1. Consider a field theory of a real pseudoscalar field coupled to a fermion field. The interaction Lagrangian is:

$$\mathscr{L}_{\rm int} = -i\lambda\,\overline{\psi}(x)\,\gamma_5\,\psi(x)\phi(x)\,,$$

where λ is a real coupling constant (called the Yukawa coupling). Using functional techniques, derive the Feynman rule for the interaction vertex of this theory.

In class, we derived expressions for the generating functional for a free scalar field theory

$$Z_0[J] = \exp\left\{-\tfrac{1}{2}i\int d^4x\,d^4y\,J(x)\Delta_F(x-y)J(y)\right\}\,,$$

and the generating functional for a free Dirac fermion field theory,

$$Z_0[\overline{\zeta},\zeta] = \exp\left\{-i\int d^4x\,d^4y\,\overline{\zeta}(x)S_F(x-y)\zeta(y)\right\}\,, \tag{1}$$

where Δ_F and S_F are the free-field propagators of the scalar and Dirac fermion fields, respectively. J is a commuting source and $\overline{\zeta}$ and ζ are anticommuting sources. For a free field theory consisting of both a scalar and a Dirac fermion field, the generating functionals given above can be combined,

$$\begin{aligned} Z_0[J,\overline{\zeta},\zeta] &= \mathcal{N}_0\int\mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi\,\exp\left\{i\left[S_0+\int d^4x\big[J(x)\Phi(x)+\overline{\zeta}(x)\psi(x)+\overline{\psi}(x)\zeta(x)\big]\right]\right\} \\ &= \exp\left\{-\tfrac{1}{2}i\int d^4x\,d^4y\,J(x)\Delta_F(x-y)J(y)\right\}\exp\left\{-i\int d^4z\,d^4w\,\overline{\zeta}(z)S_F(z-w)\zeta(w)\right\}\,, \end{aligned} \tag{2}$$

where S_0 is the action of the free field theory, and $\mathcal{N}_0$ is a normalization constant chosen such that $Z[0,0,0]=1$.

For the interacting theory,

$$Z[J,\overline{\zeta},\zeta] = \mathcal{N}\int\mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi\,\exp\left\{i\left[S_0+S_{\rm int}+\int d^4x\big[J(x)\Phi(x)+\overline{\zeta}(x)\psi(x)+\overline{\psi}(x)\zeta(x)\big]\right]\right\}\,, \tag{3}$$

where

$$S_{\rm int}\equiv\int d^4x\,\mathscr{L}_{\rm int}\,,$$

and $\mathcal{N}$ is a normalization constant chosen such that $Z[0,0,0]=1$. We can rewrite eq. (3) as

$$Z[J,\overline{\zeta},\zeta] = \mathcal{N}\,\exp\left\{iS_{\rm int}\left(\frac{1}{i}\,\frac{\delta}{\delta J(x)}\,,\,i\,\frac{\delta}{\delta\zeta(x)}\,,\,\frac{1}{i}\,\frac{\delta}{\delta\overline{\zeta}(x)}\right)\right\}Z_0[J,\zeta,\overline{\zeta}]\,. \tag{4}$$

Note the appearance of $i\delta/\delta\zeta$ in eq. (4). The reason for the factor of i instead of $1/i$ is due to the anticommutative properties of ζ and $\overline{\zeta}$. In particular, note that

$$\frac{\delta}{\delta\zeta(x)}\int d^4y\big[\overline{\zeta}(y)\psi(y)+\overline{\psi}(y)\zeta(y)\big] = -\overline{\psi}(x)\,, \tag{5}$$

$$\frac{\delta}{\delta\overline{\zeta}(x)}\int d^4y\big[\overline{\zeta}(y)\psi(y)+\overline{\psi}(y)\zeta(y)\big] = \quad \psi(x)\,, \tag{6}$$

after using the delta functions obtained via

$$\frac{\delta\zeta(y)}{\delta\zeta(x)} = \delta^4(y-x)\,, \qquad\qquad \frac{\delta\overline{\zeta}(y)}{\delta\overline{\zeta}(x)} = \delta^4(y-x)\,,$$

to integrate over y. The minus sign on the right hand side of eq. (5), which arises when we move $\delta/\delta\zeta(x)$ past $\overline{\psi}(y)$, is properly compensated for by employing $i\delta/\delta\zeta(x)$ in eq. (4).

It is convenient to write the interaction Lagrangian with the spinor indices made explicit,

$$\mathscr{L}_{\rm int} = -i\lambda\,\overline{\psi}_\alpha(x)\,(\gamma_5)_{\alpha\beta}\,\psi_\beta(x)\phi(x)\,, \tag{7}$$

where repeated indices are summed over. Then, to first order in perturbation theory,

$$\exp\left\{iS_{\rm int}\left(\frac{1}{i}\frac{\delta}{\delta J}\,,\,i\frac{\delta}{\delta\zeta_\alpha}\,,\,\frac{1}{i}\frac{\delta}{\delta\overline{\zeta}_\beta}\right)\right\} = 1+\lambda(\gamma_5)_{\alpha\beta}\int d^4x\,\frac{1}{i}\frac{\delta}{\delta J(x)}\,i\frac{\delta}{\delta\zeta_\alpha(x)}\,\frac{1}{i}\frac{\delta}{\delta\overline{\zeta}_\beta(x)}\,.$$

It follows that to $\mathcal{O}(\lambda)$,

$$Z[J,\overline{\zeta},\zeta] = \left\{1+\lambda(\gamma_5)_{\alpha\beta}\int d^4x\,\frac{1}{i}\frac{\delta}{\delta J(x)}\,i\frac{\delta}{\delta\zeta_\alpha(x)}\,\frac{1}{i}\frac{\delta}{\delta\overline{\zeta}_\beta(x)}\right\}Z_0[J,\overline{\zeta},\zeta]\,.$$

The three-point Green function is given by[1]

$$G^{(3)}(y,z,w)_{\rho\sigma} = \langle\Omega|T\big[\phi(y)\psi_\rho(z)\overline{\psi}_\sigma(w)\big]|\Omega\rangle = \frac{1}{i}\frac{\delta}{\delta J(y)}\,\frac{1}{i}\frac{\delta}{\delta\overline{\zeta}_\rho(z)}\,i\frac{\delta}{\delta\zeta_\sigma(w)}Z[J,\overline{\zeta},\zeta]\bigg|_{J=\zeta=\overline{\zeta}=0}\,. \tag{8}$$

There is no $\mathcal{O}(\lambda^0)$ contribution to $G^{(3)}$, since a factor of J arises when one takes a functional derivative with respect to J. Thus, the end result vanishes when taking $J=0$. Thus, at $\mathcal{O}(\lambda)$,

$$G^{(3)}(y,z,w)_{\rho\sigma} = -\lambda(\gamma_5)_{\alpha\beta}\frac{\delta}{\delta J(y)}\,\frac{\delta}{\delta\overline{\zeta}_\rho(z)}\,\frac{\delta}{\delta\zeta_\sigma(w)}\left\{\int d^4x\,\frac{\delta}{\delta J(x)}\,\frac{\delta}{\delta\zeta_\alpha(x)}\,\frac{\delta}{\delta\overline{\zeta}_\beta(x)}Z_0[J,\overline{\zeta},\zeta]\right\}\bigg|_{J=\zeta=\overline{\zeta}=0}\,.$$

[1]The order of the functional derivatives in eq. (8) is determined by the order of the fermion fields inside the time-ordered product [cf. eqs. (5) and (6)]. Different orderings can yield a different overall sign since the fermion fields anticommute.

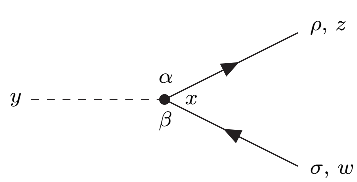
Using eqs. (1) and (2),

$$\begin{aligned}
G^{(3)}(y,z,w)_{\rho\sigma} &= \lambda(\gamma_5)_{\alpha\beta}\frac{\delta}{\delta\overline{\zeta}_\rho(z)}\,\frac{\delta}{\delta\zeta_\sigma(w)}\left\{\int d^4x\, i\Delta_F(y-x)\,\frac{\delta}{\delta\zeta_\alpha(x)}\,\frac{\delta}{\delta\overline{\zeta}_\beta(x)}Z_0[\overline{\zeta},\zeta]\right\}\Bigg|_{\zeta=\overline{\zeta}=0} \\
&= \lambda(\gamma_5)_{\alpha\beta}\frac{\delta}{\delta\overline{\zeta}_\rho(z)}\,\frac{\delta}{\delta\zeta_\sigma(w)}\left\{\int d^4x\, i\Delta_F(y-x)\Big[-iS_F(0)_{\beta\alpha}Z_0[\overline{\zeta},\zeta]\right. \\
&\qquad \left.-\int d^4x_1\, d^4x_2\, S_F(x-x_2)_{\beta\tau}\zeta_\tau(x_2)S_F(x_1-x)_{\gamma\alpha}\overline{\zeta}_\gamma(x_1)Z_0[\overline{\zeta},\zeta]\Big]\right\}\Bigg|_{\zeta=\overline{\zeta}=0} \\
&= \lambda(\gamma_5)_{\alpha\beta}\int d^4x\, i\Delta_F(y-x)\left[iS_F(x-w)_{\beta\sigma}\, iS_F(z-x)_{\rho\alpha} - iS_F(0)_{\beta\alpha} iS_F(z-w)_{\rho\sigma}\right],
\end{aligned} \tag{9}$$

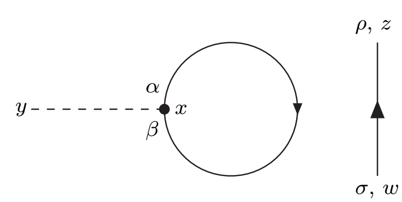
after using the delta functions obtained via

$$\frac{\delta\zeta_\tau(x_2)}{\delta\zeta_\sigma(w)} = \delta_{\tau\sigma}\,\delta^4(x_2-w)\,, \qquad\qquad \frac{\delta\overline{\zeta}_\gamma(x_1)}{\delta\overline{\zeta}_\rho(z)} = \delta_{\gamma\rho}\,\delta^4(x_1-z)\,,$$

to integrate over x_1 and x_2, and noting that $Z[0,0]=1$. Diagrammatically, the two terms represented by eq. (9) are:



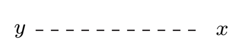
and



The second diagram above is disconnected. Thus, the connected three-point Green function is given by

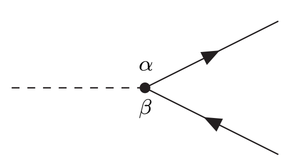
$$G_c^{(3)}(y,z,w)_{\rho\sigma} = \lambda(\gamma_5)_{\alpha\beta}\int d^4x\, i\Delta_F(y-x)\, iS_F(x-w)_{\beta\sigma}\, iS_F(z-x)_{\rho\alpha}\,.$$

We can easily read off the Feynman rules in coordinate space:



$i\Delta_F(x-y)$

$iS_F(x-y)_{\alpha\beta}$



$\lambda(\gamma_5)_{\alpha\beta}\int d^4x$

Note that the order of the spinor indices corresponds to traversing the Feynman diagram in the direction *opposite* to the direction of the fermion line arrows.

