$

Taking a second functional derivative yields,

$$\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} = \frac{\delta\Phi(x)}{\delta J(y)}\,, \qquad\qquad \frac{\delta^2\Gamma[\Phi]}{\delta\Phi(x)\delta\Phi(y)} = -\frac{\delta J(x)}{\delta\Phi(y)}\,.$$

Hence, it follows that

$$\int d^4z\, \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\,\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)} = -\int d^4z \frac{\delta\Phi(x)}{\delta J(z)}\,\frac{\delta J(z)}{\delta\Phi(y)} = -\frac{\delta\Phi(x)}{\delta\Phi(y)} = -\delta^4(x-y)\,, \tag{11}$$

where we have used the chain rule for functional derivatives at the second step above.

Recall that the two-point 1PI Green function, $\Gamma^{(2)}(x_1,x_2)$, and the two-point connected Green function $G_c^{(2)}(x_1,x_2)$, are defined as

$$\Gamma^{(2)}(x_1,x_2) = \left(\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(x)\delta\Phi(y)}\right)\Bigg|_{\Phi=0}\,, \qquad\qquad G^{(2)}(x_1,x_2) = -i\left(\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}\right)\Bigg|_{J=0}\,,$$

We shall assume that the quantum field $\phi(x)$ has no vacuum expectation value,[2] in which case

$$G_c^{(1)}(x) \equiv \langle\Omega|\phi(x)|\Omega\rangle = \frac{\delta W[J]}{\delta J(x)}\bigg|_{J=0} = \Phi(x)\big|_{J=0} = 0\,. \tag{12}$$

That is, setting $J=0$ implies that $\Phi=0$ and vice versa.

Using the above results, eq. (11) implies that

$$\int d^4z\, \Gamma^{(2)}(x,z)G_c^{(2)}(z,y) = i\delta^4(x-z)\,, \tag{13}$$

In momentum space, eq. (13) yields

$$\Gamma^{(2)}(p,-p)G_c^{(2)}(p,-p) = i\,.$$

Since $G_c^{(2)}(p,-p)$ is the momentum space propagator, it follows that $i\Gamma^{(2)}(p,-p)$ is the *negative* of the inverse propagator in momentum space.

[2]If $\langle\Omega|\phi(x)|\Omega\rangle = v \neq 0$, then one can redefine the quantum field by redefining $\phi \to \phi + v$.

(b) By taking one further functional derivative, show that $\Gamma$ generates the amputated connected three-point function.

We shall take a functional derivative of eq. (11). On the right hand side of eq. (11), we have

$$\frac{\delta}{\delta J(w)}\,\delta^4(x-y)=0\,,$$

since $\delta^4(x-y)$ is the analog of the Kronecker delta, $\delta_{ij}$, for an infinite dimensional function space. On the left hand side of eq. (11),

$$\frac{\delta}{\delta J(w)}\left[\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\,\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)}\right]=\frac{\delta^3 W[J]}{\delta J(w)\delta J(x)\delta J(z)}\,\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)}+\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\,\frac{\delta}{\delta J(w)}\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)}\,.$$

In the second term on the right hand side above, we use the chain rule,

$$\frac{\delta}{\delta J(w)}=\int d^4v\,\frac{\delta\phi(v)}{\delta J(w)}\,\frac{\delta}{\delta\phi(v)}=\int d^4v\,\frac{\delta^2 W[J]}{\delta J(w)\delta J(v)}\,\frac{\delta}{\delta\phi(v)}\,,$$

after using the definition of the classical field $\phi(v)$ given in eq. (10). Hence, eq. (11) yields

$$\int d^4z\,\frac{\delta^3 W[J]}{\delta J(w)\delta J(x)\delta J(z)}\,\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)}+\int d^4z\,d^4v\,\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\,\frac{\delta^2 W[J]}{\delta J(w)\delta J(v)}\,\frac{\delta^3\Gamma[\Phi]}{\delta\Phi(v)\delta\Phi(z)\delta\Phi(y)}=0\,. \tag{14}$$

We now multiply eq. (14) by

$$\frac{\delta^2 W[J]}{\delta J(y)\delta J(u)}\,,$$

and integrate over $d^4y$. Using the result of eq. (11), we obtain

$$\begin{aligned}\int d^4z\,d^4y&\frac{\delta^3 W[J]}{\delta J(w)\delta J(x)\delta J(z)}\,\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)}\,\frac{\delta^2 W[J]}{\delta J(y)\delta J(u)}\\ &=-\int d^4z\frac{\delta^3 W[J]}{\delta J(w)\delta J(x)\delta J(z)}\,\delta^4(z-u)\\ &=-\frac{\delta^3 W[J]}{\delta J(w)\delta J(x)\delta J(u)}\,.\end{aligned}$$

Applying this result to eq. (14) yields

$$\frac{\delta^3 W[J]}{\delta J(w)\delta J(x)\delta J(u)}=\int d^4v\,d^4y\,d^4z\,\frac{\delta^3\Gamma[\Phi]}{\delta\Phi(v)\delta\Phi(z)\delta\Phi(y)}\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\,\frac{\delta^2 W[J]}{\delta J(w)\delta J(v)}\,\frac{\delta^2 W[J]}{\delta J(y)\delta J(u)}\,. \tag{15}$$

Recall the definition of the connected $n$-point Green function,

$$G_c^{(n)}(x_1,x_2,\ldots,x_n)=i^{1-n}\,\frac{\delta^n W[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}\bigg|_{J=0}\,, \tag{16}$$

and the $n$-point 1PI Green function,

$$\Gamma^{(n)}(x_1, x_2, \ldots, x_n) = \left.\frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1)\delta\phi(x_2)\cdots\delta\phi(x_n)}\right|_{\phi=0} . \tag{17}$$

In light of eq. (12), we can set $J = \phi = 0$ in eq. (15). Using eqs. (16) and (17), it then follows that

$$G_c^{(3)}(w, x, u) = i\int d^4v\, d^4y\, d^4z\, \Gamma^{(3)}(v, z, y) G_c^{(2)}(v, w) G_c^{(2)}(z, x) G_c^{(2)}(y, u) . \tag{18}$$

