1. In class, I defined the matrix-valued covariant derivative operator in the adjoint representation, $\mathcal{D}_\mu$, by

$$\mathcal{D}_\mu V \equiv (D_\mu V)_a T^a = \partial_\mu V + ig[A_\mu , V],$$

where $V \equiv V^a T^a$ is a matrix-valued adjoint field and $(D_\mu)_{ab} \equiv \delta_{ab} \partial_\mu + gf_{cab} A_\mu^c$ is the covariant derivative acting on a field in the adjoint representation. The commutation relations satisfied by the generators of the Lie group $G$ are given by $[T_a , T_b] = if_{abc} T^c$, and the indices $a$, $b$ and $c$ take on $d_G$ possible values, where $d_G$ is the dimension of $G$.

(a) Prove that for any pair of matrix-valued adjoint fields $V$ and $W$,

$$[\mathcal{D}_\mu , V] W = (\mathcal{D}_\mu V)W,$$

where $[,]$ is the usual matrix commutator. This means that $\mathcal{D}_\mu V = [\mathcal{D}_\mu , V]$ holds as an operator equation.

By definition of the commutator, for adjoint fields $V$ and $W$,

$$[\mathcal{D}_\mu , V] W \equiv (\mathcal{D}_\mu V - \mathcal{D}_\mu W)W = \mathcal{D}_\mu (VW) - V \mathcal{D}_\mu W$$

$$= \partial_\mu (VW) + ig[A_\mu , VW] - V \partial_\mu W - igV[A_\mu , W]$$

$$= (\partial_\mu V)W + ig[A_\mu , V]W = \{\partial_\mu V + ig[A_\mu , V]\}W$$

$$= (\mathcal{D}_\mu V)W.$$ 

This is true for an arbitrary adjoint field $W$. Hence,

$$\mathcal{D}_\mu V = [\mathcal{D}_\mu , V],$$

holds as an operator identity.

(b) Prove that for any matrix-valued adjoint field $V$,

$$[\mathcal{D}_\mu , \mathcal{D}_\nu]V = ig[F_{\mu\nu} , V],$$

where $F_{\mu\nu} \equiv F^a_{\mu\nu} T^a$ is the matrix-value field strength tensor of the non-abelian gauge theory.

Using the definition of $\mathcal{D}_\mu$ given in eq. (1),

$$[\mathcal{D}_\mu , \mathcal{D}_\nu]V = \mathcal{D}_\mu (\partial_\nu V + ig[A_\nu , V]) - \mathcal{D}_\nu (\partial_\mu V + ig[A_\mu , V])$$

$$= \partial_\mu (\partial_\nu V + ig[A_\nu , V]) + ig[A_\mu , \partial_\nu V + ig[A_\nu , V]]$$

$$- \partial_\nu (\partial_\mu V + ig[A_\mu , V]) - ig[A_\nu , \partial_\mu V + ig[A_\mu , V]]$$

$$= ig\left\{[\partial_\mu A_\nu - \partial_\nu A_\mu , V] + ig[A_\mu , [A_\nu , V]] - ig[A_\nu , [A_\mu , V]]\right\}.$$
Using the Jacobi identity,
\[
[A_\mu, [A_\nu, V]] + [V, [A_\mu, A_\nu]] + [A_\nu, [V, A_\mu]] = 0.
\]
and the antisymmetry of the commutator, e.g. \([V, A_\mu] = -[A_\mu, V]\), it follows that,
\[
[D_\mu, D_\nu]V = ig[\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], V].
\]
using the definition of the matrix field-strength tensor,
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu],
\]
we end up with
\[
[D_\mu, D_\nu]V = ig[F_{\mu\nu}, V].
\]
(3)

An alternative derivation:
Note that by using the definition of the commutator, for adjoint fields \(V\) and \(W\),
\[
([D_\mu, D_\nu]V)W = (D_\mu D_\nu - D_\nu D_\mu)W
= (D_\mu[D_\nu, V] - D_\nu[D_\mu, V])W
= ([D_\mu, [D_\nu, V]] - [D_\nu, [D_\mu, V]])W,
\]
after using eq. (2). The Jacobi identity implies that the following operator identity holds:
\[
[D_\mu, [D_\nu, V]] + [V, [D_\mu, D_\nu]] + [D_\nu, [V, D_\mu]] = 0.
\]
Thus, using the Jacobi identity and the antisymmetry property of the commutator, eq. (4) yields
\[
([D_\mu, D_\nu]V)W = [[D_\mu, D_\nu], V]W.
\]
This is true for an arbitrary adjoint field \(W\). Hence,
\[
[D_\mu, D_\nu]V = [[D_\mu, D_\nu], V],
\]
(5)
holds as an operator identity.

The matrix field strength tensor was initially defined in class via the operator identity
\[
[D_\mu, D_\nu] = igF_{\mu\nu},
\]
(6)
where \(D_\mu\) is the covariant derivative defined by its action on a field that transforms according to an arbitrary representation of the Lie group \(G\). In particular, eq. (6) must hold in the adjoint representation, which implies that
\[
[D_\mu, D_\nu] = igF_{\mu\nu}
\]
holds as an operator identity. Inserting this result into eq. (5) yields
\[
[D_\mu, D_\nu]V = ig[F_{\mu\nu}, V],
\]
which again confirms eq. (3).
2. Consider the spontaneous breaking of a gauge group $G$ down to $U(1)$. The unbroken generator $Q = c_a T^a$ is some real linear combination of the generators of $G$.

(a) Prove that $x_b \equiv c_b/g_b$ is an (unnormalized) eigenvector of the vector boson squared-mass matrix, $M^2_{ab}$, with zero eigenvalue.

It is convenient to use a real basis for the scalar fields, in which case the $iT^a$ are real antisymmetric matrices. Suppose that the $T^a$ are represented by $n \times n$ matrices, which act on a multiplet of $n$ scalar fields. The group $G$ is spontaneously broken down to $U(1)$ when the vacuum expectation values of the scalar fields is given by $v$, which can be represented by a column vector of $n$ rows. The unbroken generator,

$$Q = c_a T^a,$$

is some real linear combination of the generators of $G$ that satisfies

$$Q_{jk}v_k = 0,$$

where there is an implicit sum over $k = 1, 2, \ldots, n$.

In eq. (64) of solution set 2, the squared-mass matrix of the gauge bosons was obtained,

$$M^2_{ab} = g^2 v_j (T^a T^b)_{jk} v_k,$$

where the $T^a$ are the generators of a simple compact gauge group $G$. If the gauge group $G$ is a direct product of the form $G = G_1 \times G_2 \cdots G_r$, where the $G_i$ are either simple compact Lie groups or $U(1)$, then we associate an independent gauge coupling $g_1, g_2, \ldots, g_r$ with each factor. In this case, we generalize eq. (9) slightly,

$$M^2_{ab} = g_a g_b v_j (T^a T^b)_{jk} v_k,$$

where there is no sum over the repeated indices $a$ and $b$. Here, we associate $g_a$ with the generators that belong to the appropriate $G_i$ of the direct product. In particular, if $T^a$ and $T^b$ are generators of $G_i$, then $g_a = g_b = g_i$ and

$$[T^a, T^b] = if_{abc} T^c,$$

where $f_{abc} = 0$ if $T^c$ is a generator of $G_j$ with $j \neq i$, and $f^{abc}$ are the structure constants of $G_i$ if $j = i$. Likewise, if $T^a$ is a generator of $G_i$ and $T^b$ is a generator of $G_j$ with $i \neq j$, then $g_a = g_i$, $g_b = g_j$ and $[T^a, T^b] = 0$ (or equivalently, $f_{abc} = 0$) as a consequence of the direct product structure of $G$. One consequence of these observations is that

$$(g_a - g_b)f_{abc} = 0,$$

where there is no implicit sum over $a$ and $b$.

We now consider the spontaneous breaking of a gauge group $G$ down to $U(1)$. The unbroken generator $Q = c_a T^a$ is some real linear combination of the generators of $G$. Then,

$$\sum_b M^2_{ab} c_b g_b = \sum_b g_a g_b v_j (T^a T^b)_{jk} v_k c_b g_b = g_a v_j T^a_{j\ell} \left( \sum_b c_b T^b_{\ell k} \right) v_k = g_a v_j T^a_{j\ell} Q_{\ell k} v_k = 0,$$
after using eqs. (8) and (10). That is, \( c_b/g_b \) is an eigenvector of \( M_{ab}^2 \) with eigenvalue zero. This corresponds to the massless U(1) gauge boson which remains massless due to the residual unbroken U(1) gauge symmetry.