The Feynman rules in momentum space are obtained following the same procedure given in problem 3(b). We define

$$G_c^{(3)}(p_1,p_2,p_3)_{\rho\sigma}(2\pi)^4\delta^4(p_1+p_2+p_3) = \int d^4x_1\, d^4x_2\, d^4x_3\, e^{i(p_1x_1+p_2x_2+p_3x_3)} G_c^{(4)}(x_1,x_2,x_3)$$

$$= \lambda(\gamma_5)_{\alpha\beta}\int d^4x\, d^4x_1\, d^4x_2\, d^4x_3\, e^{i(p_1x_1+p_2x_2+p_3x_3)}\, i\Delta_F(x_1-x)\, iS_F(x-x_3)_{\beta\sigma}\, iS_F(x_2-x)_{\rho\alpha}$$

$$= -i\lambda\int d^4x\, d^4x_1\, d^4x_2\, d^4x_3\, e^{ix(p_1+p_2+p_3)}e^{ip_1(x_1-x)}\Delta_F(x-x_1)$$

$$\times e^{ip_2(x_2-x)}iS_F(x_2-x)_{\rho\alpha}\, e^{ip_3(x_3-x)}iS_F(x-x_3)_{\beta\sigma}\,.$$

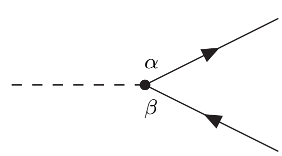
We can now perform the integration over x_1, x_2 and x_3 using the expression for the free-field propagators in momentum space,

$$\frac{1}{p^2-m_s^2+i\epsilon} = \int d^4x\, e^{-ipx}\Delta_F(x)\,, \qquad \frac{(\not{p}+m_f)_{\alpha\beta}}{p^2-m_f^2+i\epsilon} = \int d^4x\, e^{-ipx}S_F(x)_{\alpha\beta}\,,$$

where m_s and m_f are the masses of the pseudoscalar and fermion, respectively. Finally, using the integral representation of the momentum conserving delta function, the end result is

$$G_c^{(3)}(p_1,p_2,p_3)_{\rho\sigma} = \frac{i}{p_1^2-m_s^2+i\epsilon}\,\frac{i(\not{p}_2+m_f)_{\rho\alpha}}{p_2^2-m_f^2+i\epsilon}\,\frac{i(\not{p}_3+m_f)_{\beta\sigma}}{p_3^2-m_f^2+i\epsilon}\,\lambda(\gamma_5)_{\alpha\beta}\,.$$

If we now amputate the three external propagators, we arrive at the momentum space Feynman rule for the pseudoscalar–fermion Yukawa interaction:



$$\lambda(\gamma_5)_{\alpha\beta}$$

The momentum-space Feynman rule is simply obtained by removing the fields from $i\mathscr{L}_{\rm int}$ given in eq. (7). As previously noted, the order of the spinor indices corresponds to traversing the Feynman diagram in the direction *opposite* to the direction of the fermion line arrows.

(b) Calculate the $\mathcal{O}(\lambda^2)$ contribution of the pseudoscalar to the anomalous magnetic moment of the electron.

In class, we showed that the anomalous magnetic moment of the electron of mass m was given by

$$\tfrac{1}{2}(g-2) = F_2(0)\,,$$

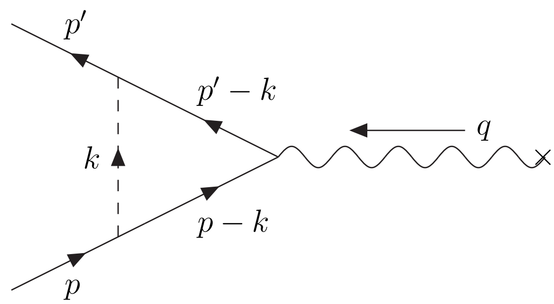
where the form factor $F_2(q^2)$ is given by

$$F_2(q^2) = \text{Tr}\left[(\not{p}+m)\left(g_1\gamma_\mu + \frac{g_2}{2m}(p+p')_\mu\right)(\not{p}'+m)\Gamma^\mu(p,p')\right], \tag{10}$$

and

$$g_1 = \frac{m^2}{q^2(4m^2-q^2)}, \qquad g_2 = \frac{-2m^2(q^2+2m^2)}{q^2(4m^2-q^2)^2}. \tag{11}$$

Here, p and p' are the four-momenta of the initial and final electrons (which are taken to be on-shell; i.e., $p^2 = p'^2 = m^2$), $q \equiv p' - p$ and $\Gamma^\mu(p,p')$ is the $ee\gamma$ vertex due to an external static electromagnetic field (represented by the cross in the diagram below). In this problem, we are asked for the contribution at one-loop to the anomalous magnetic moment of the electron due to a pseudoscalar interaction of the electron with a scalar boson of mass M. That is, we must evaluate the contribution of the following diagram



Using the Feynman rule for the pseudoscalar-fermion vertex obtained in part (a),

$$\Gamma^\mu(p,p') = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i\epsilon}(g\gamma_5)\frac{i(\not{p}'-\not{k}+m)}{(p'-k)^2-m^2+i\epsilon}\gamma^\mu\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2+i\epsilon}(g\gamma_5).$$

That is,

$$\overline{u}(p')\Gamma^\mu(p,p')u(p) = -ig^2\int\frac{d^4k}{(2\pi)^4}\frac{\overline{u}(p')\gamma_5(\not{p}'-\not{k}+m)\gamma^\mu(\not{p}-\not{k}+m)\gamma_5 u(p)}{(k^2-M^2+i\epsilon)[(p'-k)^2-m^2+i\epsilon][(p-k)^2-m^2+i\epsilon]}. \tag{12}$$

We can simplify the numerator of eq. (12) by using the fact the the electrons are on-shell, which means that the Dirac equation is satisfied,

$$(\not{p}-m)u(p) = 0, \qquad \overline{u}(p')(\not{p}'-m) = 0. \tag{13}$$

Hence, using the anticommutation relation, $\{\gamma_\mu, \gamma_5\} = 0$, the numerator of eq. (12) can be rewritten as

$$\begin{aligned}
&\overline{u}(p')\gamma_5(\not{p}'-\not{k}+m)\gamma^\mu(\not{p}-\not{k}+m)\gamma_5 u(p) \\
&\quad = \overline{u}(p')\gamma_5\big[(\not{p}'-\not{k})\gamma^\mu(\not{p}-\not{k}) + m[\gamma^\mu(\not{p}-\not{k}) + (\not{p}'-\not{k})\gamma^\mu + m^2\gamma^\mu\big]\gamma_5 u(p) \\
&\quad = \overline{u}(p')\big[(-(\not{p}'-\not{k})\gamma^\mu(\not{p}-\not{k}) + m[\gamma^\mu(\not{p}-\not{k}) + (\not{p}'-\not{k})\gamma^\mu - m^2\gamma^\mu\big]u(p) \\
&\quad = \overline{u}(p')\big[-m^2\gamma^\mu + m(\gamma^\mu\not{k}+\not{k}\gamma^\mu) - \not{k}\gamma^\mu\not{k} + 2m^2\gamma^\mu - m(\gamma^\mu\not{k}+\not{k}\gamma^\mu) - m^2\gamma^\mu\big]u(p) \\
&\quad = -\overline{u}(p')\not{k}\gamma^\mu\not{k}u(p) = -\overline{u}(p')\not{k}(2k^\mu - \not{k}\gamma^\mu)u(p) \\
&\quad = (k^2 g^{\mu\nu} - 2k^\mu k^\nu)\overline{u}(p')\gamma_\nu u(p),
\end{aligned} \tag{14}$$

where we used the Dirac equation [cf. eq. (13)] to eliminate factors of $\not{p}$ and $\not{p}'$, and we have employed the anticommutator of two gamma matrices, $\{\gamma^\mu\,,\,\gamma^\nu\}=2g^{\mu\nu}$ at the penultimate step above. Hence, it follows that

$$\Gamma^\mu(p,p')=-ig^2\gamma_\nu\int\frac{d^4k}{(2\pi)^4}\frac{k^2g^{\mu\nu}-2k^\mu k^\nu}{(k^2-M^2+i\epsilon)[(p'-k)^2-m^2+i\epsilon][(p-k)^2-m^2+i\epsilon]}\,. \tag{15}$$

We now insert eq. (15) into eq. (10). That is, we must compute,

$$(k^2g^{\mu\nu}-2k^\mu k^\nu)\,\mathrm{Tr}\left[(\not{p}+m)\left(g_1\gamma_\mu+\frac{g_2}{2m}(p+p')_\mu\right)(\not{p}'+m)\gamma_\nu\right]\,.$$

Consider first the $g^{\mu\nu}$ piece.

$$\begin{aligned}
&g^{\mu\nu}\,\mathrm{Tr}\left[(\not{p}+m)\left(g_1\gamma_\mu+\frac{g_2}{2m}(p+p')_\mu\right)(\not{p}'+m)\gamma_\nu\right]\\
&\qquad=g_1\,\mathrm{Tr}\big[(\not{p}+m)\gamma_\mu(\not{p}'+m)\gamma^\mu\big]+\frac{g_2}{2m}(p+p')_\mu\,\mathrm{Tr}\big[(\not{p}+m)(\not{p}'+m)\gamma^\mu\big]\\
&\qquad=g_1\,\mathrm{Tr}\big[(\not{p}+m)(-2\not{p}'+4m)\big]+\frac{g_2}{2m}(p+p')_\mu\,\mathrm{Tr}\big[m(\not{p}+\not{p}')\gamma^\mu\big]\\
&\qquad=8g_1(2m^2-p\!\cdot\!p')+2g_2(p+p')^2\\
&\qquad=4(q^2+2m^2)g_1+2(4m^2-q^2)g_2=0\,,
\end{aligned}$$

after using eq. (11). In the penultimate step above, we used the fact that

$$2p\!\cdot\!p'=p^2+p'^2-(p'-p)^2=2m^2-q^2\,,$$

in light of the fact that the mass-shell conditions are $p^2=p'^2=m^2$.