To invert this equation, we make use of the inverse propagator, which satisfies

$$\int d^4z\, G_c^{(2)}(x, z) G_c^{(2)\,-1}(z, y) = \delta^4(x - y) .$$

Then, we can rewrite eq. (18) as

$$i\Gamma^{(3)}(v, z, y) = \int d^4w\, d^4x\, d^4u\, G_c^{(2)\,-1}(w, v) G_c^{(2)\,-1}(x, z) G_c^{(2)\,-1}(u, y) G_c^{(3)}(w, x, u) .$$

The effect of the factors of $G_c^{(2)\,-1}$ is to remove the explicit propagators that appear on the external legs of the three-point Green function. That is, $i\Gamma^{(3)}$ is obtained from $G_c^{(3)}$ by amputating the full propagators on the three external legs.

3. Consider the quantum field theory of a real scalar field governed by the Lagrangian,

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 . \tag{19}$$

(a) Evaluate the generating functional $Z[J]$ perturbatively, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$ as follows. First, show that $Z[J]$ can be written in the following form,

$$Z[J] = \mathcal{N}\left[1 - \frac{i\lambda}{4!}\int d^4y \left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^4 + \mathcal{O}(\lambda^2)\right] \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1)\Delta_F(x_1 - x_2)J(x_2)\right\} , \tag{20}$$

where $\mathcal{N}$ is the $J$-independent constant. Then, carry out the functional derivatives with respect to $J$, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$. Using the result just obtained for $Z[J]$, obtain an expression for the generating functional for the connected Green functions, $W[J]$, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$.

The generating functional for the connected Green functions, $W[J]$, is determined by

$$Z[J] = \exp\{iW[J]\} , \tag{21}$$

where

$$Z[J] = \frac{\displaystyle\int \mathcal{D}\phi\, \exp\left\{i\int d^4x \left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi\right]\right\}}{\displaystyle\int \mathcal{D}\phi\, \exp\left\{i\int d^4x \left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right]\right\}} . \tag{22}$$

Expanding the numerator of eq. (22) to $\mathcal{O}(\lambda)$,

$$\int \mathcal{D}\phi \, \exp\left\{i\int d^4x\left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 + J\phi\right]\right\}\left[1 - i\int d^4y \frac{\lambda}{4!}\phi^4(y)\right]$$

$$= \left[1 - \frac{i\lambda}{4!}\int d^4y\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^4\right]\int \mathcal{D}\phi \exp\left\{i\int d^4x\left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 + J\phi\right]\right\},$$

since each $i^{-1}\delta/\delta J(x)$ operator brings down a factor of $\phi(x)$.

Next we use the result obtained in class,

$$Z_0[J] = \frac{\displaystyle\int \mathcal{D}\phi \, \exp\left\{i\int d^4x\left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 + J\phi\right]\right\}}{\displaystyle\int \mathcal{D}\phi \, \exp\left\{i\int d^4x\left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2\right]\right\}}$$

$$= \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1)\Delta_F(x_1 - x_2)J(x_2)\right\},$$

where $i\Delta_F$ is the free-field propagator. It then follows that

$$Z[J] = \mathcal{N}\left[1 - \frac{i\lambda}{4!}\int d^4y\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^4\right]\exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1)\Delta_F(x_1 - x_2)J(x_2)\right\}, \quad (23)$$

where $\mathcal{N}$ is the $J$-independent constant,

$$\mathcal{N} \equiv \frac{\displaystyle\int \mathcal{D}\phi \, \exp\left\{i\int d^4x\left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2\right]\right\}}{\displaystyle\int \mathcal{D}\phi \, \exp\left\{i\int d^4x\left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right]\right\}}. \quad (24)$$

There is no need to evaluate $\mathcal{N}$ using eq. (24), since it can be determined at the end of our computation using $Z[0] = 1$.

To evaluate eq. (23), we first compute

$$\frac{1}{i}\frac{\delta}{\delta J(y)} \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1 - x_2)J(x_2)\right\}$$

$$= -\int d^4x\, \Delta_F(y - x)J(x) \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1)\Delta_F(x_1 - x_2)J(x_2)\right\},$$

where we have used the fact that $\Delta_F(x - y) = \Delta_F(y - x)$. Taking a second functional derivative yields

$$\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^2 \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1 - x_2)J(x_2)\right\}$$

$$= \left\{i\Delta_F(0) + \left[\int d^4x\, \Delta_F(y - x)J(x)\right]^2\right\}\exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1 - x_2)J(x_2)\right\}.$$

Taking a third functional derivative yields

$$\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^3 \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1-x_2)J(x_2)\right\}$$
$$= \left\{-3i\Delta_F(0)\int d^4x\,\Delta_F(y-x)J(x) - \left[\int d^4x\,\Delta_F(y-x)J(x)\right]^3\right\}$$
$$\times \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1-x_2)J(x_2)\right\}.$$

Finally, taking a fourth functional derivative yields

$$\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^4 \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1-x_2)J(x_2)\right\}$$
$$= \left\{-3[\Delta_F(0)]^2 + 6i\Delta_F(0)\left[\int d^4x\,\Delta_F(y-x)J(x)\right]^2 + \left[\int d^4x\,\Delta_F(y-x)J(x)\right]^4\right\}$$
$$\times \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1-x_2)J(x_2)\right\}.$$

The end result is

$$Z[J] = \mathcal{N}\left\{1 + \frac{i\lambda}{8}\left[\int d^4y[\Delta_F(0)]^2 - 2i\Delta_F(0)\int d^4y\, d^4x_1\, d^4x_2\,\Delta_F(y-x_1)\Delta_F(y-x_2)J(x_1)J(x_2)\right.\right.$$
$$\left.\left. -\frac{1}{3}\int d^4y\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\,\Delta_F(y-x_1)\cdots\Delta_F(y-x_4)J(x_1)\cdots J(x_4)\right]\right\}$$
$$\times \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1-x_2)J(x_2)\right\}.$$