(b) Suppose that \( A_\mu \) is the massless gauge field that corresponds to the generator \( Q \). Show that the covariant derivative can be expressed in the following form:

\[
D_\mu = \partial_\mu + ieQA_\mu + \ldots,
\]

where we have omitted terms in eq. (12) corresponding to all the other gauge bosons and

\[
e = \left[ \sum_a \left( \frac{c_a}{g_a} \right)^2 \right]^{-1/2}.
\]

Employing a real basis for the scalar fields, we define real antisymmetric generators via

\[
L^a \equiv ig_a T^a,
\]

where there is no implicit sum over \( a \) [cf. the comment following eq. (10)]. Following the class handout on gauge theories, we can rewrite eq. (10) as

\[
M_{ab}^2 = (L_a v, L_b v)
\]

is the gauge boson squared-mass matrix. Here, we have employed a convenient notation where:

\[
(x, y) \equiv \sum_i x_i y_i.
\]

The gauge boson squared-mass matrix is real symmetric, so it can be diagonalized with an orthogonal similarity transformation:

\[
\mathcal{O}M^2\mathcal{O}^T = \text{diag} \left( 0, 0, \ldots, 0, m_1^2, m_2^2, \ldots \right).
\]

The corresponding gauge boson mass-eigenstates are:\footnote{Indeed, one can easily check that \( M_{ab}^2 A_\mu^a A_\mu^b = \sum_a m_a^2 \tilde{A}_\mu^a \tilde{A}^{\mu a} \).}

\[
\tilde{A}_\mu^a \equiv \mathcal{O}_{ab}A_\mu^b.
\]

Likewise, we may define a new basis for the Lie algebra:

\[
\tilde{L}_a \equiv \mathcal{O}_{ab}L_b.
\]

It then follows that:

\[
(\mathcal{O}M^2\mathcal{O}^T)_{ab} = (\tilde{L}_a^\nu, \tilde{L}_b^\nu) = m_a^2 \delta_{ab},
\]

is the diagonalized vector boson squared-mass matrix, and the covariant derivative is given by

\[
D_\mu = \partial_\mu + L_a A_\mu^a = \partial_\mu + \tilde{L}_a \tilde{A}_\mu^a = \partial_\mu + ieQA_\mu + \ldots,
\]

where \( A_\mu \) is the gauge boson corresponding to the unbroken U(1) generator.
Let us choose \( O \) such that \( m_1 = 0 \) is the mass of the gauge boson that corresponds to the unbroken U(1). Then, we can identify the unbroken generator as

\[
\tilde{L}_1 = ieQ = O_{1b}L_b,
\]

(21)

where there is an implicit sum over \( b \). Moreover, eq. (20) yields \( O_{1a}M^2_{ab}O_{1b} = 0 \). The rows of the diagonalizing matrix \( O \) correspond to the normalized eigenvectors of \( M^2 \). Thus,

\[
O_{1b} = \frac{1}{N} \frac{c_b}{g_b}
\]

(22)

is the normalized eigenvector corresponding to the zero eigenvalue found in part (a). The normalization constant \( N \) is easily obtained,

\[
N^2 = \sum_b \left( \frac{c_b}{g_b} \right)^2
\]

(23)

Eqs. (21) and (22) yield

\[
_ieQ = \frac{1}{N} \sum_b \frac{c_b}{g_b}L_b = \frac{i}{N} \sum_b c_bT^b = \frac{i}{N} Q,
\]

after making use of eqs. (7) and (14). It immediately follows that \( e = 1/N \). In light of eq. (22),

\[
e = \left[ \sum_b \left( \frac{c_b}{g_b} \right)^2 \right]^{-1/2}
\]

(24)

By convention, we take \( e > 0 \).

(c) Evaluate \( Q \) in the adjoint representation (that is, \( Q = c_aT^a \), where the \((T^a)_{bc} = -if_{abc}\) are the generators of the gauge group in the adjoint representation). Show that \( Q_{bc}x_c = 0 \), where \( x_c \) is defined in part (a). What is the physical interpretation of this result?

Using \( x_c = c_c/g_c \) and eq. (7), we obtain

\[
Q_{bc}x_c = c_aT^a_{bc}x_c = -if_{abc} \frac{c_ac_c}{g_c} = if_{acb} \frac{c_ac_c}{g_c},
\]

(25)

using the antisymmetry properties of the \( f_{abc} \). Employing eq. (11),

\[
\sum_{a,c} f_{acb} \frac{c_ac_c}{g_c} = \sum_{a,c} g_a f_{acb} \frac{c_ac_c}{g_a g_c} = \sum_{a,c} g_c f_{acb} \frac{c_ac_c}{g_a g_c} = \sum_{a,c} f_{acb} \frac{c_ac_c}{g_a} = \sum_{a,c} f_{cab} \frac{c_ac_c}{g_c} = - \sum_{a,c} f_{acb} \frac{c_ac_c}{g_c},
\]

where in the penultimate step we relabeled \( a \rightarrow c \) and \( c \rightarrow a \), and in the last step we used \( f_{cab} = -f_{acb} \). Hence,

\[
\sum_{a,c} f_{acb} \frac{c_ac_c}{g_c} = 0,
\]

which means that \( Q_{bc}x_c = 0 \). The physical interpretation of this statement is that the U(1) gauge boson is neutral with respect to the unbroken generator \( Q \).
(d) Prove that the commutator \([Q, M^2]\) = 0, where \(Q\) is the unbroken U(1) generator in the adjoint representation and \(M^2\) is the gauge boson squared-mass matrix. Conclude that one can always choose the eigenstates of the gauge boson squared-mass matrix to be states of definite unbroken U(1)-charge.

In the adjoint representation [cf. eq. (25)], \(Q_{bc} = i \sum_e f_{ecb} c_e\), where there is an implicit sum over the repeated index \(a\). Using eq. (10),
\[
[Q, M^2]_{ac} = \sum_b \left( Q_{ab} M_{bc}^2 - M_{ab}^2 Q_{bc} \right) = iv^T \sum_{b,e} \left( g_b g_c c_e f_{eba} T^b T^c - g_a g_b c_e f_{ecb} T^a T^b \right) v,
\]
where \(v^T T^b T^c v \equiv v_j (T^b T^c)_{jk} v_k\), etc. Note that all sums are explicitly exhibited; there are no implicit sums over repeated indices in eq. (26). Employing eq. (11),
\[
\sum_b g_b f_{eba} T^b = \sum_b g_b f_{abe} T^b = g_a \sum_b f_{bae} T^b,
\]
\[
\sum_b g_b f_{ecb} T^b = \sum_b g_b f_{ecb} T^b = g_c \sum_b f_{cbe} T^b.
\]

Using the commutation relations of the generators,
\[
if_{bae} T^b = if_{aeb} T^b = [T^a, T^e] = T^a T^e - T^e T^a,
\]
\[
if_{ecb} T^b = if_{ecb} T^b = [T^c, T^e] = T^e T^c - T^c T^e.
\]

Inserting these results back into eq. (26) yields
\[
[Q, M^2]_{ac} = g_a g_c v^T \left\{ c_e (T^a T^e - T^e T^a) T^c - c_e T^a (T^e T^c - T^c T^e) \right\} v. \tag{27}
\]

The \(T^a\) are the generators in the representation that acts on the scalar fields. In this representation the charge operator, which will be denoted by \(Q\) to distinguish it from the charge operator in the adjoint representation \(Q\), is defined by
\[
Q = \sum_e c_e T^e.
\]

Hence, eq. (27) yields
\[
[Q, M^2]_{ac} = g_a g_c v^T \left\{ (T^a Q - QT^a) T^c - T^a (QT^c - T^c Q) \right\} v
\]
\[
= g_a g_c v^T (T^a T^c Q - QT^a T^c) v. \tag{28}
\]

Using the fact that \(Q\) is an unbroken generator corresponding to the unbroken U(1) subgroup of \(G\), it follows that \(v^T Q = Q v = 0\). Employing this result in eq. (28) yields
\[
[Q, M^2] = 0.
\]

Thus, one can simultaneously diagonalize \(M^2\) and \(Q\). The corresponding simultaneous eigenstates are gauge boson states of definite mass and unbroken U(1)-charge.
3. In class, we examined in detail the structure of a spontaneously broken SU(2) \times U(1) gauge theory, in which the symmetry breaking was due to the vacuum expectation value of a $Y = 1$, SU(2) doublet of complex scalar fields. In this problem, we will replace this multiplet of scalar fields with a different representation.

(a) Consider a spontaneously broken SU(2) \times U(1) gauge theory with a $Y = 0$, SU(2) triplet of real scalar fields. Assume that the electrically neutral ($Q = 0$) member of the scalar triplet acquires a vacuum expectation value (where $Q = T_3 + Y/2$). After symmetry breaking, identify the subgroup that remains unbroken. Compute the vector boson masses and the physical Higgs scalar masses in this model. Deduce the Feynman rules for the three-point interactions among the Higgs and vector bosons.

Consider a model where the SU(2) \times U(1) gauge symmetry is broken by a $Y = 0$ triplet of real scalar fields, whose neutral member acquires a vacuum expectation value. The generators of SU(2) in the triplet (adjoint) representation are $(T^a)_{bc} = -i \epsilon_{abc}$. Explicitly,

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (29)$$

Using eq. (14),

$$L_1 = g \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad L_2 = g \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L_3 = g \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which act on the scalar field multiplet,

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{pmatrix},$$

where the $\phi_i(x)$ are real scalar fields. In addition, the generator corresponding to U(1) is

$$L_4 = i \frac{1}{g'} Y,$$

where $Y$ is the hypercharge operator. Since $\Phi(x)$ is a triplet of scalar fields with zero hypercharge, it follows that $L_4 \Phi = 0$.