Next, consider the $k^\mu k^\nu$ piece.

$$\begin{aligned}
&k^\mu k^\nu\,\mathrm{Tr}\left[(\not{p}+m)\left(g_1\gamma_\mu+\frac{g_2}{2m}(p+p')_\mu\right)(\not{p}'+m)\gamma_\nu\right]\\
&\qquad=g_1\,\mathrm{Tr}\big[(\not{p}+m)\not{k}(\not{p}'+m)\not{k}\big]+\frac{g_2}{2m}k\!\cdot\!(p+p')\,\mathrm{Tr}\big[(\not{p}+m)(\not{p}'+m)\not{k}\big]\\
&\qquad=4g_1\big[2k\!\cdot\!p\,k\!\cdot\!p'+k^2(m^2-p\!\cdot\!p')\big]+2g_2\big[k\!\cdot\!(p+p')\big]^2\,.
\end{aligned} \tag{16}$$

Using the following identities (which are a consequence of $q=p'-p$ and $p^2=p'^2=m^2$)

$$2k\!\cdot\!p\,k\!\cdot\!p'=\tfrac{1}{2}\big[k\!\cdot\!(p+p')\big]^2-\tfrac{1}{2}(k\!\cdot\!q)^2\,,\qquad\qquad m^2-p\!\cdot\!p'=\tfrac{1}{2}q^2\,,$$

we can rewrite eq. (16) as

$$k^\mu k^\nu\,\mathrm{Tr}\left[(\not{p}+m)\left(g_1\gamma_\mu+\frac{g_2}{2m}(p+p')_\mu\right)(\not{p}'+m)\gamma_\nu\right]=2\big[k\!\cdot\!(p+p')\big]^2(g_1+g_2)+2\big[k^2q^2-(k\!\cdot\!q)^2\big]g_1\,. \tag{17}$$

Note that eq. (11) yields

$$g_1+g_2=\frac{-3m^2}{(4m^2-q^2)^2}\,.$$

Hence,

$$k^\mu k^\nu \,\mathrm{Tr}\left[(\not p + m)\left(g_1\gamma_\mu + \frac{g_2}{2m}(p+p')_\mu\right)(\not p' + m)\gamma_\nu\right] = \frac{-6m^2\big[k\cdot(p+p')\big]^2}{(4m^2-q^2)^2} + \frac{2m^2\big[k^2q^2-(k\cdot q)^2\big]}{q^2(4m^2-q^2)}\,. \tag{18}$$

Thus, we conclude that

$$\begin{aligned}&(k^2 g^{\mu\nu} - 2k^\mu k^\nu)\,\mathrm{Tr}\left[(\not p + m)\left(g_1\gamma_\mu + \frac{g_2}{2m}(p+p')_\mu\right)(\not p' + m)\gamma_\nu\right]\,. \\ &= \frac{4m^2}{4m^2-q^2}\left[\frac{(k\cdot q)^2}{q^2} - k^2\right] + \frac{12m^2}{(4m^2-q^2)^2}\big[k\cdot(p+p')\big]^2\,. \end{aligned} \tag{19}$$

The form factor $F_2(q^2)$ is now determined [cf. eq. (10)],

$$\begin{aligned}F_2(q^2) = \frac{-4ig^2m^2}{4m^2-q^2}\int\frac{d^4k}{(2\pi)^4}&\frac{1}{(k^2-M^2+i\epsilon)[(p'-k)^2-m^2+i\epsilon][(p-k)^2-m^2+i\epsilon]} \\ &\times\left\{\frac{(k\cdot q)^2}{q^2} - k^2 + \frac{3\big[k\cdot(p+p')\big]^2}{4m^2-q^2}\right\}\,. \end{aligned} \tag{20}$$

We now take the limit of $q\to 0$, In this limit, $p\to p'$, and we obtain

$$F_2(0) = -ig^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-M^2+i\epsilon)[(p-k)^2-m^2+i\epsilon]^2}\left\{\frac{(k\cdot q)^2}{q^2} - k^2 + \frac{3(k\cdot p)^2}{m^2}\right\}\,.$$

Note that in the limit of $q\to 0$, $(k\cdot q)^2/q^2$ approaches a finite quantity, which we will determine shortly.

Let us define the integral,

$$I_{\mu\nu} = \int\frac{d^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{(k^2-M^2+i\epsilon)[(p-k)^2-m^2+i\epsilon]^2}\,,$$

so that

$$F_2(0) = -ig^2\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} + \frac{3p^\mu p^\nu}{m^2}\right)I_{\mu\nu}\,. \tag{21}$$

Using Feynman's trick, we write

$$\frac{1}{A^2B} = \int_0^1\frac{2x\,dx}{[xA+(1-x)B]^3}\,,$$

where $A\equiv(p-k)^2-m^2+i\epsilon$ and $B^2 = k^2-M^2+i\epsilon$. Then,

$$Ax+(1-x)B+x[(p-k)^2-m^2+i\epsilon]+(1-x)(k^2-M^2+i\epsilon) = k^2-2xk\cdot p-(1-x)M^2+i\epsilon\,,$$

after using $p^2=m^2$. Thus,

$$I_{\mu\nu} = \int_0^1 2x\,dx\int\frac{d^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{\big[k^2-2xk\cdot p-(1-x)M^2+i\epsilon\big]^3}\,.$$

If we now change the integration variable $k \to k + px$, the denominator above becomes

$$(k+px)^2 - 2xp\cdot(k+px) - (1-x)M^2 + i\epsilon = k^2 - m^2x^2 - M^2(1-x) + i\epsilon\,.$$

Hence,

$$I_{\mu\nu} = \int_0^1 2x\,dx \int \frac{d^4k}{(2\pi)^4}\,\frac{k_\mu k_\nu + x(k^\mu p^\nu + k^\nu p^\mu) + x^2 p^\mu p^\nu}{\bigl[k^2 - m^2x^2 - M^2(1-x) + i\epsilon\bigr]^3}\,. \tag{22}$$

The linear term in k^μ integrates to zero by symmetry. Moreover, symmetry dictates that

$$\int d^4k\, k_\mu k_\nu F(k^2) = \tfrac{1}{4}g_{\mu\nu}\int d^4k\, k^2\, F(k^2)\,,$$

which means that we can replace $k_\mu k_\nu \to \frac{1}{4}g_{\mu\nu}k^2$ in eq. (22). That is,

$$I_{\mu\nu} = \int_0^1 2x\,dx \int \frac{d^4k}{(2\pi)^4}\,\frac{\frac{1}{4}g_{\mu\nu}k^2 + x^2 p^\mu p^\nu}{\bigl[k^2 - m^2x^2 - M^2(1-x) + i\epsilon\bigr]^3}\,.$$

Thus to evaluate eq. (21), we need to evaluate the $q \to 0$ limit of

$$\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} + \frac{3p^\mu p^\nu}{m^2}\right) I_{\mu\nu}\,.$$

Then, using

$$\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} + \frac{3p^\mu p^\nu}{m^2}\right)\left(\tfrac{1}{4}g_{\mu\nu}k^2 + x^2 p^\mu p^\nu\right) = x^2\left[2m^2 - \frac{(p\cdot q)^2}{q^2}\right]\,,$$

after noting that $p^2 = m^2$, and taking the $q \to 0$ limit, we end up with

$$\lim_{q\to 0}\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} + \frac{3p^\mu p^\nu}{m^2}\right)\left(\tfrac{1}{4}g_{\mu\nu}k^2 + x^2 p^\mu p^\nu\right) = 2m^2x^2\,,$$

after noting that $p\cdot q = p\cdot(p'-p) = p\cdot p' - m^2 = -\frac{1}{2}q^2$. Hence, it follows that

$$\lim_{q\to 0}\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} + \frac{3p^\mu p^\nu}{m^2}\right) I_{\mu\nu} = 4m^2\int_0^1 x^3\,dx \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{\bigl[k^2 - m^2x^2 - M^2(1-x) + i\epsilon\bigr]^3}\,.$$

In class, I showed that

$$\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{\bigl(k^2 - \mathcal{M}^2 + i\epsilon\bigr)^3} = \frac{-i}{32\pi^2\mathcal{M}^2}\,,$$

after dropping the $i\epsilon$ term in the denominator. Identifying $\mathcal{M}^2 \equiv m^2x^2 + M^2(1-x)$, we end up with

$$F_2(0) = -\frac{g^2m^2}{8\pi^2}\int_0^1 \frac{x^3\,dx}{m^2x^2 + M^2(1-x)}\,. \tag{23}$$