Using $Z[0]=1$, it follows that to $\mathcal{O}(\lambda)$,

$$\mathcal{N} = 1 - \frac{i\lambda}{8}\int d^4y[\Delta_F(0)]^2\,.$$

Thus,

$$Z[J] = \left\{1 - \frac{i\lambda}{4!}\left[6i\Delta_F(0)\int d^4y\, d^4x_1\, d^4x_2\,\Delta_F(y-x_1)\Delta_F(y-x_2)J(x_1)J(x_2)\right.\right.$$
$$\left.\left. + \int d^4y\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\,\Delta_F(y-x_1)\cdots\Delta_F(y-x_4)J(x_1)\cdots J(x_4)\right]\right\}$$
$$\times \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1-x_2)J(x_2)\right\}.$$

Since we are only keeping terms of $\mathcal{O}(\lambda)$, we can also rewrite $Z[J]$ in the following form,

$$Z[J] = \exp\biggl\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1-x_2)J(x_2)$$
$$-\frac{i\lambda}{4!}\biggl[6i\Delta_F(0)\int d^4y\, d^4x_1\, d^4x_2\,\Delta_F(y-x_1)\Delta_F(y-x_2)J(x_1)J(x_2)$$
$$+\int d^4y\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\,\Delta_F(y-x_1)\cdots\Delta_F(y-x_4)J(x_1)\cdots J(x_4)\biggr]\biggr\}.$$

Hence, using eq. (21) it follows that

$$W[J] = -\frac{1}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1-x_2)J(x_2) \tag{25}$$
$$-\frac{i\lambda}{4}\Delta_F(0)\int d^4y\, d^4x_1\, d^4x_2\,\Delta_F(y-x_1)\Delta_F(y-x_2)J(x_1)J(x_2)$$
$$-\frac{\lambda}{4!}\int d^4y\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\,\Delta_F(y-x_1)\cdots\Delta_F(y-x_4)J(x_1)\cdots J(x_4)\,.$$

(b) Using the result of part (a) for $W[J]$, compute the four-point connected Green function. Check that the same result is obtained by making use of Coleman's lemma derived in class to obtain the $\mathcal{O}(\lambda)$ contribution to $G^{(4)}(x_1,x_2,x_3,x_4)$. By taking the appropriate Fourier transform, verify that you obtain the momentum space Feynman rule for the four-point scalar interaction obtained in class.

Using eqs. (16) and (25), it immediately follows that

$$G_c^{(4)}(x_1,x_2,x_3,x_4) = -i\lambda\int d^4y\Delta_F(y-x_1)\Delta_F(y-x_2)\Delta_F(y-x_3)\Delta_F(y-x_4)\,. \tag{26}$$

In particular, note that the coefficient of $1/4!$ is canceled due to the fact that there are 4! ways to take the functional derivatives in eq. (16).

We can also obtain eq. (26) directly by making use of Coleman's lemma, which states that

$$F\left(-i\frac{\delta}{\delta J}\right)G[J] = G\left(-i\frac{\delta}{\delta\phi}\right)\left\{F[\phi]\exp\left[i\int d^4z\,\phi(z)J(z)\right]\right\}\bigg|_{\phi=0}. \tag{27}$$

Choose $F[\phi]=\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)$ and $G[J]=W[J]$. Then, eq. (27) yields

$$\frac{\delta^4 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} = W\left(-i\frac{\delta}{\delta\phi}\right)\left\{\phi(y_1)\phi(y_2)\phi(y_3)\phi(y_4)\exp\left[i\int d^4z\,\phi(z)J(z)\right]\right\}\bigg|_{\phi=0}. \tag{28}$$

Using the expression for $W[J]$ given in eq. (25), it is clear that the only way to get a nonzero result on the right hand side of eq. (28) is from a term that involves at least four functional derivatives with respect to the field $\phi$. Thus, we make use of eq. (25) to write

$$W\left(-i\frac{\delta}{\delta\phi}\right) = -\frac{\lambda}{4!}\int d^4y\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, \Delta_F(y-x_1)\cdots\Delta_F(y-x_4)\frac{\delta}{\delta\phi(x_1)}\,\frac{\delta}{\delta\phi(x_2)}\,\frac{\delta}{\delta\phi(x_3)}\,\frac{\delta}{\delta\phi(x_4)}\,,$$

where terms involving fewer than four functional derivatives have been omitted. Inserting this result into eq. (28) and carrying out the functional derivatives, we note that if any of the derivatives act on the exponential term in eq. (28), then there will be a scalar field factor left over, which when set to zero will yield a zero result. Consequently, the four functional derivatives must act on the four scalar fields inside the pair of braces. There are 4! ways that the functional derivatives can act, so the end result for the right hand side of eq. (28) after setting $\phi = 0$ is,

$$-\frac{\lambda}{4!}\int d^4y\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4 \Delta_F(y-x_1)\cdots\Delta_F(y-x_4)\frac{\delta}{\delta\phi(x_1)}\,\frac{\delta}{\delta\phi(x_2)}\,\frac{\delta}{\delta\phi(x_3)}\,\frac{\delta}{\delta\phi(x_4)}$$

$$\times\left\{\phi(y_1)\phi(y_2)\phi(y_3)\phi(y_4)\exp\left[i\int d^4z\,\phi(z)J(z)\right]\right\}\Bigg|_{\phi=0}\,.$$

$$= -\frac{\lambda}{4!}\int d^4y\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4 \Delta_F(y-x_1)\cdots\Delta_F(y-x_4)$$

$$\times\sum_P \delta^4(x_1-y_{i_1})\delta^4(x_2-y_{i_2})\delta^4(x_3-y_{i_3})\delta^4(x_4-y_{i_4})$$

$$= -\lambda\int d^4y \Delta_F(y-y_1)\Delta_F(y-y_2)\Delta_F(y-y_3)\Delta_F(y-y_4)\,, \tag{29}$$

where $\sum_P$ refers to the sum over $\{i_1, i_2, i_3, i_4\}$ corresponding to the 4! distinct permutations of $\{1, 2, 3, 4\}$. Thus, eq. (28) yields

$$\frac{\delta^4 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} = -\lambda\int d^4y \Delta_F(y-y_1)\Delta_F(y-y_2)\Delta_F(y-y_3)\Delta_F(y-y_4)\,. \tag{30}$$

Since the right hand side of eq. (30) is independent of $J$, we are free to set $J = 0$ on the left hand side. Then in light of eq. (16), we conclude that

$$G_c(y_1, y_2, y_3, y_4) = -i\lambda\int d^4y \Delta_F(y-y_1)\Delta_F(y-y_2)\Delta_F(y-y_3)\Delta_F(y-y_4)\,.$$

which reproduces eq. (26) as advertised.