We now compute the squared-mass matrix of the gauge bosons using eq. (15), where $v$ is the vacuum expectation value of the electrically neutral member of the scalar triplet. The electric charge operator is given by

$$Q = T^3 + \frac{1}{2} Y.$$  

In particular, when acting on the scalar triplet (which has hypercharge zero),

$$Q \Phi = (T^3 + \frac{1}{2} Y) \Phi = T^3 \Phi. \quad (30)$$
This implies that the electrically neutral member of the scalar triplet must be an eigenstate of \( T^3 \) with zero eigenvalue. Thus, we choose the vector \( v \) to have the form
\[
v = \begin{pmatrix}
0 \\
0 \\
v
\end{pmatrix}, \tag{31}
\]
in order to ensure that after spontaneous symmetry breaking, the unbroken gauge group preserves the U(1) of electromagnetism. We now can compute \( L_a v \) for \( a = 1, 2, 3, 4 \). We already know that \( L_4 \Phi = 0 \), so we need only consider \( a = 1, 2, 3 \).

\[
L_1 v = g \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
v
\end{pmatrix} = \begin{pmatrix}
0 \\
gv \\
0
\end{pmatrix},
\]
\[
L_2 v = g \begin{pmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
v
\end{pmatrix} = \begin{pmatrix}
-gv \\
0 \\
0
\end{pmatrix},
\]
\[
L_3 v = g \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
v
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

Thus, eq. (15) yields the squared-mass matrix of the gauge bosons,
\[
M^2 = \begin{pmatrix}
g^2 v^2 & 0 & 0 & 0 \\
0 & g^2 v^2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{32}
\]

We conclude that the \( W^\pm \) has gained a mass \( m_{W^\pm} = gv \), whereas \( W^3 \) and \( B \) remain massless. This means that the SU(2) × U(1) gauge symmetry has broken down to U(1) × U(1). One of the U(1)’s can be identified with the gauge group of electromagnetism. In light of eq. (30), we can choose the electromagnetic charge operator to be \( Q = T^3 \). Using eq. (24), we can therefore identify the photon field as \( A_\mu = W^3_\mu \) and \( e = g \).

The physical Higgs bosons of the model are obtained from the Higgs scalar potential. The most general quartic gauge invariant scalar potential is
\[
V(\Phi) = -\frac{1}{2} m^2 \Phi^T \Phi + \lambda (\Phi^T \Phi)^2 = -\frac{1}{2} m^2 (\phi_1^2 + \phi_2^2 + \phi_3^2)^2 + \lambda (\phi_1^2 + \phi_2^2 + \phi_3^2)^4. \tag{33}
\]
The minimum of the scalar potential corresponds to \( \Phi = v \) given by eq. (31). Imposing the minimum condition,
\[
\left( \frac{\partial V}{\partial \phi_i} \right)_{\Phi = v} = 0,
\]
yields
\[
-m^2 v + 4 \lambda v^3 = 0. \tag{34}
\]
Assuming that the symmetry is broken, \( v \neq 0 \), and we obtain

\[
v = \frac{m}{2\sqrt{\lambda}}.
\]

Note that this implies that \( V(v) = -m^2/(16\lambda) \). A second extremum corresponding to \( \Phi = 0 \) (the symmetry conserving minimum) yields \( V(0) = 0 \). Hence, if \( m^2 > 0 \), it follows that the symmetry-breaking minimum is the global minimum of the scalar potential.

Next, we identify the Goldstone bosons, which were given in class by

\[
G_a = \frac{1}{m_a} \sum_j (\bar{L}_a v)_j \eta_j,
\]

where the \( \eta_j \) are the shifted scalar fields defined by

\[
\Phi = \begin{pmatrix}
\eta_a \\
\eta_2 \\
v + \eta_3
\end{pmatrix},
\]

and the \( \bar{L}_a \) are defined in eq. (19). Since \( M^2 \) given in eq. (32) is already diagonal, the diagonalization matrix \( O = 1 \) and \( \bar{L}_a = L_a \). Since \( L_a v = 0 \) for \( a = 3 \) and \( 4 \), it follows that there are precisely two Goldstone bosons, \( \eta_1 \) and \( \eta_2 \). Thus, \( H = \eta_3 \) is the physical Higgs boson.

In the unitary gauge, we set the Goldstone fields to zero. Then,

\[
V(H) = -\frac{1}{2} m^2 (H + v)^2 + \lambda (H + v)^4.
\]

Using eq. (34), the term linear in \( H \) vanishes. The constant term can be removed by redefining the energy of the vacuum to be zero. Thus,

\[
V(H) = H^2 (-\frac{1}{2} m^2 + 6\lambda v^2) + \mathcal{O}(H^3) + \mathcal{O}(H^4).
\]

The coefficient of \( H^2 \) is identified as \( \frac{1}{2} m_H^2 \), where \( m_H \) is the Higgs mass. Using eq. (34) to eliminate \( m^2 \), we obtain

\[
m_H^2 = -m^2 + 12\lambda v^2 = 8\lambda v^2.
\]

Hence,

\[
m_H = 2\sqrt{2\lambda v}.
\]

The three-point interactions among the Higgs and Gauge bosons arise from the kinetic energy term,

\[
\mathcal{L}_{KE} = \frac{1}{2} (D_\mu \Phi)^T D^\mu \Phi,
\]

where the covariant derivative acting on the scalar field is given by

\[
(D_\mu \Phi)_i = \partial_\mu \phi_i + ig T^a_{ij} W^a_\mu \phi_j + i\frac{g}{2} g' BY \phi_i.
\]

Since \( Y \Phi = 0 \), \( A_\mu = W^3_\mu \) and \( g = e \), we have

\[
D_\mu = \partial_\mu + \frac{ie}{\sqrt{2}} (T^+ W^+_\mu + T^- W^-_\mu) + ie QA_\mu,
\]
where $T^\pm \equiv T_1 \pm iT^2$ and

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp iT^2_\mu). \quad (35)$$

In the unitary gauge,

$$\Phi = \begin{pmatrix} 0 \\ 0 \\ v + H \end{pmatrix}.$$ 

Using

$$ieT^+ = e \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 1 \\ i & -1 & 0 \end{pmatrix}, \quad ieT^- = e \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & -1 & 0 \end{pmatrix}, \quad ieQ = e \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

it follows that

$$\left\{ \partial_\mu + \frac{ie}{\sqrt{2}}(T^+ W^+ + T^- W^-) + ieQA_\mu \right\} \begin{pmatrix} 0 \\ 0 \\ v + H \end{pmatrix} = \begin{pmatrix} -\frac{ie}{\sqrt{2}}(W^+_\mu - W^-_\mu)(v + H) \\ \frac{ie}{\sqrt{2}}(W^+_\mu + W^-_\mu)(v + H) \\ \partial_\mu H \end{pmatrix}.$$ 

Therefore, we end up with

$$\frac{1}{2}(D_\mu \Phi)^T D^\mu \Phi = \frac{1}{2}(\partial_\mu H)^2 + \frac{1}{2}(v + H)^2\left[ (W^+_\mu + W^-_\mu)^2 - (W^+_\mu - W^-_\mu)^2 \right]$$

$$= \frac{1}{2}(\partial_\mu H)^2 + e^2(v^2 + 2vH + H^2)W^+_\mu W^-_\mu$$

$$= \frac{1}{2}(\partial_\mu H)^2 + (m^2_W + 2emW^0 H + e^2H^2)W^+_\mu W^-_\mu,$$

after using $m_W = gv = ev$. We can therefore identify the Feynman rule for the trilinear $HW^+W^-$ interaction,

\[
\begin{array}{c}
\begin{array}{c}
H
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\nu
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
2iemWg^{\mu\nu}
\end{array}
\end{array}
\]

(b) Consider a spontaneously broken SU(2) $\times$ U(1)$_Y$ gauge theory with a $Y = 2$, SU(2) triplet of complex scalar fields. Again, assume that the electrically neutral ($Q = 0$) member of the scalar triplet acquires a vacuum expectation value (where $Q = T_3 + Y/2$). After symmetry breaking, identify the subgroup that remains unbroken. Compute the vector boson masses and the physical Higgs scalar masses in this model.

Consider a model where the SU(2) $\times$ U(1) gauge symmetry is broken by a $Y = 2$ triplet of complex scalar fields $\Phi(x)$, whose neutral member acquires a vacuum expectation value $\nu \equiv \langle \Phi \rangle$. 