The integral in eq. (23) can be performed analytically. The end result is

$$F_2(0) = -\frac{g^2}{16\pi^2}\left\{1 + 2z + z(1-z)\ln z + (z^2-3z)\left(1-\frac{4}{z}\right)^{-1/2}\ln\left(\frac{1+\left(1-\frac{4}{z}\right)^{1/2}}{1-\left(1-\frac{4}{z}\right)^{1/2}}\right)\right\},$$

where $z \equiv M^2/m^2$. Note that since the squared-masses are non-negative, the denominator in eq. (23) is manifestly positive (which provides the justification for dropping the $i\epsilon$ term), and we conclude that $F_2(0)$ is real for $0 \leq z < \infty$. To exhibit this explicitly, we first note that an equivalent expression for $F_2(0)$ which is valud for $z \geq 4$ is,

$$F_2(0) = -\frac{g^2}{16\pi^2}\left\{1 + 2z + z(1-z)\ln z + 2(z^2-3z)\left(1-\frac{4}{z}\right)^{-1/2}\tanh^{-1}\left(1-\frac{4}{z}\right)^{1/2}\right\},$$

In the case of $0 \leq z \leq 4$, the above result can be rewritten as

$$F_2(0) = -\frac{g^2}{16\pi^2}\left\{1 + 2z + z(1-z)\ln z + 2(z^2-3z)\left(\frac{4}{z}-1\right)^{-1/2}\tan^{-1}\left(\frac{4}{z}-1\right)^{1/2}\right\},$$

The limit as $z \to 4$ from either below or above is smooth. Finally, note that for $z = 0$, one obtains $F_2(0) = -g^2/(16\pi^2)$. Compared to the Schwinger result for QED, $F_2(0) = e^2/(8\pi^2)$, the contribution of the massless pseudoscalar exchange (for $g = e$) is opposite in sign and smaller in magnitude by a factor of 2.

2. Consider the function of a *real* parameter z

$$F(z) \equiv \int_0^1 dx\, \ln\big[1 - zx(1-x) - i\epsilon\big]\,, \tag{24}$$

which appeared in the computation of the one-loop correction to the four-point function in scalar field theory.

(a) Evaluate Im $F(z)$. For what values of z does Im F vanish?

We shall denote the argument of the logarithm in eq. (24) by the function,

$$f(x) \equiv zx^2 - zx + 1 \geq 0\,.$$

First, we note that $f(0) = f(1) = 1$. Next, we compute the first and second derivatives,

$$f'(x) = z(2x-1)\,, \qquad f''(x) = 2z\,,$$

Thus, $f(x)$ has an extremum at $x = \frac{1}{2}$. Since $f''(\frac{1}{2}) = 2z$, it follows that $x = \frac{1}{2}$ is a maximum if $z < 0$ and $x = \frac{1}{2}$ is a minimum if $z > 0$. At $z = 0$, we have $f(x) = 1$ for all x. Moreover, for $z > 0$, the minimum value of $f(x)$ is equal to $f(\frac{1}{2}) = 1 - \frac{1}{4}z$. Thus, for values of $z \leq 0$, we have $f(x) \geq 1$ in the region $0 \leq x \leq 1$ and for values of $0 < z \leq 4$, the minimum value of $f(x)$ is non-negative (for all x).

Observe that $\operatorname{Im} F(z) = 0$ if $f(z) \geq 0$ for $0 \leq x \leq 1$, which implies that $\operatorname{Im} F(z) = 0$ if $z \leq 4$. When $z > 4$, the minimum value of $f(x)$ at $x = \frac{1}{2}$ is negative. Since $f(0) = f(1) = 1$, it follows that $f(x) < 0$ for values of $x_- < x < x_+$, where $x_\pm$ are the roots of $f(x)$,

$$x_\pm = \tfrac{1}{2}\left[1 \pm \sqrt{1 - \frac{4}{z}}\,\right] . \tag{25}$$

Thus,

$$\operatorname{Im} F(z) = \Theta(z-4) \int_{x_-}^{x_+} dx \, \operatorname{Im} \ln\bigl[1 - zx(1-x) - i\epsilon\bigr] \,, \tag{26}$$

where we have explicitly included the step function to enforce the condition that $\operatorname{Im} F(z) = 0$ if $z \leq 4$. To evaluate the imaginary part of the logarithm, we employ the principal value of the complex-valued logarithm, with the branch cut taken along the negative real axis. In particular, assuming that y is a non-zero real number and ϵ is a *positive* infinitesimal,

$$\ln(y - i\epsilon) = \ln|y| - i\pi\Theta(-y) \,.$$

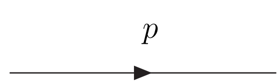
It follows that $\operatorname{Im} \ln(y - i\epsilon) = -\pi\Theta(-y)$. Employing this result in eq. (26),

$$\operatorname{Im} F(z) = -\Theta(z-4)\pi \int_{x_-}^{x_+} dx = -\Theta(z-4)\pi\bigl(x_+ - x_-\bigr) = -\Theta(z-4)\pi\sqrt{1 - \frac{4}{z}} \,, \tag{27}$$

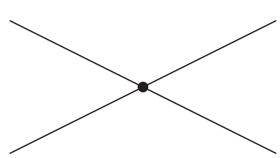
after using the explicit form for $x_\pm$ given in eq. (25).

(b) Consider the 1PI 4-point Green function, $\Gamma^{(4)}$, in a field theory of a real scalar field with mass m and an interaction Lagrangian density given by $\mathscr{L}_I = -\lambda\phi^4/4!$. Using the Feynman rules for this theory, write down an integral expression for the full $\mathcal{O}(\lambda^2)$ contribution to $\Gamma^{(4)}$. From the integral expression, evaluate $\operatorname{Im} \Gamma^{(4)}$ up to order λ^2 by making use of the *cutting rules* given in Section 24.1.2 [pp. 456–459] of Schwartz.

The Feynman rules for the scalar propagator and the 4-point scalar interaction are

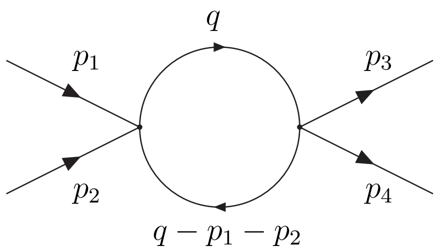


$$\frac{i}{p^2 - m^2 + i\epsilon}$$

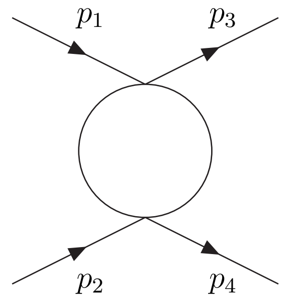


$$-i\lambda$$

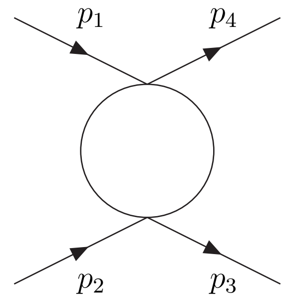
The Feynman rules are used to compute $i\Gamma^{(4)}$, where $\Gamma^{(4)}$ is the 1PI 4-point Green function. At tree level, $i\Gamma^{(4)} = -i\lambda$. The one-loop contributions to $i\Gamma^{(4)}$ are obtained by using the Feynman rules to evaluate the one-loop diagrams that are exhibited at the top of the following page.



s-channel



t-channel



u-channel

Thus, employing the Feynman rules (and recalling the symmetry factor of $\frac{1}{2}$ for each of the diagrams above), it follows that including all terms up to $\mathcal{O}(\lambda^2)$,

$$i\Gamma^{(4)} = -i\lambda + \tfrac{1}{2}(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4}\left\{\frac{i}{q^2-m^2+i\epsilon}\,\frac{i}{(q-p_1-p_2)^2-m^2+i\epsilon} + (p_2 \to p_3) + (p_2 \to p_4)\right\},$$

where the second and third terms above in the integrand are given by the first term with the momentum substitutions indicated. That is, the three terms exhibited in $i\Gamma^{(4)}$ correspond to the s-channel, t-channel and u-channel diagrams, respectively. Thus,

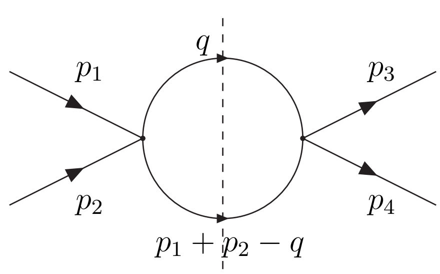
$$\Gamma^{(4)} = -\lambda - \tfrac{1}{2}i\lambda^2 \int \frac{d^4q}{(2\pi)^4}\left\{\frac{1}{q^2-m^2+i\epsilon}\,\frac{1}{(q-p_1-p_2)^2-m^2+i\epsilon} + (p_2 \to p_3) + (p_2 \to p_4)\right\}.$$

We shall focus first on the s-channel diagram. We expect that the singularity structure in the complex s plane to have a branch point at the threshold for the $2 \to 2$ scattering process at threshold, $s = 4m^2$, and a branch cut extending to ∞ along the positive real axis.[2]

By definition, the discontinuity of $\Gamma^{(4)}(s)$ across the branch cut is

$$\operatorname{Disc}\Gamma^{(4)}(s) \equiv \Gamma^{(4)}(s+i\epsilon) - \Gamma^{(4)}(s-i\epsilon)\,,$$

where ϵ is a positive infinitesimal. The cutting rules state that $\operatorname{Disc}\Gamma^{(4)}(s)$ is obtained by cutting the Feynman diagram



and replacing the "cut" propagators by:

$$\frac{1}{q^2-m^2+i\epsilon} \longrightarrow -2\pi i\delta(q^2-m^2)\Theta(q_0)\,.$$

[2]Note that $s = (p_1+p_2)^2 = 2(m^2 + p_1\cdot p_2) = 2(m^2 + E_1E_2 - \vec{\boldsymbol{p}}_1\cdot\vec{\boldsymbol{p}}_2)$. At threshold, $\vec{\boldsymbol{p}}_1 = \vec{\boldsymbol{p}}_2 = 0$ and $E_1 = E_2 = m$, which implies that $s = 4m^2$ at threshold.