The connected Green function in momentum space is obtained by taking the following Fourier transform,

$$G_c^{(4)}(p_1, p_2, p_3, p_4)(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)$$

$$= \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, e^{i(p_1x_1+\cdots+p_4x_4)}G_c^{(4)}(x_1, x_2, x_3, x_4)$$

$$= -i\lambda\int d^4y\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, e^{i(p_1x_1+\cdots+p_4x_4)}\Delta_F(y-x_1)\cdots\Delta_F(y-x_4)$$

$$= -i\lambda\int d^4y\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, e^{iy(p_1+\cdots+p_4)}e^{ip_1(x_1-y)}\Delta_F(y-x_1)\cdots e^{ip_4(x_4-y)}\Delta_F(y-x_4)\,.$$

We can now perform the integration over $x_1, \ldots, x_4$ using the expression for the free-field propagator in momentum space,

$$\frac{1}{p^2 - m^2 + i\epsilon} = \int d^4x \, e^{-ipx} \Delta_F(x) \,,$$

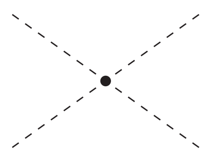
where $m$ is the mass of the scalar field. Employing the integral representation of the momentum conserving delta function,

$$\int d^4y \, e^{iy(p_1 + \cdots + p_4)} = (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \,,$$

the end result is

$$G_c^{(4)}(p_1, p_2, p_3, p_4) = -i\lambda \, \frac{i}{p_1^2 - m^2 + i\epsilon} \cdots \frac{i}{p_4^2 - m^2 + i\epsilon} \,.$$

If we now amputate the four external propagators, we arrive at the Feynman rule for the four-point scalar interaction shown below.



$-i\lambda$

## An alternative method of employing Coleman's lemma

Coleman's lemma allows us to establish directly the perturbation series for the $n$-point Green function. In class, we showed that if we rewrite eq. (19) as

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{2} m^2 \phi^2 + \mathscr{L}_I \,,$$

where

$$\mathscr{L}_I = -\frac{\lambda}{4!}\phi^4 \,, \qquad (31)$$

then the $n$-point Green function is given by[3]

$$G^{(n)}(x_1, x_2, \ldots, x_n) = \exp\left\{ \tfrac{1}{2} i \int d^4y \, d^4z \, \Delta_F(y - z) \frac{\delta}{\delta\phi(y)} \frac{\delta}{\delta\phi(z)} \right\}$$
$$\times \phi(x_1)\phi(x_2) \cdots \phi(x_n) \exp\left( i \int d^4x \, \mathscr{L}_I \right) \Bigg|_{\phi=0} \,. \qquad (32)$$

[3]Strictly speaking, the expression given by eq. (32) includes disconnected vacuum bubble graphs, since it was obtained from an expression for $Z[J]$ where the normalization condition $Z[0] = 1$ was not imposed. However, such terms are easily identified and removed in any explicit computation.

Expanding the last exponential in eq. (32) generates the perturbative expansion for the $n$-point Green function. Keeping terms up to and including terms of $\mathcal{O}(\lambda)$,

$$G^{(n)}(x_1, x_2, \ldots, x_n) = G_0^{(n)}(x_1, x_2, \ldots, x_n) + \exp\left\{\tfrac{1}{2}i\int d^4y\, d^4z\, \Delta_F(y-z)\frac{\delta}{\delta\phi(y)}\frac{\delta}{\delta\phi(z)}\right\}\left[\phi(x_1)\phi(x_2)\cdots\phi(x_n)\, i\int d^4x\, \mathscr{L}_I\right]\Bigg|_{\phi=0}, \tag{33}$$

where $G_0^{(n)}(x_1, x_2, \ldots, x_n)$ is the $n$-point Green function of the free scalar field theory. It is straightforward to verify that eq. (32) yields the free scalar field theory result obtained in class,

$$G_0^{(4)}(x_1, x_2, x_3, x_4) = \Delta_F(x_1-x_2)\Delta_F(x_3-x_4)+\Delta_F(x_1-x_3)\Delta_F(x_2-x_4)+ \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)\,,$$

which consists entirely of disconnected pieces, and thus does not contribute to the connected 4-point Green function $G_c^{(4)}(x_1, x_2, x_3, x_4)$. Using eq. (31), it is convenient to write

$$\int d^4x\, \mathscr{L}_I(x) = -\frac{\lambda}{4!}\int d^4w_1\, d^4w_2\, d^4w_3\, d^4w_4 \phi(w_1)\phi(w_2)\phi(w_3)\phi(w_4)\delta^4(w_1-w_2)\delta^4(w_1-w_3)\delta^4(w_1-w_4)\,.$$