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One could use the methods of part (a) by rewriting the complex scalar fields in terms of their real and imaginary parts. In this case, the real antisymmetric generators \( iT^a \) are \( 6 \times 6 \) matrices,

\[
iT^a = \begin{pmatrix} -\text{Im} \ T^a & -\text{Re} \ T^a \\ \text{Re} \ T^a & -\text{Im} \ T^a \end{pmatrix},
\]

which have been expressed in terms of the \( 3 \times 3 \) hermitian generators \( T^a \). However, following the class handout on gauge theories, we shall work directly with the complex fields and the basis of hermitian generators.

It is convenient to employ a basis of hermitian generators where \( T^3 \) is diagonal. For example, we can employ that standard spin-1 matrices defined in the \( |j m \rangle \) basis in quantum mechanics.\(^2\)

Explicitly,

\[
T^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Using eq. (14),

\[
\mathcal{L}_1 = \frac{g}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad \mathcal{L}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{L}_3 = g \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix},
\]

which act on the scalar field multiplet,

\[
\Phi(x) = \begin{pmatrix} \phi^{++}(x) \\ \phi^+(x) \\ \phi^0(x) \end{pmatrix},
\]

where \( \phi^{++}(x), \phi^+(x) \) and \( \phi^0(x) \) are complex scalar fields. In addition, the generator corresponding to the hypercharge \( U(1) \) is

\[
\mathcal{L}_4 = \frac{1}{2} ig' Y = ig' \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

where the hypercharge operator is normalized such that \( \mathcal{L}_4 \Phi = ig' \Phi \).

The electric charge operator is given by

\[
Q = T^3 + \frac{1}{2} Y = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

This implies that the electrically neutral member of the scalar triplet can be identified with \( \phi^0 \). Thus, we choose the vector \( \nu = \langle \Phi \rangle \) to have the form

\[
\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix},
\]

in order to ensure that after spontaneous symmetry breaking, the unbroken gauge group preserves U(1)_{EM}.^{3}

In eq. (67) of solution set 2, the squared-mass matrix of the gauge bosons was obtained in the case where a complex basis of scalar fields were employed,

\[ M^2_{ab} = g_a g_b \nu^\dagger (T^a T^b + T^b T^a) \nu = -\nu^\dagger (L^a L^b + L^b L^a) \nu, \]

(40)

where the \( T^a \) are hermitian generators. Using eq. (40), the gauge boson squared-mass matrix is easily computed. Employing the explicit matrices given in eqs. (37) and (38), we obtain

\[
M^2 = \begin{pmatrix}
\frac{1}{2} g^2 v^2 & 0 & 0 & 0 \\
0 & \frac{1}{2} g^2 v^2 & 0 & 0 \\
0 & 0 & g^2 v^2 & -gg' v^2 \\
0 & 0 & -gg' v^2 & g'^2 v^2
\end{pmatrix}.
\]

(41)

We see that there are two degenerate gauge bosons, which we identify as \( W^\pm \) [defined in eq. (35)] with \( m_{W^\pm}^2 = \frac{1}{2} g^2 v^2 \).

The diagonalization of the lower \( 2 \times 2 \) block is nearly identical to the computation of the Standard Model (with a complex, hypercharge-one Higgs doublet). Indeed, the only difference is the minus sign that appears in the off-diagonal term. The \( Z \) corresponds to the eigenvector,

\[
\frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix}
0 \\
g \\
-g'
\end{pmatrix}, \quad m_Z^2 = (g^2 + g'^2) v^2,
\]

(43)

and the massless photon corresponds to

\[
\frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix}
0 \\
g' \\
g
\end{pmatrix}, \quad m_\gamma = 0.
\]

(44)

If we define \( \sin \theta_W \equiv g' / \sqrt{g^2 + g'^2} \), then we can identify,

\[ Z_\mu = W^3_\mu \cos \theta_W - B_\mu \sin \theta_W; \]
\[ A_\mu = W^3_\mu \sin \theta_W + B_\mu \cos \theta_W. \]

Note that that SU(2) \( \times \) U(1) has spontaneously broken down to U(1), which we identify as the gauge group of electromagnetism.

---

3The factor of \( 1/\sqrt{2} \) in eq. (39) is conventional. If we write \( \phi^0 = (\phi_R^0 + i\phi_I^0) / \sqrt{2} \), then the kinetic energy term for \( \phi_R^0 \) will be canonically normalized. We can choose the vacuum expectation value to be real without loss of generality, in which case \( \langle \phi_R^0 \rangle = v. \)

4One can repeat this calculation using the generators given in eq. (29). In this case, in order to preserve U(1)_{EM}, one must choose the vacuum expectation value of the form \( \nu = \frac{1}{\sqrt{2}} v (1, -i, 0) \), so that \( Q \nu = 0 \). One can check that with this choice, eq. (40) yields the gauge boson squared-mass matrix obtained in eq. (41).
To analyze the scalar sector of this model, we must specify the Higgs scalar potential. In this case, the most general quartic gauge invariant scalar potential is

\[ V(\Phi) = -m^2\Phi^\dagger\Phi + \lambda_1(\Phi^\dagger\Phi)^2 + \lambda_2 \sum_a (\Phi^\dagger T^a \Phi)(\Phi^\dagger T^a \Phi). \] (45)

Note that this is somewhat more complicated than eq. (33), since there are two independent gauge invariant quartic interactions for the case of a complex hypercharge-two scalar field.\(^5\)

One can minimize eq. (45) and demonstrate that for \(m^2 > 0\), there exists a global minimum corresponding to eq. (39). Here, let us assume that such a global minimum exists. Following the class handout on gauge theories, the Goldstone boson fields can be identified as

\[ G_a = \frac{1}{m_a} \left[ \overline{\Phi} \tilde{\mathcal{L}}_a \nu + (\tilde{\mathcal{L}}_a \nu)^\dagger \overline{\Phi} \right], \] (46)

where \( \overline{\Phi} \equiv \nu + \Phi \) and \( \tilde{\mathcal{L}}_a \equiv O_{ab} \mathcal{L}_b \) is determined by the diagonalization of the gauge boson squared-mass matrix,

\( (OM^2O^T)_{ab} = (\tilde{\mathcal{L}}_a \nu)^\dagger(\tilde{\mathcal{L}}_b \nu) + (\tilde{\mathcal{L}}_b \nu)^\dagger(\tilde{\mathcal{L}}_a \nu) = m_a^2 \delta_{ab}. \)

In light of eqs. (43) and (44), we identify \( \tilde{\mathcal{L}}_1 = \mathcal{L}_1, \tilde{\mathcal{L}}_2 = \mathcal{L}_2, \) and\(^6\)

\[ \tilde{\mathcal{L}}_3 = \frac{1}{\sqrt{g^2 + g'^2}}(g\mathcal{L}_3 - g'\mathcal{L}_4) = \frac{ig}{\cos \theta_W} (\mathcal{T}^3 - Q \sin^2 \theta_W), \]

\[ \tilde{\mathcal{L}}_4 = \frac{1}{\sqrt{g^2 + g'^2}}(g'\mathcal{L}_3 + g\mathcal{L}_4) = ieQ. \]

Note that \( \tilde{\mathcal{L}}_a \nu \neq 0 \) for \( a = 1, 2, 3 \) and \( \tilde{\mathcal{L}}_4 \nu = ieQ\nu = 0, \) which implies that three Goldstone bosons are present and provide masses for the \( W^\pm \) and \( Z. \)

It is straightforward to evaluate,

\[ \tilde{\mathcal{L}}_1 \nu = \frac{1}{2}igv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{\mathcal{L}}_1 \nu = \frac{1}{2}gv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{\mathcal{L}}_3 \nu = -\frac{iv}{\sqrt{2}} \sqrt{g^2 + g'^2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]

Then, eq. (46) yields

\[ G_1 = \sqrt{2} \text{Im } \Phi^+, \quad G_2 = \sqrt{2} \text{Re } \Phi^+, \quad G_3 = -\sqrt{2} \text{Im } \Phi^0, \]

where we have used eqs. (42) and (43) to simplify our results. The physical Higgs states are orthonormal to the \( G_a \) and can be determined by inspection,

\[ H^{++} = \phi^{++}, \quad H^{--} = [\phi^{++}]^\dagger, \quad H = \sqrt{2} \text{Re } \Phi^0 - \nu. \]

\(^5\)In the case of a real triplet of scalar fields, \((\Phi^T \Phi)^2\) is the only quartic invariant. Note that in the case of a real multiplet of scalar fields, the generators \( T^a \) are antisymmetric matrices, and it follows that \( \Phi^T T^a \Phi = \phi_i T^a_0 \phi_j = 0. \)

\(^6\)These results are precisely the same as those obtained in the Standard Model with a complex hypercharge-one Higgs doublet.
Thus, the complex scalar triplet takes the form
\[
\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} H^{++} \\ G_2 + iG_1 \\ v + H - iG_3 \end{pmatrix} \equiv \begin{pmatrix} H^{++} \\ G^+ \\ \frac{1}{\sqrt{2}} [v + H + iG^0] \end{pmatrix},
\]
which defines the Goldstone states of definite charge, \(G^\pm\) and \(G^0\), where \(G^- \equiv (G^+)^\dagger\) and \(G^0 = -G_3\).