The discontinuity $\mathrm{Disc}\,\Gamma^{(4)}(s)$ is related to $\mathrm{Im}\,\Gamma^{(4)}(s)$ as follows. First, we observe that the reflection principle of complex analysis implies that[3]

$$\Gamma^{(4)}(s-i\epsilon)=\Gamma^{(4)}(s+i\epsilon)^*\,. \tag{28}$$

It then follows that

$$\mathrm{Disc}\,\Gamma^{(4)}(s)\equiv\Gamma^{(4)}(s+i\epsilon)-\Gamma^{(4)}(s+i\epsilon)^*=2i\,\mathrm{Im}\,\Gamma^{(4)}(s)\,,$$

where we have defined

$$\Gamma^{(4)}(s)\equiv\lim_{\epsilon\to 0}\Gamma^{(4)}(s+i\epsilon)\,.$$

Applying the cutting rules to the s-channel one-loop diagram (shown above),

$$2i\,\mathrm{Im}\,\Gamma^{(4)}(s)=-\tfrac{1}{2}i\lambda^2(-2\pi i)^2\int\frac{d^4q}{(2\pi)^4}\,\delta(q^2-m^2)\Theta(q_0)\delta\big((q-p_1-p_2)^2-m^2\big)\Theta(p_{10}+p_{20}-q_0)\,. \tag{29}$$

It should be noted that the form of the Θ-function corresponds to placing a cut propagator line *on mass shell.* To evaluate the integral in eq. (29), note that

$$\begin{aligned}\int d^4q\,\delta(q^2-m^2)\Theta(q_0)&=\int d^3q\,dq_0\,\delta(q_0^2-|\vec{\boldsymbol{q}}|^2-m^2)\Theta(q_0)\\&=\int d^3q\,dq_0\,\frac{1}{2\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}}\left[\delta\big(q_0-\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}+\delta\big(q_0+\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}\right]\Theta(q_0)\\&=\int\frac{d^3q}{2\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}}\,.\end{aligned}$$

It follows that

$$\begin{aligned}&\int\frac{d^4q}{(2\pi)^4}\,\delta(q^2-m^2)\Theta(q_0)\delta\big((q-p_1-p_2)^2-m^2\big)\Theta(p_{10}+p_{20}-q_0)\\&\qquad=\frac{1}{(2\pi)^4}\int\frac{d^3q}{2\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}}\delta\big((q-p_1-p_2)^2-m^2\big)\Theta(p_{10}+p_{20}-q_0)\bigg|_{q_0=\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}}\,,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&\int\frac{d^4q}{(2\pi)^4}\,\delta(q^2-m^2)\Theta(q_0)\delta\big((q-p_1-p_2)^2-m^2\big)\Theta(p_{10}+p_{20}-q_0)\\&\qquad=\frac{1}{(2\pi)^4}\int\frac{d^3q}{2\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}}\delta\big(s-2q\cdot(p_1+p_2)\big)\Theta(p_{10}+p_{20}-q_0)\bigg|_{q_0=\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}}\,,\end{aligned} \tag{30}$$

after using $s\equiv(p_1+p_2)^2$ and noting that $q^2-m^2=0$ is equivalent to $q_0=\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}$.

[3]See the Appendix to this solution set. In addition, a very nice discussion can be found in Paul Roman, *Introduction to Quantum Field Theory* (John Wiley & Sons, Inc., New York, NY, 1969) pp. 440–441.

The simplest way to evaluate the integral above is to work in the center-of-mass frame of the system, where

$$p_1 + p_2 = \left(\sqrt{s}\,;\,\vec{\boldsymbol{0}}\right).$$

In this case,

$$2q\cdot(p_1+p_2))\Big|_{q_0=\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}} = 2\sqrt{s}\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}\,,$$

and eq. (30) reduces to

$$\begin{aligned}\int \frac{d^4q}{(2\pi)^4}\,&\delta(q^2-m^2)\Theta(q_0)\delta\big((q-p_1-p_2)^2-m^2\big)\Theta(p_{10}+p_{20}-q_0)\\ &=\frac{1}{(2\pi)^4}\int\frac{d^3q}{2\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}}\delta\big(s-2\sqrt{s}\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}\big)\Theta\big(s-\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}\big)\,. \qquad (31)\end{aligned}$$

The delta function enforces $\sqrt{s}=2\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}$, which means that the argument of the Θ function is positive so that $\Theta\big(s-\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}\big)=1$. Hence,

$$\begin{aligned}\int \frac{d^4q}{(2\pi)^4}\,&\delta(q^2-m^2)\Theta(q_0)\delta\big((q-p_1-p_2)^2-m^2\big)\Theta(p_{10}+p_{20}-q_0)\\ &=\frac{1}{(2\pi)^4\sqrt{s}}\int d^3q\,\delta\big(s-2\sqrt{s}\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}\big)\,. \qquad (32)\end{aligned}$$

To evaluate the above integral, use spherical coordinates, $d^3q=|\vec{\boldsymbol{q}}|^2d|\vec{\boldsymbol{q}}|\,d\Omega=4\pi|\vec{\boldsymbol{q}}|^2d|\vec{\boldsymbol{q}}|$, since there is no dependence on the direction of $\vec{\boldsymbol{q}}$ in the integrand above. It is convenient to change the integration variable to $E\equiv\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}$, in which case $|\vec{\boldsymbol{q}}|d|\vec{\boldsymbol{q}}|=EdE$. It follows that

$$\begin{aligned}\int d^3q\,\delta\big(s-2\sqrt{s}\sqrt{|\vec{\boldsymbol{q}}|^2+m^2}\big) &= 4\pi\int_m^\infty |\vec{\boldsymbol{q}}|\,E\,dE\,\delta\big(s-2\sqrt{s}E\big)\\ &=\frac{2\pi}{\sqrt{s}}\int_m^\infty E(E^2-m^2)^{1/2}\delta\big(E-\tfrac{1}{2}\sqrt{s}\big)\,dE\\ &=\frac{\pi\sqrt{s}}{2}\left(1-\frac{4m^2}{s}\right)^{1/2}\Theta\big(\sqrt{s}-2m\big)\,.\end{aligned}$$

Note that the Θ-function appears, since if $\sqrt{s}<2m$, then the argument of the delta function is never zero over the range of integration $m\leq E<\infty$, in which case the delta function must be set to zero.

Inserting the above result into eq. (32), we end up with

$$\int \frac{d^4q}{(2\pi)^4}\,\delta(q^2-m^2)\Theta(q_0)\delta\big((q-p_1-p_2)^2-m^2\big)\Theta(p_{10}+p_{20}-q_0)=\frac{1}{32\pi^3}\left(1-\frac{4m^2}{s}\right)^{1/2}\Theta\big(\sqrt{s}-2m\big)\,. \qquad (33)$$

Note that the δ-function and Θ-function conditions are satisfied only when $s\geq 4m^2$, This is true because the cut propagator lines are both on-shell, which means that to conserve both energy and three-momentum requires that $s \geq 4m^2$. Hence, we can replace $\Theta(\sqrt{s}-2m)$ above with $\Theta(s-4m^2)$. In fact, our analysis in the center-of-mass frame implicitly assumed that s was positive since $\sqrt{s} = p_{10} + p_{20}$ is real. Thus, our analysis above does not apply to the case of $s < 0$; in this latter case a second computation would be required. However, in practice we do not have to perform this second computation since the physical argument based on the mass-shell conditions imply that one cannot satisfy the δ-function and Θ-function conditions if $s < 0$. Hence, we can rewrite eq. (33) as

$$\int \frac{d^4q}{(2\pi)^4}\,\delta(q^2-m^2)\Theta(q_0)\delta\big((q-p_1-p_2)^2-m^2\big)\Theta(p_{10}+p_{20}-q_0) = \frac{1}{32\pi^3}\left(1-\frac{4m^2}{s}\right)^{1/2}\Theta\big(s-4m^2\big)\,.$$

Inserting this expression into eq. (29) yields our final result,

$$\operatorname{Im}\Gamma^{(4)}(s) = \frac{\lambda^2}{32\pi}\left(1-\frac{4m^2}{s}\right)^{1/2}\Theta\big(s-4m^2\big)\,. \tag{34}$$

So far, we have only examined the s-channel piece of the above expression. If we now include the t-channel and u-channel diagrams, it is clear that the only change in our analysis is to replace s with t and u, respectively. Thus,

$$\begin{aligned}\operatorname{Im}\Gamma^{(4)}(p_1,p_2,p_3,p_4) = \frac{\lambda^2}{32\pi}&\left[\left(1-\frac{4m^2}{s}\right)^{1/2}\Theta\big(s-4m^2\big)\right.\\ &\left.+\left(1-\frac{4m^2}{t}\right)^{1/2}\Theta\big(t-4m^2\big)+\left(1-\frac{4m^2}{u}\right)^{1/2}\Theta\big(u-4m^2\big)\right].\end{aligned} \tag{35}$$

The physical region of scattering corresponds to $s \geq 4m^2$, $t < 0$ and $u < 0$. Thus, the last two terms on the right hand side of eq. (35) do not survive in the physical scattering amplitude, in which case

$$\operatorname{Im}\Gamma^{(4)}(p_1,p_2,p_3,p_4) = \frac{\lambda^2}{32\pi}\sqrt{1-\frac{4m^2}{s}}\,. \tag{36}$$

(c) An explicit one-loop computation of $\Gamma^{(4)}$ yields

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = -\lambda - \frac{\lambda^2}{32\pi^2}\left[F\left(\frac{s}{m^2}\right)+F\left(\frac{t}{m^2}\right)+F\left(\frac{u}{m^2}\right)+G(m^2)\right]\,, \tag{37}$$

where $s \equiv (p_1+p_2)^2$, $t \equiv (p_1-p_3)^2$, $u \equiv (p_1-p_4)^2$ are Lorentz-invariant kinematic variables, the function F is defined in part (a), and the function G is a real function.[4] Using eq. (37) and the results of part (a), compute $\operatorname{Im}\Gamma^{(4)}$ and check that your calculation in part (b) is correct.