Then, the $\mathcal{O}(\lambda)$ term in $G^{(4)}(x_1, x_2, \ldots, x_n)$, denoted below by $G_1^{(4)}(x_1, x_2, \ldots, x_n)$, arises entirely from the fourth term of the expansion of the exponential in eq. (33), which involves eight functional derivatives,

$$\begin{aligned} G_1^{(4)}(x_1, x_2, x_3, x_4) = -\frac{i\lambda}{4!}\frac{1}{4!}\left(\frac{i}{2}\right)^4 \int & d^4w_1\cdots d^4w_4\, d^4y_1\cdots d^4y_4\, d^4z_1\cdots d^4z_4 \\ & \times\delta^4(w_1-w_2)\delta^4(w_1-w_3)\delta^4(w_1-w_4) \\ & \times\Delta_F(y_1-z_1)\Delta_F(y_2-z_2)\Delta_F(y_3-z_3)\Delta_F(y_4-z_4) \\ & \times\frac{\delta}{\delta\phi(y_1)}\cdots\frac{\delta}{\delta\phi(y_4)}\frac{\delta}{\delta\phi(z_1)}\cdots\frac{\delta}{\delta\phi(z_4)}\Big\{\phi(x_1)\phi(x_2) \\ & \quad\times\phi(x_3)\phi(x_4)\phi(w_1)\phi(w_2)\phi(w_3)\phi(w_4)\Big\}. \end{aligned} \tag{34}$$

Note that the expression obtained after evaluating the eight functional derivatives contain no factors of the scalar field. Indeed, when we set $\phi = 0$ in eq. (33), the $\mathcal{O}(\lambda)$ appearing in eq. (34) is the only term that survives. Evaluating the eight functional derivatives in eq. (34) using Leibniz's rule leads to a sum of products of eight delta-functions,

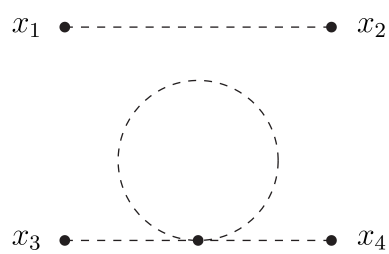
$$\begin{aligned} G_1^{(4)}(x_1, x_2, x_3, x_4) = -\frac{i\lambda}{4!}\frac{1}{4!}\left(\frac{i}{2}\right)^4 \int & d^4w_1\cdots d^4w_4\, d^4y_1\cdots d^4y_4\, d^4z_1\cdots d^4z_4 \\ & \times\delta^4(w_1-w_2)\delta^4(w_1-w_3)\delta^4(w_1-w_4) \\ & \times\Delta_F(y_1-z_1)\Delta_F(y_2-z_2)\Delta_F(y_3-z_3)\Delta_F(y_4-z_4) \\ & \times\left[\delta^4(x_1-y_1)\cdots\delta^4(x_4-y_4)\,\delta^4(w_1-z_1)\cdots\delta^4(w_4-z_4)+\ldots\right], \end{aligned} \tag{35}$$

where we do not explicitly exhibit all possible 8! permutations of $\{x_1, x_2, x_3, x_4, w_1, w_2, w_3, w_4\}$ in the sum of products of delta-functions.

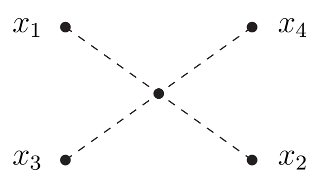
Integrating over $y_1, \ldots, y_4$ and $z_1, \ldots z_4$ in eq. (35) using the eight delta functions yields,

$$G_1^{(4)}(x_1, x_2, x_3, x_4) = -\frac{i\lambda}{4!}\frac{1}{4!}\left(\frac{i}{2}\right)^4 \int d^4w_1 \cdots d^4w_4\, \delta^4(w_1 - w_2)\delta^4(w_1 - w_3)\delta^4(w_1 - w_4)$$
$$\times\big[\Delta_F(x_1 - w_1)\Delta_F(x_2 - w_2)\Delta_F(x_3 - w_3)\Delta_F(x_4 - w_4) + \ldots\big]. \quad (36)$$

where the terms not shown above are obtained by allowing for all possible permutations of $\{x_1, x_2, x_3, x_4, w_1, w_2, w_3, w_4\}$. The possible contributions to $G_1^{(4)}(x_1, x_2, x_3, x_4)$ fall into two classes. In one class of terms, at least one propagator factor of the form $\Delta_F(w_i - w_j)$ will appear in the product of four propagator factors for some $i, j = 1, 2, 3, 4$. When the integration over $w_2$, $w_3$ and $w_4$ is carried out, a factor of $\Delta_F(0)$ will appear due to the presence of the delta functions in eq. (36). Such a term corresponds to a disconnected piece of the 4-point Green function. Diagrammatically, a term of this type is represented by, e.g.,[4]




In the second class of terms, all propagator factors are of the form $\Delta_F(x_i - w_j)$. When the integration over $w_2$, $w_3$ and $w_4$ is carried out, the resulting contributions to $G_1^{(4)}(x_1, x_2, x_3, x_4)$ are diagrammatically represented by the connected diagram,




It is straightforward to count the number of terms appearing in eq. (36) of this type. Starting from the product of propagator factors shown in eq. (36), we add all possible terms related to it by the 4! possible permutations of $\{x_1, x_2, x_3, x_4\}$, the 4! possible permutations of $\{w_1, w_2, w_3, w_4\}$, and an additional $2^4 = 16$ terms (for each term previously identified) that are obtained by interchanging $x_i \leftrightarrow w_i$ for $i = 1, 2, 3$ and 4. Since $\Delta_F(x_i - w_i) = \Delta_F(w_i - x_i)$, it follows that there are $2^4 \times (4!)^2$ identical terms after integrating over $w_2$, $w_3$ and $w_4$. Renaming the variable $w_1 = x$, the end result is the connected 4-point Green function at $\mathcal{O}(\lambda)$,

$$G_c^{(4)}(x_1, x_2, x_3, x_4) = -i\lambda \int d^4x \Delta_F(x - x_1)\Delta_F(x - x_2)\Delta_F(x - x_3)\Delta_F(x - x_4)\,,$$

which again reproduces the result of eq. (26).

[4]Contributions arising from terms with two propagator factors of the form $\Delta_F(w_i - w_j)$, when integrated over $w_2$, $w_3$ and $w_4$, yield two factors of $\Delta_F(0)$. These contributions are represented by disconnected diagrams that include a two-loop vacuum bubbles containing an interaction vertex, which can be discarded (cf. footnote 3).