In the unitary gauge, we can set \(G^\pm = G^0 = 0\). Then,
\[
\Phi^\dagger \Phi = |H^{++}|^2 + \frac{1}{2}|H|^2 + vH + \frac{1}{2}v^2,
\]
\[
\Phi^\dagger \mathcal{T}^1 \Phi = 0,
\]
\[
\Phi^\dagger \mathcal{T}^2 \Phi = 0,
\]
\[
\Phi^\dagger \mathcal{T}^3 \Phi = |H^{++}|^2 - \frac{1}{2}|H|^2 - vH - \frac{1}{2}v^2.
\]

Hence, eq. (45) yields
\[
V(H^{++}, H) = -m^2(|H^{++}|^2 + \frac{1}{2}|H|^2 + vH + \frac{1}{2}v^2) + \lambda_1 (|H^{++}|^2 + \frac{1}{2}|H|^2 + vH + \frac{1}{2}v^2)^2 + \lambda_2 (|H^{++}|^2 - \frac{1}{2}|H|^2 - vH - \frac{1}{2}v^2)^2
\]
\[
= \text{constant} + v[(\lambda_1 + \lambda_2)v^2 - m^2] + \left[\frac{3}{2}(\lambda_1 + \lambda_2)v^2 - \frac{1}{2}m^2\right]H^2
\]
\[
+ [(\lambda_1 - \lambda_2)v^2 - m^2]|H^{++}|^2 + \text{cubic terms + quartic terms}. \quad (47)
\]
The terms linear in \(H\) must vanish at the minimum of the scalar potential, which implies that
\[
m^2 = (\lambda_1 + \lambda_2)v^2. \quad (48)
\]
I leave it as an exercise for the reader to check that an extremum of the scalar potential given by eq. (45) exists with the vacuum expectation value of \(\Phi\) given by eq. (48). Inserting the result of eq. (48) into eq. (47), we obtain
\[
V(H^{++}, H) = \text{constant} + (\lambda_1 + \lambda_2)v^2H^2 - 2\lambda_2v^2|H^{++}|^2 + \text{cubic terms + quartic terms}.
\]
Comparing the terms quadratic in the Higgs field with \(\frac{1}{2}m_H^2H^2 + m_{H^{++}}^2|H^{++}|^2\), we conclude that
\[
m_H^2 = 2(\lambda_1 + \lambda_2)v^2, \quad m_{H^{++}}^2 = m_{H^{--}}^2 = -2\lambda_2v^2.
\]
Since the squared-masses of the physical Higgs bosons must be positive, we must demand that
\[
\lambda_1 + \lambda_2 > 0, \quad \lambda_2 < 0,
\]
in order to guarantee that the extremum of the scalar potential corresponding to eq. (39) is a local minimum. These conditions also require that \(m^2 > 0\), in light of eq. (48).

\footnote{Since \(H\) is a real field and \(H^{++}\) is a complex field, the correct normalization of the scalar fields, which yields canonically normalized kinetic energy terms, also yields squared-mass terms of the form \(\frac{1}{2}m_H^2H^2 + m_{H^{++}}^2|H^{++}|^2\).}
(c) If both doublet and triplet Higgs fields exist in nature, what does this exercise imply about the parameters of the Higgs Lagrangian?

In the Standard Model with a complex hypercharge-one Higgs doublet with \( \langle \Phi^0 \rangle = v/\sqrt{2} \), one finds
\[
m_W^2 = \frac{1}{4} g^2 v^2, \quad m_Z^2 = \frac{1}{4} (g^2 + g'^2) v^2 \quad \Rightarrow \quad \rho = \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = 1.
\]
This can be compared with the results of parts (a) and (b). In a model with a real hypercharge-zero Higgs triplet with \( \langle \Phi^0 \rangle = v \),
\[
m_W^2 = g^2 v^2, \quad m_Z^2 = 0 \quad \Rightarrow \quad \rho = \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = \infty.
\]
In a model with a complex hypercharge-two Higgs triplet with \( \langle \Phi^0 \rangle = v/\sqrt{2} \),
\[
m_W^2 = \frac{1}{2} g^2 v^2, \quad m_Z^2 = (g^2 + g'^2) v^2 \quad \Rightarrow \quad \rho = \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = \frac{1}{2}.
\]

In a model with multiple Higgs bosons, each vacuum expectation value contributes to the \( W \) and the \( Z \) mass. Since experimental observation confirms that \( \rho \approx 1 \), the conclusion of this analysis is that if Higgs triplet fields also exist, then there are two possibilities. Either, the vacuum expectation values of the triplet fields are much smaller than that of the doublet field, in which case we would expect that the relation \( \rho = 1 \) would be minimally disturbed. A second possibility is that the vacuum expectation values are arranged such that the contribution of the triplet fields to the \( W \) and \( Z \) masses cancels almost exactly. An example of such a model was proposed by H. Georgi and M. Machacek in 1985.\(^8\)

For your amusement, I provide a general formula for \( \rho \) in a model with an arbitrary number of Higgs multiplets of isospin \( T \) and hypercharge \( Y \) (note that a scalar field with weak isospin \( T \) has \( 2T + 1 \) components),\(^9\)
\[
\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = \frac{\sum_{T,Y} [4T(T+1) - Y^2] |v_{T,Y}|^2 c_{T,Y}}{\sum_{T,Y} 2Y^2 |v_{T,Y}|^2}, \quad (49)
\]
where \( \langle \Phi^0(T,Y) \rangle \equiv v_{T,Y} \) defines the vacuum expectation value of each neutral Higgs field of weak isospin \( T \) and hypercharge \( Y \). In addition, we have introduced the notation,
\[
c_{T,Y} = \begin{cases} 
1, & (T,Y) \in \text{complex representation}, \\
\frac{1}{2}, & (T,Y) \in \text{real representation}.
\end{cases}
\]
Here, we employ a rather narrow definition of a real representation, as consisting of a real multiplet of scalar fields with integer weak isospin and \( Y = 0 \).

It is a simple matter to check that eq. (49) reproduces the three cases treated above. Note that Higgs doublet and triplet fields have weak isospins \( T = \frac{1}{2} \) and \( T = 1 \), respectively.


4. (a) Compute the differential cross section at $O(\alpha_s^2)$ for $q\bar{q} \to t\bar{t}$ (where $q \neq t$ is any light quark and $t$ is the top quark), in terms of the center-of-mass energy $\sqrt{s}$ and the squared four-momentum transfer $t$. Integrate your result over $t$ to obtain the total cross section as a function of the squared center-of-mass energy $s$. In your calculation, average over initial colors and spins and sum over final colors and spins. You may assume that the initial quark and anti-quark are massless, but do not neglect the mass of the top-quark.

Only one Feynman diagram contributes,

\[ q \rightarrow t \bar{t} \]

where the direction of the four-momenta are indicated on the diagram (i.e. the incoming four-momenta are $k_1$ and $k_2$ and the outgoing four-momenta are $p_1$ and $p_2$).

The invariant matrix element for $q \bar{g} \rightarrow t \bar{t}$ is

\[
i\mathcal{M} = \bar{v}(p_2)(-ig_{\mu\nu}\delta_{ab})v(k_2)(-ig_{\alpha\beta}\gamma^\mu T^a)\bar{v}(k_1),\tag{50}\]

where $g_s$ is the strong coupling constant and the square of the center-of-mass energy,

\[
s = (k_1 + k_2)^2 = (p_1 + p_2)^2 = 2k_1 \cdot k_2 = 2(M^2 + p_1 \cdot p_2),\tag{51}\]

where we have neglected the masses of $q$ and $\bar{q}$, and we have denoted the top quark mass by $M$.

It is also convenient to introduce the kinematic invariants,

\[
t = (k_1 - p_1)^2 = (p_2 - k_2)^2 = M^2 - 2p_1 \cdot k_1 = M^2 - 2p_2 \cdot k_2,\tag{52}\]

\[
u = (k_1 - p_2)^2 = (p_2 - k_1)^2 = M^2 - 2p_1 \cdot k_2 = M^2 - 2p_2 \cdot k_1.\tag{53}\]

In particular, note the identity,

\[
s + t + \nu = 4M^2 + 2p_1 \cdot (p_2 - k_1 - k_2) = 2M^2,\tag{54}\]

after applying the conservation of momentum, $p_1 + p_2 = k_1 + k_2$, and using $p_1^2 = M^2$.

Squaring the matrix element and performing an average over initial colors and spins and sum over final colors and spins, we first focus on the color sum and average. It is instructive to perform the color sum and average for an SU($N$) gauge theory of strong interactions. (One can set $N = 3$ which is relevant for QCD at the end of the computation.) Consider the $N \times N$ matrix generators $T^a$ in the fundamental representation of SU($N$). The standard normalization for these generators are:

\[
\text{Tr}(T^a T^b) = \frac{1}{2}\delta_{ab}.\tag{55}\]

The following identity is especially useful,

\[
T_{ij}^a T_{k\ell}^a = \frac{1}{2} \left( \delta_{i\ell} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{k\ell} \right),\tag{56}\]

where there is an implicit sum over $a = 1, 2, \ldots, N^2 - 1$. Note that $i, j, k, \ell = 1, 2, \ldots, N$. 