[4]In fact, the function G is infinite, but this infinity can be removed by renormalization. Since we are only interested here in $\operatorname{Im}\Gamma^{(4)}$, we can safely ignore any details associated with the renormalization procedure.

Using eq. (27) with $z = s/m^2$ yields

$$\operatorname{Im} F\left(\frac{s}{m^2}\right) = -\theta(s-4m^2)\pi\sqrt{1-\frac{4m^2}{s}}\,.$$

In eq. (37), only $F(s/m^2)$ has an imaginary part in the physical region corresponding to $s \geq 4m^2$, $t < 0$ and $u < 0$. Taking the imaginary part of eq. (37) therefore yields

$$\operatorname{Im}\Gamma^{(4)}(p_1,p_2,p_3,p_4) = -\frac{\lambda^2}{32\pi^2}\operatorname{Im} F\left(\frac{s}{m^2}\right) = \frac{\lambda^2}{32\pi}\sqrt{1-\frac{4m^2}{s}}\,.$$

Indeed, we have reproduced the result of the cutting rules [cf. eq. (36)].

3. The Lagrangian of QED is given by:

$$\mathscr{L}_{\rm QED} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ + eA\!\!\!/)\psi - m\overline{\psi}\psi - \frac{1}{2a}(\partial_\mu A^\mu)^2\,. \qquad (38)$$

(a) Compute the tree-level photon propagator (in momentum space).

To compute the tree-level photon propagator, we focus on the the free field theory of photons,

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2a}(\partial_\mu A^\mu)^2\,.$$

We can obtain the propagator from the generating functional,

$$Z[J] = \mathcal{N}\int \mathcal{D}A_\mu\, \exp\left\{i\int d^4x\left[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2a}(\partial_\mu A^\mu)^2 + J_\mu A^\mu\right]\right\}\,,$$

where J^μ are the sources corresponding to A_μ and $\mathcal{N}$ is chosen such that $Z[0] = 1$. It is convenient to write

$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2a}(\partial_\mu A^\mu)^2 = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2a}(\partial_\mu A^\mu)^2$$

$$= \tfrac{1}{2}A^\mu\left[g_{\mu\nu}\Box - \left(1-\frac{1}{a}\right)\partial_\mu\partial_\nu\right]A^\mu + \text{total divergence}\,,$$

where $\Box \equiv \partial_\mu\partial^\mu$. The total divergence integrates to zero inside the exponential of the action (assuming that the surface terms at infinity vanish) and hence can be dropped. Thus,

$$Z[J] = \mathcal{N}\int \mathcal{D}A_\mu\, \exp\left\{i\int d^4x\, \tfrac{1}{2}A^\mu\left[g_{\mu\nu}\Box - \left(1-\frac{1}{a}\right)\partial_\mu\partial_\nu\right]A^\mu + J_\mu A^\mu\right\}\,.$$

This is equivalent to the path integral over four identical scalar fields. Thus, we can use the known result for the generating functional of a scalar field theory to obtain

$$Z[J] = \exp\left\{-\frac{i}{2}\int d^4x\, d^4y\, J^\mu(x)\left[g_{\mu\nu}\Box - \left(1-\frac{1}{a}\right)\partial_\mu\partial_\nu\right]^{-1} J^\nu(y)\right\}\,.$$

The photon propagator in coordinate space is

$$G^{(2)}(x,y) = \langle\Omega|T\big[A_\mu(x)A_\nu(y)\big]|\Omega\rangle = \left(\frac{1}{i}\right)^2 \frac{\delta^2 Z[J]}{\delta J^\mu(x)\delta J^\nu(y)}\bigg|_{J^\mu=0}$$

$$= i\left[g_{\mu\nu}\Box - \left(1-\frac{1}{a}\right)\partial_\mu\partial_\nu\right]^{-1}\delta^4(x-y)\,. \qquad (39)$$

Note that the derivative operators in eq. (39) should be regarded as $\partial_\mu = \partial/\partial x^\mu$ and $\Box \equiv \Box_x$. To invert the operator above, we note that for arbitrary operators F and G,

$$\left[F\left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right) + G\left(\frac{\partial_\mu\partial_\nu}{\Box}\right)\right]^{-1} = F^{-1}\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right) + G^{-1}\left(\frac{\partial^\mu\partial^\nu}{\Box}\right).$$

This result is a consequence of

$$\left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)\left(g^{\nu\alpha} - \frac{\partial^\nu\partial^\alpha}{\Box}\right) + \left(\frac{\partial_\mu\partial_\nu}{\Box}\right)\left(\frac{\partial^\nu\partial^\alpha}{\Box}\right) = g_\mu^\alpha\,,$$

and

$$\left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)\left(\frac{\partial^\nu\partial^\alpha}{\Box}\right) = 0\,.$$

Hence, it follows that

$$G^{(2)}(x,y) = i\left[g_{\mu\nu}\Box - \left(1-\frac{1}{a}\right)\partial_\mu\partial_\nu\right]^{-1}\delta^4(x-y)$$

$$= i\left[\Box\left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right) + \frac{1}{a}\Box\left(\frac{\partial_\mu\partial_\nu}{\Box}\right)\right]^{-1}\delta^4(x-y)$$

$$= i\left[\frac{1}{\Box}\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right) + \frac{a}{\Box}\left(\frac{\partial^\mu\partial^\nu}{\Box}\right)\right]\delta^4(x-y)\,.$$

In momentum space,

$$G^{(2)}(p,q)(2\pi)^4\delta^4(p+q) = \int d^4x\, d^4y\, e^{i(px+qy)}G^{(2)}(x,y)$$

$$= i\int d^4x\, d^4y\, e^{i(px+qy)}\left[\frac{1}{\Box}\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right) + \frac{a}{\Box}\left(\frac{\partial^\mu\partial^\nu}{\Box}\right)\right]\delta^4(x-y)$$

$$= i\int d^4x\, d^4y\, \delta^4(x-y)\left[\frac{1}{\Box}\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right) + \frac{a}{\Box}\left(\frac{\partial^\mu\partial^\nu}{\Box}\right)\right]e^{i(px+qy)}$$

$$= -i\int d^4x\, d^4y\, \delta^4(x-y)\left[\frac{1}{p^2}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) + \frac{a}{p^2}\left(\frac{p^\mu p^\nu}{p^2}\right)\right]e^{i(px+qy)}$$

$$= -i(2\pi)^4\delta^4(p+q)\left[\frac{1}{p^2}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) + \frac{a}{p^2}\left(\frac{p^\mu p^\nu}{p^2}\right)\right].$$

Thus, the momentum space propagator is

$$G^{(2)}(p,-p) = -\frac{i}{p^2}\left[g^{\mu\nu} - (1-a)\frac{p^\mu p^\nu}{p^2}\right] .$$

To be technically correct, we should write

$$G^{(2)}(p,-p) = -\frac{i}{p^2+i\epsilon}\left[g^{\mu\nu} - (1-a)\frac{p^\mu p^\nu}{p^2}\right] ,$$

The justification for this is given in section 14.4 of Matt Schwartz's textbook (see pp. 264–266).

(b) Show that this Lagrangian is not invariant under the infinitesimal gauge transformations,

$$\delta\psi = ie\Lambda(x)\psi(x)\,, \tag{40}$$
$$\delta A_\mu = \partial_\mu\Lambda(x)\,, \tag{41}$$

where $\Lambda(x)$ is an arbitrary real function of x that vanishes (sufficiently fast) as $|\vec{\boldsymbol{x}}| \to \infty$.

Under the infinitesimal gauge transformations given in eqs. (40) and (41),

$$\delta F_{\mu\nu} = \delta(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu = (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\Lambda = 0\,,$$

since the partial derivatives commute under the assumption that $\Lambda(x)$ is a smooth function. Likewise, since λ is a real function, we have

$$\begin{aligned}
\delta(\overline{\psi}\psi) &= \overline{\delta\psi}\psi + \overline{\psi}\delta\psi = -ie\Lambda\overline{\psi}\psi + ie\Lambda\overline{\psi}\psi = 0\,,\\
\delta(\overline{\psi}\partial\!\!\!/\psi) &= -ie\Lambda\overline{\psi}\partial\!\!\!/\psi + ie(\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi + ie\Lambda\overline{\psi}\partial\!\!\!/\psi = ie(\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi\,,\\
\delta(\overline{\psi}A\!\!\!/\psi) &= -ie\Lambda\overline{\psi}A\!\!\!/\psi + (\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi + ie\Lambda\overline{\psi}A\!\!\!/\psi = (\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi\,.
\end{aligned}$$

Hence, it follows that

$$\delta\big(\overline{\psi}(i\partial\!\!\!/ + eA\!\!\!/)\psi\big) = -e(\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi + e(\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi = 0\,,$$

as expected. Finally, working to first order in the field variations,[5]

$$\delta(\partial_\mu A^\mu)^2 = 2(\partial_\mu A^\mu)\delta(\partial_\mu A^\mu) = 2(\partial_\mu A^\mu)\partial_\mu(\delta A^\mu) = 2(\Box\Lambda)\partial_\mu A^\mu\,.$$

Thus, the variation of the QED Lagrangian given in eq. (38) is

$$\delta\mathscr{L}_{\rm QED} = -\frac{1}{a}\,(\Box\Lambda)(\partial_\mu A^\mu)\,, \tag{42}$$

which is non-vanishing.

[5]Alternatively, we can write

$$\delta(\partial_\mu A^\mu)^2 = \partial_\mu(A^\mu + \partial^\mu\Lambda)\partial_\nu(A^\nu + \partial^\nu\Lambda) - (\partial_\mu A^\mu)^2 = 2(\Box\Lambda)\partial_\mu A^\mu$$

after dropping terms that are quadratic in Λ.