(c) Evaluate the classical field $\phi_c(x)$ and the generating functional for the 1PI Green functions, $\Gamma[\phi_c]$, perturbatively, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$. Then, repeat part (b) for the four-point 1PI Green function.

The effective action is given by eq. (9), where the classical field is defined by eq. (10). Using eq. (25), it follows that

$$\Phi(x) = -\int d^4x_1\,\Delta_F(x-x_1)J(x_1) - \tfrac{1}{2}i\lambda\Delta_F(0)\int d^4y\,d^4x_1\,\Delta_F(y-x)\Delta_F(y-x_1)J(x_1)$$
$$-\frac{\lambda}{6}\int d^4y\,d^4x_1\,d^4x_2\,d^4x_3\,\Delta_F(y-x)\Delta_F(y-x_1)\Delta_F(y-x_2)\Delta_F(y-x_3)J(x_1)J(x_2)J(x_3)\,. \tag{37}$$

We must invert this equation and solve for $J(x)$. This can be done using an iterative process. Operate on eq. (37) with the operator $\Box_x + m^2 - i\epsilon$. Using

$$(\Box_x + m^2 - i\epsilon)\Delta_F(x-y) = -\delta^4(x-y)\,, \tag{38}$$

it follows that

$$(\Box_x + m^2 - i\epsilon)\Phi(x) = J(x) + \tfrac{1}{2}i\lambda\Delta_F(0)\int d^4x_1\,\Delta_F(x-x_1)J(x_1)$$
$$+\frac{\lambda}{6}\int d^4x_1\,d^4x_2\,d^4x_3\,\Delta_F(x-x_1)\Delta_F(x-x_2)\Delta_F(x-x_3)J(x_1)J(x_2)J(x_3)\,. \tag{39}$$

At $\mathcal{O}(\lambda^0)$, we have $J(x) = (\Box_x + m^2 - i\epsilon)\Phi(x)$. Thus, in the $\mathcal{O}(\lambda)$ term in eq. (39), we can replace $J(x_k)$ with $(\Box_{x_k} + m^2 - i\epsilon)\Phi(x_k)$, for $k = 1, 2, 3$. We can then move the operators $(\Box_{x_k} + m^2 - i\epsilon)$ so that they operate on the $\Delta_F(x - x_k)$ by two successive integrations by parts. Using eq. (38), we produce three delta functions, after which the integrals over $x_1$, $x_2$ and $x_3$ are trivially done. The end result is

$$(\Box_x + m^2 - i\epsilon)\Phi(x) = J(x) - \tfrac{1}{2}i\lambda\Delta_F(0)\Phi(x) - \tfrac{1}{6}\lambda\big[\Phi(x)\big]^3\,.$$

Hence, to $\mathcal{O}(\lambda)$,

$$J(x) = (\Box_x + m^2 - i\epsilon)\Phi(x) + \tfrac{1}{2}i\lambda\Delta_F(0)\Phi(x) + \tfrac{1}{6}\lambda\big[\Phi(x)\big]^3\,. \tag{40}$$

We can use the same procedure to rewrite $W[J]$ in terms of the classical field $\Phi(x)$. We simply insert eq. (40) into eq. (25), and keep only terms up to $\mathcal{O}(\lambda)$. This yields

$$W[J] = \tfrac{1}{2}\int d^4x\,\Phi(x)\Big\{(\Box_x + m^2)\Phi(x) + \tfrac{1}{2}i\lambda\Delta_F(0)\Phi(x) + \tfrac{1}{6}[\Phi(x)]^3\Big\}$$
$$-\tfrac{1}{4}i\lambda\Delta_F(0)\int d^4x\,\big[\Phi(x)\big]^3 - \frac{\lambda}{4!}\int d^4x\,\big[\Phi(x)\big]^4\,,$$

after taking the $\epsilon \to 0$ limit. Using eq. (9) to obtain the effective action, we note that

$$\int d^4x\, J(x)\Phi(x) = \int d^4x\, \Phi(x)\Big\{(\Box_x + m^2)\Phi(x) + \tfrac{1}{2}i\lambda\Delta_F(0)\Phi(x) + \tfrac{1}{6}\lambda[\Phi(x)]^3\Big\}\,,$$

where we have again used eq. (40) and have kept only terms up to $\mathcal{O}(\lambda)$. Hence, we end up with

$$\Gamma[\Phi] = -\tfrac{1}{2}\int d^4x\, \Phi(x)(\Box_x + m^2)\Phi(x) - \tfrac{1}{4}i\lambda\Delta_F(0)\int d^4x\, \big[\Phi(x)\big]^3 - \frac{\lambda}{4!}\int d^4x\, \big[\Phi(x)\big]^4\,. \qquad (41)$$

Finally, we make use of eq. (17) to compute the 1PI four-point function,

$$\Gamma^{(4)}(x_1,\ldots,x_4) = \frac{\delta^n\Gamma[\Phi]}{\delta\Phi(x_1)\cdots\delta\Phi(x_4)}\bigg|_{\Phi=0}\,.$$

Using eq. (41),

$$\Gamma^{(4)}(x_1,\ldots,x_4) = -\lambda\int d^4x\, \delta^4(x-x_1)\delta^4(x-x_2)\delta^4(x-x_3)\delta^4(x-x_4)\,.$$

In momentum space,

$$\begin{aligned}\Gamma^{(4)}(p_1,p_2,p_3,p_4)&(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\\ &= \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, e^{i(p_1x_1+\cdots+p_4x_4)}\Gamma^{(4)}(x_1,x_2,x_3,x_4)\\ &= -\lambda\int d^4x\, e^{ix(p_1+\cdots+p_4)}\\ &= -(2\pi)^4\lambda\,\delta^4(p_1+p_2+p_3+p_4)\,.\end{aligned}$$

That is,

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = -\lambda\,.$$

The Feynman rule for the four-point scalar interaction corresponds to $i\Gamma^{(4)}(p_1,p_2,p_3,p_4)$.