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Focusing only on the color degrees of freedom and suppressing all other factors,

\[ |\mathcal{M}|^2_{\text{ave}} = \frac{1}{N^2} \left[ T_{ij}^a \delta_{ab} T_{kl}^b \right] \left[ T_{ij}^c \delta_{cd} T_{kl}^d \right]^* = \frac{1}{N^2} \left[ T_{ij}^a \delta_{ab} T_{kl}^b \right] \left[ T_{ij}^c \delta_{cd} T_{kl}^d \right] \]

where we have used the fact that the generators are hermitian to write \((T_{ij}^c)^* = T_{ji}^c\). Averaging over the colors of the incoming quark and antiquark yields two factors of 1/\(N\). Returning to the full expression given in eq. (50), the average over initial spins (which yields two factors of 1/2) and the sum over final spins yields

\[ |\mathcal{M}|^2_{\text{ave}} = \frac{1}{4} \frac{N^2 - 1}{4N^2} \frac{g^4}{s^2} \text{Tr} \left[ \gamma_\mu (\not{p}_2 - M) \gamma_\alpha (\not{p}_1 + M) \right] \text{Tr} \left( \gamma^\mu \not{k}_1 \gamma^\alpha \not{k}_2 \right). \]

Working out the traces,

\[ |\mathcal{M}|^2_{\text{ave}} = \frac{N^2 - 1}{4N^2} \frac{g^4}{4s^2} \cdot 16 \left[ p_{2\mu} p_{1\alpha} + p_{2\alpha} p_{1\mu} - g_{\mu\alpha} (p_1 \cdot p_2 + M^2) \right] \left[ k^{1\mu} k^{2\alpha} + k^{1\alpha} k^{2\mu} - g^{\mu\alpha} k_1 \cdot k_2 \right] \]

\[ = \frac{N^2 - 1}{4N^2} \frac{8g^4}{s^2} \left( p_1 \cdot k_1 p_2 \cdot k_2 + p_1 \cdot k_2 p_2 \cdot k_1 + M^2 k_1 \cdot k_2 \right) \]

\[ = \frac{N^2 - 1}{4N^2} \frac{2g^4}{s^2} \left[ (M^2 - t)^2 + (M^2 - u)^2 + 2M^2 s \right], \]

where we have used eqs. (51)–(53) to express the matrix element in terms of the invariants \(s\), \(t\), and \(u\). Defining \(\alpha_s \equiv g^2_s/(4\pi)\) and setting \(N = 3\),

\[ |\mathcal{M}|^2_{\text{ave}} = \frac{64\pi^2 \alpha_s^2}{9s^2} \left[ (M^2 - t)^2 + (M^2 - u)^2 + 2M^2 s \right]. \]

The differential cross section for the scattering of two massless particles is given by

\[ \frac{d\sigma}{dt} = \frac{1}{16\pi s^2} |\mathcal{M}|^2_{\text{ave}}. \]

It then follows that

\[ \frac{d\sigma}{dt} = \frac{4\pi \alpha_s^2}{9s^4} \left[ (M^2 - t)^2 + (M^2 - u)^2 + 2M^2 s \right]. \]

We now integrate over \(t\) to get the total cross section. To obtain the limits of integration, we use eq. (52) to write

\[ t = M^2 - 2p_1 \cdot k_1 = M^2 - 2E_t E_q + 2\vec{p}_1 \cdot \vec{k}_1 \]

The corresponding energies are \(E_t = E_q = \frac{1}{2} \sqrt{s}\), whereas the magnitudes of the corresponding three momenta are

\[ |\vec{k}_1| = \frac{1}{2} \sqrt{s}, \quad |\vec{p}_1| = \frac{1}{2} \sqrt{s} \beta, \]

where

\[ \beta \equiv \sqrt{1 - \frac{4M^2}{s}}. \] (58)
Thus
\[ t = M^2 - \frac{1}{2}s(1 - \beta \cos \theta), \] (59)
where \( \theta \) is the angle between the three-momenta \( \vec{p}_1 \) and \( \vec{k}_1 \). The minimum and maximum of \( t \) correspond to \( \cos \theta = -1 \) and +1, respectively. It is convenient to define
\[ t_1 \equiv t - M^2, \quad u_1 \equiv u - M^2. \]
Then,
\[ \sigma = \int \frac{1}{-\frac{1}{2}s(1+\beta)} \frac{d\sigma}{dt_1} \cdot dt_1 = \frac{4\pi \alpha_s^2}{9s^4} \int \frac{1}{-\frac{1}{2}s(1+\beta)} (t_1^2 + u_1^2 + 2M^2s) \cdot dt_1 \]
In light of eq. (54),
\[ t_1^2 + u_1^2 + 2M^2s = t_1^2 + (s + t_1)^2 + 2M^2s = 2t_1^2 + 2t_1s + s(s + 2M^2). \]
Hence,
\[ \sigma = \frac{4\pi \alpha_s^2}{9s^4} \int \frac{1}{-\frac{1}{2}s(1+\beta)} \left[ 2t_1^2 + 2t_1s + s(s + 2M^2) \right] dt_1 \]
\[ = \frac{4\pi \alpha_s^4}{9s^4} \left\{ -\frac{1}{12}s^3 \left[ (1 - \beta)^3 - (1 + \beta)^3 \right] + \frac{1}{4}s^3 \left[ (1 - \beta)^2 - (1 + \beta)^2 \right] + \beta s^2(s + 2M^2) \right\}. \]
\[ = \frac{4\pi \alpha_s^4}{9s} \beta \left\{ \frac{1}{6}(3 + \beta^2) + \frac{2M^2}{s} \right\} \]
Using eq. (58), we end up with
\[ \sigma(q\bar{q} \to t\bar{t}) = \frac{8\pi \alpha_s^4}{27s} \left( 1 + \frac{2M^2}{s} \right) \left( 1 - \frac{4M^2}{s} \right)^{1/2}. \]

(b) Compute the differential cross section at \( \mathcal{O}(\alpha_s^2) \) for \( gg \to t\bar{t} \), where \( g \) is a gluon, in terms of the squared center-of-mass energy \( \sqrt{s} \) and the squared four-momentum transfer \( t \). Integrate your result over \( t \) to obtain the total cross section as a function of \( s \). In your calculation, average over initial colors and spins and sum over final colors and spins.

Consider the process \( gg \to t\bar{t} \). The incoming gluon momenta are denoted by \( k_1 \) and \( k_2 \), respectively, and the outgoing momenta of the \( t \) and \( \bar{t} \) are denoted [as in part (a) of this problem] by \( p_1 \) and \( p_2 \), respectively. Once again we introduce the three kinematic invariants,
\[ s = (k_1 + k_2)^2 = (p_1 + p_2)^2 = 2k_1 \cdot k_2 = 2(M^2 + p_1 \cdot p_2), \] (60)
\[ t = (k_1 - p_1)^2 = (p_2 - k_2)^2 = M^2 - 2p_1 \cdot k_1 = M^2 - 2p_2 \cdot k_2, \] (61)
\[ u = (k_1 - p_2)^2 = (p_2 - k_1)^2 = M^2 - 2p_1 \cdot k_2 = M^2 - 2p_2 \cdot k_1, \] (62)
where $M$ is the top quark mass. The identity given in eq. (54) still holds, since the gluon is massless. Three Feynman diagrams contribute at tree-level to $gg \rightarrow t\bar{t}$,

and the corresponding QCD Feynman rules, 

\[ i\mathcal{M}_a = \overline{u}(p_1)(-ig_s\gamma_\mu T^a)\frac{i(\gamma_1 - \gamma_{1'} + M)}{t - M^2}( -ig_s\gamma_\nu T^b)v(p_2)\epsilon^\mu_a(k_1, \lambda_1)\epsilon^\nu_b(k_2, \lambda_2) , \]

where $\epsilon^\mu_a(k, \lambda)$ is the polarization vector for a gluon of color $a$, helicity $\lambda$ and four-momentum $k$. Slightly simplifying the above expression yields,

\[ \mathcal{M}_a = \frac{g_s^2}{M^2 - t} \epsilon^\mu_a(k_1, \lambda_1)\epsilon^\nu_b(k_2, \lambda_2)T^aT^b \overline{u}(p_1)\gamma_\mu(\gamma_1 - \gamma_{1'} + M)\gamma_\nu v(p_2) . \] (63)