(c) Consider the modified Lagrangian:

$$\mathscr{L} = \mathscr{L}_{\rm QED} + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi\,, \tag{43}$$

where ϕ is a free scalar field. Show that the action is invariant under the generalized (infinitesimal) gauge transformation:

$$\delta\psi = ie\epsilon\,\phi(x)\psi(x)\,, \tag{44}$$

$$\delta A_\mu = \epsilon\,\partial_\mu\phi(x)\,, \tag{45}$$

$$\delta\phi = -\frac{\epsilon}{a}\,\partial_\mu A^\mu\,, \tag{46}$$

where ϵ is an infinitesimal parameter. This has a name: it is called the BRST-transformation. The action is therefore said to be BRST-invariant.

The results of part (b) still apply where $\Lambda(x)$ is replaced by $\epsilon\,\phi(x)$. It then follows from eq. (42) that

$$\delta\mathscr{L}_{\rm QED} = -\frac{\epsilon}{a}\,(\Box\phi)(\partial_\mu A^\mu)\,.$$

Using eq. (46) and working to first order in the field variations,

$$\delta(\partial_\mu\phi\partial^\mu\phi) = \partial_\mu(\delta\phi)\partial^\mu\phi + \partial_\mu\phi\partial^\mu(\delta\phi) = -\frac{2\epsilon}{a}(\partial_\mu\partial_\nu A^\nu)(\partial^\mu\phi)\,.$$

Hence,

$$\delta\mathscr{L} = -\frac{\epsilon}{a}\big[(\Box\phi)(\partial_\mu A^\mu) + (\partial_\mu\partial_\nu A^\nu)(\partial^\mu\phi)\big] = -\frac{\epsilon}{a}\,\partial^\mu\big[(\partial_\mu\phi)(\partial_\nu A^\nu)\big]\,,$$

which we recognize as a total divergence. Hence, the variation of the action,

$$\delta S = \int d^4x\,\delta\mathscr{L} = 0\,,$$

under the usual assumption that the fields at infinity vanish so that the surface terms vanish. Hence, the action is invariant under the BRST transformations.

4. Consider the Lagrangian for a non-abelian gauge theory, with gauge field A^a_μ and gauge field strength tensor $F^a_{\mu\nu}$,

$$\mathscr{L}_{\rm YM} = \tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}\,,$$

which is invariant under the gauge transformation:

$$\delta A^a_\mu(x) = \epsilon D^{ab}_\mu\omega_b(x)\,, \tag{47}$$

where ϵ is infinitesimal, D_μ is the covariant derivative, and $\omega_b(x)$ is an arbitrary function of x.

(a) In order to be able to define a propagator for the gauge field, we must add a gauge-fixing term:

$$\mathscr{L}_{\rm GF} = -\frac{1}{2a}(\partial^\mu A^a_\mu)^2\,. \tag{48}$$

Show that under the gauge transformation of eq. (47), the gauge invariance is broken due to an extra term generated:

$$\delta\mathscr{L}_{\rm GF} = -\frac{\epsilon}{a}(\partial^\mu A_\mu^a)(\partial^\nu D_\nu^{ab}\omega_b)\,.$$

Under an infinitesimal gauge transformation given in eq. (47), and working to first order in the field variations,

$$\delta(\partial^\mu A_\mu^a)^2 = 2(\partial^\mu A_\mu^a)\delta(\partial^\nu A_\nu^a) = 2(\partial^\mu A_\mu^a)(\partial^\nu \delta A_\nu^a) = \epsilon(\partial^\mu A_\mu^a)\big(\partial^\nu D_\nu^{ab}\omega_b\big)\,.$$

Hence,

$$\delta\mathscr{L}_{\rm GF} = -\frac{\epsilon}{a}(\partial^\mu A_\mu^a)(\partial^\nu D_\nu^{ab}\omega_b)\,. \tag{49}$$

(b) Attempt to restore the symmetry by adding a new field $\eta(x)$ and a new term to the Lagrangian:

$$\mathscr{L}_{\rm G} = -\eta_a(\partial^\mu D_\mu^{ab}\omega_b)\,, \tag{50}$$

and by postulating the transformation law:

$$\delta\eta_a = -\frac{\epsilon}{a}(\partial_\mu A_a^\mu)\,. \tag{51}$$

Show that this does not quite work because D_μ is field dependent and:

$$\delta(\mathscr{L}_{\rm GF} + \mathscr{L}_{\rm G}) \neq 0\,.$$

Consider a gauge group whose generators obey the commutation relations,

$$[T^a\,,\,T^b] = if^{abc}T^c\,, \tag{52}$$

where we assume that the generators have been chosen such that f^{abc} is completely antisymmetric under the interchange of any pair of indices. The covariant derivative acting on a field in the adjoint representation is then given by

$$D_\mu^{ab} = \delta^{ab}\partial_\mu + ig(T^c)^{ab}A_\mu^c\,,$$

where the Lie group generator in the adjoint representation is given by $(T^c)^{ab} = -if^{cab}$. That is,

$$D_\mu^{ab} = \delta^{ab}\partial_\mu + gf^{abc}A_\mu^c\,.$$

Thus, we can rewrite eq. (50) as

$$\mathscr{L}_{\rm G} = -\eta_a\,\partial^\mu\big(\partial_\mu\omega_a + gf^{abc}\omega_bA_\mu^c\big)\,. \tag{53}$$

Next, we apply an infinitesimal gauge transformation to eq. (53). Working to first order in the field variations,

$$\begin{aligned}\delta\mathscr{L}_{\rm G} &= -(\delta\eta_a)(\partial^\mu D_\mu^{ab}\omega_b) - \eta_a(\partial^\mu \delta D_\mu^{ab}\omega_b) = -(\delta\eta_a)(\partial^\mu D_\mu^{ab}\omega_b) - gf^{abc}\eta_a\partial^\mu\big(\omega_b\delta A_\mu^c\big) \\ &= \frac{\epsilon}{a}(\partial_\nu A_a^\nu)(\partial^\mu D_\mu^{ab}\omega_b) - gf^{abc}\eta_a\partial^\mu\big(\omega_b\epsilon D_\mu^{cd}\omega_d\big)\,. \end{aligned} \tag{54}$$

Hence,

$$\delta(\mathscr{L}_{\rm GF}+\mathscr{L}_{\rm G}) = -gf^{abc}\eta_a\partial^\mu\big(\omega_b\epsilon D_\mu^{cd}\omega_d\big) \neq 0\,.$$

(c) Save the day by promoting ω to a field and postulating the transformation law:

$$\delta\omega_a = \tfrac{1}{2}\epsilon g f^{abc}\omega_b\omega_c\,, \tag{55}$$

where g is the Yang-Mills coupling constant and the f_{abc} are the structure constants of the gauge group. Summation over repeated indices is implied. Note that since the f_{abc} are totally antisymmetric under interchange of a, b and c, the only way to have $\delta\omega\neq 0$ is to require that ω is an anticommuting field. This immediately implies that η is an anticommuting field and ϵ is an anticommuting infinitesimal constant. With this in mind, show that $\mathscr{L}_{\rm YM}+\mathscr{L}_{\rm GF}+\mathscr{L}_{\rm G}$ is invariant under the transformation laws given by eqs. (47)–(55). This enlarged gauge invariance is called BRST invariance (and δ is called an infinitesimal BRST transformation).

In light of eq. (55), eq. (54) is modified as follows:

$$\begin{aligned}\delta\mathscr{L}_{\rm G} &= -(\delta\eta_a)(\partial^\mu D_\mu^{ab}\omega_b) - \eta_a[\partial^\mu(\delta D_\mu^{ab})\omega_b] - \eta_a(\partial^\mu D_\mu^{ab}\delta\omega_b) \\ &= \frac{\epsilon}{a}(\partial_\nu A_a^\nu)(\partial^\mu D_\mu^{ab}\omega_b) - gf^{abc}\eta_a\partial^\mu\big(\omega_b\epsilon D_\mu^{cd}\omega_d\big) - \tfrac{1}{2}gf^{bcd}\eta_a\,\partial^\mu\epsilon D_\mu^{ab}(\omega_c\omega_d) \\ &= \frac{\epsilon}{a}(\partial_\nu A_a^\nu)(\partial^\mu D_\mu^{ab}\omega_b) - \epsilon g\eta_a\partial^\mu\Big\{f^{abc}\big(\omega_b D_\mu^{cd}\omega_d\big) - \tfrac{1}{2}f^{bcd}D_\mu^{ab}(\omega_c\omega_d)\Big\}\,, \end{aligned} \tag{56}$$

where we have made use of the anticommutativity properties of ϵ, η and ω.

Focusing on the term inside the braces in eq. (56),

$$\begin{aligned} f^{abc}\omega_b D_\mu^{cd}\omega_d - \tfrac{1}{2}f^{bcd}D_\mu^{ab}(\omega_c\omega_d) &= f^{abc}\omega_b(\partial_\mu\omega_c + gf^{cde}\omega_d A_\mu^e) - \tfrac{1}{2}f^{bcd}(\delta^{ab}\partial_\mu + gf^{abe}A_\mu^e)(\omega_c\omega_d) \\ &= f^{abc}\omega_b\partial_\mu\omega_c - \tfrac{1}{2}f^{acd}\big[\omega_c\partial_\mu\omega_d + (\partial_\mu\omega_c)\omega_d\big] + gA_\mu^e\big(f^{abc}f^{cde}\omega_b\omega_d - \tfrac{1}{2}f^{bcd}f^{abe}\omega_c\omega_d\big)\,. \end{aligned}$$

After an appropriate relabeling of indicies,

$$f^{abc}\omega_b\partial_\mu\omega_c - \tfrac{1}{2}f^{acd}\big[\omega_c\partial_\mu\omega_d + (\partial_\mu\omega_c)\omega_d\big] = f^{abc}\big[\omega_b\partial_\mu\omega_c - \tfrac{1}{2}\omega_b\partial_\mu\omega_c - \tfrac{1}{2}(\partial_\mu\omega_b)\omega_c\big] = 0\,,$$

after using the anticommuting properties of ω_b and ω_c and antisymmetry properties of the f^{abc}. Likewise, using the same properties and appropriate relabeling of indices,

$$\begin{aligned} f^{abc}f^{cde}\omega_b\omega_d - \tfrac{1}{2}f^{bcd}f^{abe}\omega_c\omega_d &= \omega_c\omega_d(f^{bac}f^{bde} + \tfrac{1}{2}f^{bcd}f^{bae}) \\ &= \tfrac{1}{2}\omega_c\omega_d(f^{bac}f^{bde} + f^{bad}f^{bec} + f^{bae}f^{bcd}) = 0\,, \end{aligned} \tag{57}$$

where the last step is a consequence of the Jacobi identity. Hence, the expression inside the braces in eq. (56) vanishes, and we are left with

$$\delta\mathscr{L}_{\rm G} = \frac{\epsilon}{a}(\partial_\nu A^\nu_a)(\partial^\mu D^{ab}_\mu \omega_b)\,.$$

Combining with eq. (49) yields

$$\delta(\mathscr{L}_{\rm GF} + \mathscr{L}_{\rm G}) = 0\,.$$

(d) Define δ^2 to mean the application of δ with anti-commuting parameter ϵ_1 followed by δ with anti-commuting parameter ϵ_2. Show that when δ^2 is applied to A^a_μ and ω_a, the result is zero in each case. However $\delta^2\eta_a = 0$ only if the Lagrange field equations for the ghost fields ω_a are satisfied.