4. Consider a scalar field theory defined by the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}\,\partial^\mu\phi(x)\partial_\mu\phi(x) - V(\phi(x))\,, \qquad (42)$$

and the corresponding equation of motion,

$$\Box\,\phi(x) + V'(\phi) = 0\,,$$

where $\Box \equiv \partial^\mu\partial_\mu$ and $V' \equiv dV/d\phi$.

(a) Starting from eq. (14.122) on p. 276 of Schwartz, derive the equation of motion for the Green function $\langle\Omega|T\big\{\phi(x)\phi(y)\big\}|\Omega\rangle$,

$$\Box_x\langle\Omega|T\big\{\phi(x)\phi(y)\big\}|\Omega\rangle = -\langle\Omega|T\big\{V'(\phi(x))\phi(y)\big\}|\Omega\rangle - i\delta^4(x-y)\,. \qquad (43)$$

Consider a scalar field theory where the Lagrangian density is given by,

$$\mathscr{L}[\phi] = -\tfrac{1}{2}\phi(x)\Box_x\phi(x) + \mathscr{L}_{\rm int}[\phi]\,.$$

In the case of $\mathscr{L}_{\rm int} = -V(\phi)$, this Lagrangian density differs from eq. (42) by a total divergence, which can be neglected. That is, the corresponding actions,

$$S[\phi] = \int d^4x \mathscr{L}[\phi]\,, \tag{44}$$

are the same. Then, eq. (14.122) on p. 276 of Schwartz states that[5]

$$-i\Box_x \frac{\delta Z[J]}{\delta J(x)} = \left\{\mathscr{L}'_{\rm int}\left[-i\frac{\delta}{\delta J(x)}\right] + J(x)\right\} Z[J]\,. \tag{45}$$

Take a functional derivative of eq. (45) with respect to $J(y)$,

$$-i\Box_x \frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)} = \left\{\mathscr{L}'_{\rm int}\left[-i\frac{\delta}{\delta J(x)}\right] + J(x)\right\} \frac{\delta Z[J]}{\delta J(y)} + \delta^4(x-y)Z[J]\,, \tag{46}$$

after using the product rule for differentiating and

$$\frac{\delta J(x)}{\delta J(y)} = \delta^4(x-y)\,. \tag{47}$$

We now make use of the definition of the generating functional,

$$Z[J] = \frac{\displaystyle\int \mathcal{D}\phi \exp\left\{iS[\phi] + i\int d^4x\, J(x)\phi(x)\right\}}{\displaystyle\int \mathcal{D}\phi \exp\{iS[\phi]\}}\,, \tag{48}$$

where the action $S[\phi]$ is defined in eq. (44). It follows that

$$\left(\frac{1}{i}\right)^2 \frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)} = \frac{\displaystyle\int \mathcal{D}\phi\, \phi(x)\phi(y) \exp\left\{iS[\phi] + i\int d^4x\, J(x)\phi(x)\right\}}{\displaystyle\int \mathcal{D}\phi \exp\{iS[\phi]\}}\,, \tag{49}$$

and

$$\mathscr{L}'_{\rm int}\left[-i\frac{\delta}{\delta J(y)}\right] Z[J] = \frac{\displaystyle\int \mathcal{D}\phi\, \mathscr{L}'_{\rm int}\big(\phi(x)\big) \exp\left\{iS[\phi] + i\int d^4x\, J(x)\phi(x)\right\}}{\displaystyle\int \mathcal{D}\phi \exp\{iS[\phi]\}}\,.$$

Taking another functional derivative with respect to $J(y)$ then yields,

$$\mathscr{L}'_{\rm int}\left[-i\frac{\delta}{\delta J(y)}\right] \frac{1}{i}\frac{\delta Z[J]}{\delta J(x)} = \frac{\displaystyle\int \mathcal{D}\phi\, \mathscr{L}'_{\rm int}\big(\phi(x)\big)\phi(y) \exp\left\{iS[\phi] + i\int d^4x\, J(x)\phi(x)\right\}}{\displaystyle\int \mathcal{D}\phi \exp\{iS[\phi]\}}\,. \tag{50}$$

[5] More accurately, one should employ functional derivatives in eq. (45) rather than partial derivatives.

Employing eqs. (49) and (50) in eq. (46) and then setting $J = 0$ at the end of the computation, we end up with

$$\Box_x \frac{\displaystyle\int \mathcal{D}\phi\, \phi(x)\phi(y) \exp\bigl\{iS[\phi]\bigr\}}{\displaystyle\int \mathcal{D}\phi \exp\bigl\{iS[\phi]\bigr\}} = \frac{\displaystyle\int \mathcal{D}\phi\, \mathscr{L}'_{\rm int}\bigl(\phi(x)\bigr)\phi(y) \exp\bigl\{iS[\phi]\bigr\}}{\displaystyle\int \mathcal{D}\phi \exp\bigl\{iS[\phi]\bigr\}} - i\delta^4(x-y)\,, \tag{51}$$

where we have used $Z[0] = 1$.

The $n$-point Green functions are given by

$$\langle\Omega|T\bigl[\phi(x_1)\phi(x_2)\cdots\phi(x_n)\bigr]|\Omega\rangle = i^{-n}\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots J(x_n)}\bigg|_{J=0}\,.$$

Using eq. (48), it follows that

$$\langle\Omega|T\bigl[\phi(x_1)\phi(x_2)\cdots\phi(x_n)\bigr]|\Omega\rangle = \frac{\displaystyle\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\cdots\phi(x_n) \exp\bigl\{iS[\phi]\bigr\}}{\displaystyle\int \mathcal{D}\phi \exp\bigl\{iS[\phi]\bigr\}}\,. \tag{52}$$

Since $\mathscr{L}_{\rm int} = -V(\phi)$, we see that for a potential that is polynomial in $\phi$ (or more generally, by expanding $V(\phi)$ as a functional Taylor series in $\phi$), eq. (51) is equivalent to,

$$\Box_x\langle\Omega|T\bigl\{\phi(x)\phi(y)\bigr\}|\Omega\rangle = -\langle\Omega|T\bigl\{V'(\phi(x))\phi(y)\bigr\}|\Omega\rangle - i\delta^4(x-y)\,.$$

That is, eq. (43) is proven.