Next, $\mathcal{M}_b$ is obtained from $\mathcal{M}_a$ by exchanging the two initial gluons. Hence,

\[ \mathcal{M}_b = \frac{g_s^2}{M^2 - u} \epsilon^\mu_a(k_1, \lambda_1)\epsilon^\nu_b(k_2, \lambda_2)T^bT^a \overline{u}(p_1)\gamma_\nu(\gamma_1 - \gamma_{1'} + M)\gamma_\mu v(p_2) . \] (64)

Finally, $\mathcal{M}_c$ involves the three-gluon vertex. After some minor simplification,

\[ \mathcal{M}_c = \frac{g_s^2}{s} \epsilon^\mu_a(k_1, \lambda_1)\epsilon^\nu_b(k_2, \lambda_2)(if_{abc}T^c)\overline{u}(p_1)\gamma_5^\nu v(p_2)C_{\mu\nu\beta} , \] (65)

where

\[ C_{\mu\nu\beta} = g_{\mu\nu}(k_2 - k_1)_{\beta} + g_{\beta\mu}(2k_1 + k_2)_{\nu} + g_{\nu\beta}(2k_2 - k_1)_{\mu} . \] (66)

Since these are complex calculations, it is always a good idea to check your results by some independent method. Here, I shall check gauge invariance. In class, I showed that given an invariant matrix element involving two external gluons, which is of the form

\[ \mathcal{M}_{\mu\nu}\epsilon^\mu(k_1, \lambda_1)\epsilon^\nu(k_2, \lambda_2) , \]

where the color labels have been suppressed, then one must obtain zero if either $\epsilon^\mu(k_1, \lambda_1)$ is replaced by $k_1^\mu$ or if $\epsilon^\nu(k_2, \lambda_2)$ is replaced by $k_2^\nu$. For example,

\[ k_1^\mu\mathcal{M}_{\mu\nu}\epsilon^\nu(k_2, \lambda_2) = 0 . \] (67)

Let us now verify eq. (67). We consider the effect of replacing $\epsilon^\mu(k_1, \lambda_1)$ with $k_1^\mu$ in $\mathcal{M}_a$, $\mathcal{M}_b$ and $\mathcal{M}_c$, respectively. In the case of $\mathcal{M}_a$, we must evaluate

\[ \overline{u}(p_1)k_1(\gamma_1 - \gamma_{1'} + M)\gamma_\nu v(p_2) = \overline{u}(p_1)k_1(\gamma_1 + M)\gamma_\nu v(p_2) \]

\[ = 2k_1 \cdot p_1\overline{u}(p_1)\gamma_\nu v(p_2) - \overline{u}(p_1)(\gamma_1 - M)k_1 v(p_2) \]

\[ = (M^2 - t)u(p_1)\gamma_\nu v(p_2) . \] (68)
In obtaining eq. (68), we first used \( k_1 k_1 = k_1^2 = 0 \) (since the gluon is massless). Next, we used the anticommutation relations of the gamma matrices, \( \{ \gamma_\mu, \gamma_\nu \} = 2g_{\mu\nu} \), to write

\[
k_1 \bar{\psi}_1 = 2k_1 \cdot p_1 - \bar{\psi}_1 k_1 = M^2 - t - \bar{\psi}_1 k_1.
\]

Finally, we made use of the Dirac equation to obtain \( \bar{\pi}(p_1)(\bar{\psi}_1 - M) = 0 \).

A similar computation arises when considering \( M_b \).

\[
\bar{\pi}(p_1)\gamma_\mu(\bar{\psi}_1 - k_2 + M)k_1 v(p_2) = \bar{\pi}(p_1)\gamma_\nu(\bar{\psi}_1 - k_2 + M)k_1 v(p_2)
\]

\[
= \bar{\pi}(p_1)\gamma_\nu k_1(\bar{\psi}_2 + M)v(p_2) - 2k_1 \cdot p_2 \bar{\pi}(p_1)\gamma_\nu v(p_2)
\]

\[
= (u - M^2)u(p_1)\gamma_\nu v(p_2).
\]

(69)

Note that in this calculation, it was useful to use momentum conservation, \( p_1 - k_2 = k_1 - p_2 \). At the penultimate step, we invoked the Dirac equation, \( (\bar{\psi}_2 + M)v(p_2) = 0 \).

When eq. (68) is employed in \( M_a \) and eq. (69) is employed in \( M_b \), we note that the denominators are conveniently canceled. Thus, we conclude that

\[
(M_a + M_b) \bigg|_{\epsilon_1 \rightarrow k_1} = g_s^2 \epsilon'_b(k_2, \lambda_2)(T^a T^b - T^b T^a)\bar{\pi}(p_1)\gamma_\nu v(p_2).
\]

(70)

Finally, we examine \( M_c \) with \( \epsilon^\nu(k_1, \lambda_1) \) replaced by \( k_1^\mu \). Then, using eq. (66),

\[
k_1^\mu C_\mu\nu\beta = k_{1\nu}(k_2 - k_1)_\beta + k_{1\beta}(2k_1 + k_2)_\nu - 2g_{\nu\beta}k_1 \cdot k_2,
\]

after noting that \( k_1^2 = 0 \). In what follows, we shall denote \( \epsilon'_2 \equiv \epsilon^\nu(k_2, \lambda_2) \). Then,

\[
k_1^\mu \epsilon'_2 C_\mu\nu\beta = k_1 \cdot \epsilon_2(k_1 + k_2)_\beta - 2k_1 \cdot k_2 \epsilon_2\beta,
\]

(71)

after using \( k_2 \cdot \epsilon_2 = 0 \). The factor of \( (k_1 + k_2)_\beta \) then ends up in the expression,

\[
\bar{\pi}(p_1)(k_1 + k_2)v(p_2) = \bar{\pi}(p_1)(\bar{\psi}_1 + \bar{\psi}_2)v(p_2) = 0,
\]

after using four-momentum conservation and the Dirac equation. After noting that \( s = 2k_1 \cdot k_2 \), which conveniently cancels the denominator factor in \( M_c \), we are then left with

\[
M_c \bigg|_{\epsilon_1 \rightarrow k_1} = g_s^2 \epsilon'_b(k_2, \lambda_2)(-if^{abc}T^c)\bar{\pi}(p_1)\gamma_\nu v(p_2).
\]

Combining this with eq. (70),

\[
(M_a + M_b + M_c) \bigg|_{\epsilon_1 \rightarrow k_1} = g_s^2 \epsilon'_b(k_2, \lambda_2)(T^a T^b - T^b T^a - if^{abc}T^c)\bar{\pi}(p_1)\gamma_\nu v(p_2) = 0,
\]

after employing the commutation relations of the generators, \([T^a, T^b] = if^{abc}T^c\). Thus, the gauge invariance check is successful.
It should be noted that an essential aspect of the gauge invariance check was keeping the factor of \( \epsilon_b^\nu(k_2, \lambda_2) \), since eq. (71) was obtained only after using \( k_2 \cdot \epsilon_b^\nu(k_2, \lambda_2) = 0 \). This implies that although eq. (67) is satisfied, the following relations (which do hold in QED),

\[
k_1^\mu \mathcal{M}_{\mu \nu} = 0, \quad k_2^\nu \mathcal{M}_{\mu \nu} = 0
\]

do not hold in QCD. In for example, one can easily modify the above computation to obtain

\[
k_1^\mu \mathcal{M}_{\mu \nu} = \frac{g^2}{s} i f_{abc} T^c k_{2 \mu} \bar{p}(p_1) \gamma^\nu v(p_2).
\]

Of course, eq. (67) does hold as expected, since if one multiplies the above equation by \( \epsilon_b^\nu(k_2, \lambda_2) \) and employs \( k_2 \cdot \epsilon_b^\nu(k_2, \lambda_2) = 0 \), we do get zero as expected.

Looking ahead to the calculation of the spin and color summed and averaged squared matrix element, the gluon spin sum is given by

\[
\sum_\lambda \epsilon_a^\mu(k, \lambda) \epsilon_b^\nu(k, \lambda)^* = \delta_{ab} \left(-g_{\mu \nu} + \frac{k_{\mu} \overline{k}_{\nu} + k_{\nu} \overline{k}_{\mu}}{k \cdot \overline{k}}\right),
\]

where \( k^\mu = (k_0; \overline{k}) \) and \( \overline{k}^\mu = (k_0; \overline{\lambda}) \). The consequence of eqs. (67) and (73) is that in a QCD scattering process, one is free to drop the \( (k_\mu \overline{k}_\nu + k_\nu \overline{k}_\mu) / k \cdot \overline{k} \) term in the spin sum corresponding to one (and only one) external gluon line. Thus, in the present calculation of \( gg \rightarrow t\overline{t} \), one would have to employ the full expression given by eq. (74) for one of the two gluon spin sums. In contrast, if eq. (72) were to hold (as it does in QED), then one would be justified in making the replacement,

\[
\sum_\lambda \epsilon_a^\mu(k, \lambda) \epsilon_b^\nu(k, \lambda)^* \longrightarrow -\delta_{ab} g_{\mu \nu}.
\]