We first compute

$$\begin{aligned}
\delta^2 A^a_\mu &= \delta\Big\{\epsilon_1\big[\partial_\mu\omega_a + gf^{abc}\omega_b A^c_\mu\big]\Big\} \\
&= \epsilon_1\epsilon_2\Big\{\tfrac{1}{2}gf^{abc}\partial_\mu(\eta_b\omega_c) + gf^{abc}(D^{cd}_\mu\omega_d)\omega_b + \tfrac{1}{2}gf^{abc}f^{bde}\omega_d\omega_e A^c_\mu\Big\} \\
&= \epsilon_1\epsilon_2\Big\{\tfrac{1}{2}gf^{abc}\big[(\partial_\mu\omega_b)\omega_c + \omega_b(\partial_\mu\omega_c)\big] + gf^{abc}\big[\partial_\mu\omega_c + gf^{cde}A^d_\mu\omega_e\big]\omega_b + \tfrac{1}{2}g^2 f^{abc}f^{bde}\omega_d\omega_e A^c_\mu\Big\} \\
&= \epsilon_1\epsilon_2 g^2\omega_e\omega_b A^d_\mu\big(f^{cab}f^{cde} + \tfrac{1}{2}f^{cad}f^{ceb}\big) = 0\,,
\end{aligned}$$

after an appropriate relabeling of indices. The final steps are the same as in eq. (57), where after some manipulation the Jacobi identity is invoked. Next,

$$\begin{aligned}
\delta^2\omega_a &= \tfrac{1}{2}\epsilon_1 gf^{abc}\delta(\omega_b\omega_c) \\
&= \tfrac{1}{2}\epsilon_1 gf^{abc}\big[(\delta\omega_b)\omega_c + \omega_b\delta\omega_c\big] \\
&= \tfrac{1}{4}\epsilon_1\epsilon_2 g^2(f^{abc}f^{bde}\omega_d\omega_e\omega_c - f^{abc}f^{cde}\omega_b\omega_d\omega_e) \\
&= \tfrac{1}{4}\epsilon_1\epsilon_2 g^2(f^{abc}f^{bde} - f^{acb}f^{bde}\omega_c\omega_d\omega_e \\
&= -\tfrac{1}{2}\epsilon_1\epsilon_2 g^2 f^{acb}f^{bde}\omega_c\omega_d\omega_e \\
&= -\tfrac{1}{6}\epsilon_1\epsilon_2 g^2 f^{acb}f^{bde}(\omega_c\omega_d\omega_e + \omega_e\omega_c\omega_d + \omega_d\omega_e\omega_c) \\
&= -\tfrac{1}{6}\epsilon_1\epsilon_2 g^2(f^{acb}f^{bde} + f^{adb}f^{bec} + f^{aeb}f^{bcd})\omega_c\omega_d\omega_e = 0\,,
\end{aligned}$$

after appropriate relabeling of indices and using the anticommuting properties of the ω and the antisymmetry of the f^{abc}. Finally, the Jacobi identity is used at the final step.

However, we do not obtain $\delta^2\eta_a = 0$. An explicit computation yields

$$\delta^2\eta_a = -\frac{1}{a}\epsilon_1\delta(\partial^\mu A^a_\mu) = -\frac{1}{a}\epsilon_1\epsilon_2\partial^\mu D^{ab}_\mu\omega_b \neq 0\,. \tag{58}$$

Nevertheless, in light of eq. (50), we see that the Lagrange field equations for the ghost field $\eta_a(x)$ is

$$\partial_\mu \frac{\partial \mathscr{L}_{\rm G}}{\partial(\partial_\mu \eta_a)} - \frac{\partial \mathscr{L}_{\rm G}}{\partial \eta_a} = \partial^\mu D_\mu^{ab}\omega_b = 0\,.$$

That is, if we apply the field equations for ω_b in eq. (58), we do obtain $\delta^2\eta_a = 0$.

(e) Suppose that the gauge fixing term is chosen to be

$$\mathscr{L}_{\rm GF} = B_a \partial_\mu A_a^\mu + \frac{a}{2} B_a B_a\,. \tag{59}$$

Note that the new field B_a has no kinetic energy term; it is thus an auxiliary field. Show that if one solves for B_a using the Lagrange field equations, one regains the usual gauge fixing term given by eq. (48).

Applying the Lagrange field equations for B_a,

$$\partial_\mu \frac{\partial \mathscr{L}_{\rm GF}}{\partial(\partial_\mu B_a)} - \frac{\partial \mathscr{L}_{\rm GF}}{\partial B_a} = -\partial_\mu A_\mu^a - aB_a = 0\,.$$

That is,

$$B_a = -\frac{1}{a}\partial_\mu A_\mu^a\,.$$

Inserting this result into eq. (59) yields,

$$\mathscr{L}_{\rm GF} = -\frac{1}{2a}(\partial_\mu A_a^\mu)^2\,,$$

which coincides with eq. (48).

(f) Using the new gauge fixing term given in eq. (59), we now modify the BRST transformation law of η and define:

$$\delta\eta_a = \epsilon B_a\,, \tag{60}$$

$$\delta B_a = 0\,. \tag{61}$$

Show that the full Lagrangian is still invariant under the BRST transformation. Furthermore, verify that $\delta^2 = 0$ when applied to *all* fields of the theory, independently of the field equations.

Applying the transformation given by eq. (61) to eq. (59), and using eq. (47),

$$\delta\mathscr{L}_{\rm GF} = (\delta B_a)\partial^\mu A_\mu^a + B_a \partial^\mu \delta A_\mu^a + aB_a\delta B_a = \epsilon B_a \partial^\mu D_\mu^{ab}\omega_b\,.$$

Next, in light of eq. (60), we see that eq. (56) is now given by

$$\delta\mathscr{L}_{\rm G} = -(\delta\eta_a)(\partial^\mu D_\mu^{ab}\omega_b) = -\epsilon B_a \partial^\mu D_\mu^{ab}\omega_b\,.$$

since the terms inside the braces in eq. (56) cancel by virtue of eq. (57). hence it follows that

$$\delta(\mathscr{L}_{\rm GF} + \mathscr{L}_{\rm G}) = 0\,.$$

That is, the full Lagrangian is still invariant under the BRST transformation.

Furthermore, the computations of $\delta^2 A^a_\mu = 0$ and $\delta^2\omega_a = 0$ are unchanged from those of part (d). However, now we have

$$\delta^2\eta = \epsilon_1 \delta B = 0\,,$$

and $\delta^2 B = 0$ as a consequence of eqs. (60) and (61). Thus, $\delta^2 = 0$ when applied to *all* fields of the theory, independently of the field equations.

REMARK: One can extend the above results by including a gauge invariant term in the Lagrangian density involving fermions, $\mathscr{L}_{\rm f} = i\overline{\psi}_i\gamma_\mu D^\mu_{ij}\psi_j$ with $D^\mu_{ij} = \delta_{ij}\partial^\mu + igT^a_{ij}A^\mu_a$. Note that $\mathscr{L}_{\rm f}$ is invariant with respect to the infinitesimal BRST transformation, $\delta\psi = -i\epsilon g T^a\eta_a\psi$. One can show that $\delta^2\psi = 0$ as a consequence of eq. (52).

APPENDIX: The reflection principle and its implications

If $f(z)$ is an analytic function in some region of the complex plane, then so is $f^*(z^*)$. If $f(z)$ is a real valued function in a region of the complex plane that includes part of the real axis, then $f(z) = f^*(z^*)$ along that part of the real axis (since $z = z^*$ on the real axis). Consequently, $f^*(z^*)$ and $f(z)$ are analytic continuations of one another. As long as no singularities are encountered, it follows that $f(z) = f^*(z^*)$, which implies that $f^*(z) = f(z^*)$. That is, we have proven the reflection principle of complex analysis,

Theorem (Reflection principle): If $f(z)$ is real and analytic on part of the real axis, then $f^*(z) = f(z^*)$ at all points in the complex plane where $f(z)$ is analytic.

As an application of the reflection principle, we can show that the $\mathrm{Disc}\,\Gamma^{(4)}(s)$ is related to $\mathrm{Im}\,\Gamma^{(4)}(s)$. In particular, applying the reflection principle to $\Gamma^{(4)}(s+i\epsilon)$ yields

$$\Gamma^{(4)}(s - i\epsilon) = \Gamma^{(4)}(s+i\epsilon)^*\,, \qquad (62)$$

which was quoted in eq. (28). We can therefore conclude that

$$\mathrm{Disc}\,\Gamma^{(4)}(s) \equiv \Gamma^{(4)}(s+i\epsilon) - \Gamma^{(4)}(s+i\epsilon)^* = 2i\,\mathrm{Im}\,\Gamma^{(4)}(s)\,,$$

where we have defined

$$\Gamma^{(4)}(s) \equiv \lim_{\epsilon\to 0}\Gamma^{(4)}(s+i\epsilon)\,.$$