(b) Derive eq. (43) by the following technique. Start from the path integral definition of the generating functional,

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi\, \exp\left\{i\int d^4x\,\bigl[\mathscr{L} + J(x)\phi(x)\bigr]\right\}\,, \tag{53}$$

where $\mathcal{N}$ is chosen such that $Z[0] = 1$. Perform a change of variables in the path integral, $\phi(x) \to \phi(x) + \varepsilon(x)$, where $\varepsilon(x)$ is an arbitrary infinitesimal function of $x$. Noting that a change of variables[6] does not change the value of of $Z[J]$, show that to first order in $\varepsilon(x)$,

$$\int \mathcal{D}\phi\, \exp\left\{i\int d^4x\,\bigl[\mathscr{L} + J(x)\phi(x)\bigr]\right\}\int d^4x\,\varepsilon(x)\bigl[-\Box\Phi - V'(\phi) + J(x)\bigr] = 0\,. \tag{54}$$

Since $\varepsilon(x)$ is arbitrary, we may choose $\varepsilon(x) = \epsilon\,\delta^4(x-y)$, where $\epsilon$ is an infinitesimal constant. With this choice for $\varepsilon(x)$, show that by taking the functional derivative of the eq. (54) with respect to $J(x)$ and then setting $J = 0$, one can derive eq. (43).

[6] Just as in the case of ordinary integration, a change of functional integration variables does not change the value of the functional integral.

The Jacobian corresponding to the change of field variables, $\phi(x) \to \phi(x) + \varepsilon(x)$ is unity. Applying this change of variables to eq. (53) yields

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \left[ \mathscr{L} + J(x)\phi(x) \right] \right\} \exp\left\{ i \int d^4x \left[ \partial^\mu \phi \partial_\mu \epsilon - \epsilon(x) V'(\phi) + \epsilon(x) J(x) \right] \right\} ,$$

where we have used $V(\phi+\epsilon) = V(\phi) + \epsilon V'(\phi) + \mathcal{O}(\epsilon^2)$, and we have dropped all terms of $\mathcal{O}(\epsilon^2)$. We can further expand the second exponential above, keeping only those terms up to of $\mathcal{O}(\epsilon)$. Subtracting the resulting expression from eq. (53) yields

$$i\mathcal{N} \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \left[ \mathscr{L} + J(x)\phi(x) \right] \right\} \int d^4x \, \epsilon(x) \big[ -\Box_x \Phi - V'(\phi) + J(x) \big] = 0 \,,$$

after an integration by parts. Since this expression is valid for any infinitesimal function $\epsilon(x)$, we may choose $\epsilon(x) = \epsilon \, \delta^4(x-y)$. We can then carry out the second integration above to obtain,

$$\mathcal{N} \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \left[ \mathscr{L} + J(x)\phi(x) \right] \right\} \big[ -\Box_y \Phi - V'(\phi) + J(y) \big] = 0 \,. \tag{55}$$

We now take the functional derivative of eq. (55) with respect to $J(x)$ and employ eq. (47). Setting $J = 0$ at the end of the calculation, we end up with

$$-i\mathcal{N} \int \mathcal{D}\phi \big[ \phi(x) \Box_y \phi(y) + \phi(x) V'\big(\phi(y)\big) \big] \exp\left\{ i \int d^4x \mathscr{L} \right\} + \mathcal{N} \int \mathscr{D}\phi \, \delta^4(x-y) \exp\left\{ i \int d^4x \mathscr{L} \right\} = 0 \,.$$

We can pull $\Box_y$ outside of the path integral (since it does not by itself depend on the field configurations that one is integrating over). Thus,

$$\begin{aligned} \Box_y \mathcal{N} \int \mathcal{D}\phi \, \phi(x)\phi(y) \exp\left\{ i \int d^4x \mathscr{L} \right\} &= -\mathcal{N} \int \mathcal{D}\phi \, \phi(x) V'\big(\phi(y)\big) \exp\left\{ i \int d^4x \mathscr{L} \right\} \\ &\quad - i\delta^4(x-y) \mathcal{N} \int \mathscr{D}\phi \, \exp\left\{ i \int d^4x \mathscr{L} \right\} . \end{aligned} \tag{56}$$

The constant $\mathcal{N}$ is determined from the condition $Z[0] = 1$. That is,

$$\mathcal{N}^{-1} = \int \mathscr{D}\phi \, \exp\left\{ i \int d^4x \mathscr{L} \right\} .$$

Thus, eq. (56) can be rewritten as

$$\Box_y \frac{\displaystyle\int \mathcal{D}\phi \, \phi(x)\phi(y) \exp\left\{ i \int d^4x \mathscr{L} \right\}}{\displaystyle\int \mathscr{D}\phi \, \exp\left\{ i \int d^4x \mathscr{L} \right\}} = -\frac{\displaystyle\int \mathcal{D}\phi \, \phi(x) V'\big(\phi(y)\big) \exp\left\{ i \int d^4x \mathscr{L} \right\}}{\displaystyle\int \mathscr{D}\phi \, \exp\left\{ i \int d^4x \mathscr{L} \right\}} - i\delta^4(x-y) \,. \tag{57}$$

In light of eq. (52), we see that eq. (57) is equivalent to

$$\Box_y \langle \Omega | T\{\phi(x)\phi(y)\} | \Omega \rangle = -\langle \Omega | T\{\phi(x) V'(\phi(y))\} | \Omega \rangle - i\delta^4(x-y) \,. \tag{58}$$

We now redefine the the variables $x$ and $y$ by interchanging $x \leftrightarrow y$ in eq. (58). Because the ordering of the fields that appear inside a time ordered product is irrelevant (since it is the time ordering prescription that dictates the order of the fields in a time-ordered product), and using the fact that $\delta^4(x-y)$ is an even function of its argument, we obtain eq. (43) as expected.