Let us now return to eqs. (65) and (66) and note that we can slightly simplify this result by writing

\[
\mathcal{M}_c = \frac{g^2}{s} \epsilon_a^\mu(k_1, \lambda_1) \epsilon_b^\nu(k_2, \lambda_2) (i f_{abc} T^c) \bar{p}(p_1) \gamma^\beta v(p_2) \bar{C}_{\mu \nu \beta},
\]

where

\[
\bar{C}_{\mu \nu \beta} = g_{\mu \nu}(k_2 - k_1)_\beta + 2g_{\beta \mu} k_{1 \nu} - 2g_{\beta \nu} k_{2 \mu}.
\]

Here we have eliminated two of the terms in \( \mathcal{C}_{\mu \nu \beta} \) by invoking the properties of the gluon polarization vectors, \( k_1 \cdot \epsilon_a^\mu(k_1, \lambda_1) = k_2 \cdot \epsilon_b^\nu(k_2, \lambda_2) = 0 \). Using the version of \( \mathcal{M}_c \) given in eq. (76) in the total amplitude \( \mathcal{M}_{\mu \nu} \epsilon_a^\mu(k_1, \lambda_1) \epsilon_b^\nu(k_2, \lambda_2) \equiv \mathcal{M}_a + \mathcal{M}_b + \mathcal{M}_c \), it is a simple matter to verify that eq. (67) holds! Note that we have not actually changed the value of \( \mathcal{M}_c \), but we have changed the value of \( \mathcal{M}_{\mu \nu} \). Thus, by employing the form of \( \mathcal{M}_c \) as given by eq. (76), we can now safely use eq. (75) in the gluon polarization sums for both gluons!

We are now ready to perform the calculation of the cross section for \( gg \rightarrow t\overline{t} \). In computing the absolute square of the invariant amplitude, there will be six terms,

\[
|\mathcal{M}|^2 = |\mathcal{M}_a|^2 + |\mathcal{M}_b|^2 + |\mathcal{M}_c|^2 + 2\text{Re}(\mathcal{M}_a \mathcal{M}_b^*) + 2\text{Re}(\mathcal{M}_a \mathcal{M}_c^*) + 2\text{Re}(\mathcal{M}_b \mathcal{M}_c^*).
\]

We first focus on the color factors, suppressing all other factors. As in part (a), it is instructive to perform all color computations using an SU(3) color group, and only set \( N = 3 \) at the end of the calculation. In addition to eqs. (55) and (56), the following results will also prove useful.
First, by setting \( j = k \) in eq. (56) and summing over \( N \) colors, we immediately get
\[
(T_a T_a)_{i\ell} = \frac{N^2 - 1}{2N} \delta_{i\ell}.
\]
(79)
This is the quadratic Casimir operator in the fundamental representation of SU(\( N \)). A related object is the Casimir operator in the adjoint representation of SU(\( N \)), which is given by
\[
f_{abc} f_{abd} = N \delta_{cd},
\]
(80)
where there is an implicit sum over the two pairs of repeated indices. Next, we note the relation,
\[
T^a T^b = \frac{1}{2} \left[ \frac{1}{N} \delta_{ab} 1 + (d_{abc} + i f_{abc}) T^c \right],
\]
(81)
where \( d_{abc} \equiv 2 \text{Tr}[(T^a T^b + T^b T^a)T^c] \) is completely symmetric under the interchange of any pair of its indices. We will not need an explicit expression for \( d_{abc} \) in what follows; only its symmetry properties are relevant. Taking the trace of eq. (81) and using \( \text{Tr} 1 = N \) and \( \text{Tr} T^a = 0 \), we recover eq. (55). A second consequence of eq. (81) is
\[
\text{Tr}(T^a T^b T^c) = \frac{1}{4}(d_{abc} + i f_{abc}).
\]
(82)

One other trace relation will be useful,
\[
\text{Tr}(T^a T^b T^a T^c) = \frac{1}{4N} \delta_{bc}.
\]
(83)
Using the SU(\( N \)) commutation relations, \([T^a, T^b] = i f_{abc} T^c\), one can show that eq. (83) is a consequence of eqs. (79), (80) and (82) [HINT: consider \( \text{Tr} (T^a [T^b, T^a] T^c) \) and note that \( d_{acc} f_{aeb} = 0 \) due to the fact that \( d_{acc} \) is a symmetric tensor and \( f_{aeb} \) is an antisymmetric tensor].

The color factors for each of the six terms in eq. (78) can now be obtained. We sum over the final state colors of the \( t \) and \( \bar{t} \) and average over the initial gluon colors. Since the gluon field transforms under the adjoint representation, it possesses \( N^2 - 1 \) colors in an SU(\( N \)) gauge theory. Thus, we must divide by \((N^2 - 1)^2\) to perform the color averaging of the initial gluons.

\[
|M_a|^2 : \quad \frac{(T^a T^b)_{ij} (T^b T^a)_{ji}}{(N^2 - 1)^2} = \frac{\text{Tr}(T^a T^b T^b T^a)}{(N^2 - 1)^2} = \frac{1}{(N^2 - 1)^2} \left( \frac{N^2 - 1}{2N} \right)^2 N = \frac{1}{4N},
\]
\[
|M_b|^2 : \quad \frac{(T^b T^a)_{ij} (T^a T^b)_{ji}}{(N^2 - 1)^2} = \frac{\text{Tr}(T^b T^a T^a T^b)}{(N^2 - 1)^2} = \frac{1}{4N},
\]
\[
|M_c|^2 : \quad \frac{(i f_{abc} T^i_{ji}) (-i f_{abc} T^i_{ji})}{(N^2 - 1)^2} = \frac{\text{Tr}(T^c T^c)}{(N^2 - 1)^2} = \frac{N \delta_{cc} \delta_{ee}}{2(N^2 - 1)^2} = \frac{N}{2(N^2 - 1)},
\]
\[
2 \text{Re}(M_a M^*_b) : \quad \frac{(T^a T^b)_{ij} (T^a T^b)_{ji}}{(N^2 - 1)^2} = \frac{\text{Tr}(T^a T^b T^a T^b)}{(N^2 - 1)^2} = -\frac{1}{4N(N^2 - 1)^2} \delta_{bb} = -\frac{1}{4N(N^2 - 1)},
\]
\[
2 \text{Re}(M_a M^*_c) : \quad \frac{(T^a T^b)_{ij} (-i f_{abc} T^c)_{ji}}{(N^2 - 1)^2} = -i f_{abc} \frac{\text{Tr}(T^a T^b T^c)}{(N^2 - 1)^2} = \frac{f_{abe} f_{ace}}{4(N^2 - 1)^2} = \frac{N}{4(N^2 - 1)},
\]
\[
2 \text{Re}(M_b M^*_c) : \quad \frac{(T^b T^a)_{ij} (-i f_{abc} T^c)_{ji}}{(N^2 - 1)^2} = -i f_{abc} \frac{\text{Tr}(T^b T^a T^c)}{(N^2 - 1)^2} = \frac{f_{abe} f_{bae}}{4(N^2 - 1)^2} = -\frac{N}{4(N^2 - 1)}.
\]
Note that the factors of 2 appearing in the interference terms are not incorporated in the color factors obtained above. These factors of 2 will be included separately below.

We now proceed to write out the spin and color averaged matrix element, considering separately the six terms in eq. (76). The average over gluon helicities is performed using eq. (75) [although the $\delta_{ab}$ factor has already been employed in obtaining the color factors above]. The averages over the $t$ and $\bar{t}$ spins are computed in the usual fashion, resulting in traces of gamma matrices. In the average over initial gluon helicities, we include a factor of $1/2$ for each gluon since we are averaging over two polarization states. Thus, we obtain

$$|M_a|^2_{\text{ave}} = \frac{g_s^4}{4(M^2 - t)^2} \left( \frac{1}{4N} \right) \text{Tr} \left[ \gamma_\mu (\not{p}_1 - k_1' + M) \gamma_\nu (\not{p}_2 - M) \gamma^\nu (\not{p}_1 - k_1' + M) \gamma^\mu (\not{p}_1 + M) \right],$$

$$|M_b|^2_{\text{ave}} = \frac{g_s^4}{4(M^2 - u)^2} \left( \frac{1}{4N} \right) \text{Tr} \left[ \gamma_\nu (\not{p}_1 - k_2' + M) \gamma_\mu (\not{p}_2 - M) \gamma^\mu (\not{p}_1 - k_2' + M) \gamma^\nu (\not{p}_1 + M) \right],$$

$$2\text{Re}(M_a M_b^*)_{\text{ave}} = \frac{2g_s^4}{4(M^2 - t)(M^2 - u)} \left( \frac{1}{4N} - \frac{N}{4N^2 - 1} \right) \text{Tr} \left[ \gamma_\mu (\not{p}_1 - k_1' + M) \gamma_\nu (\not{p}_2 - M) \gamma^\mu (\not{p}_1 - k_1' + M) \gamma^\nu (\not{p}_1 + M) \right],$$

$$2\text{Re}(M_a M_c^*)_{\text{ave}} = \frac{2g_s^4}{4s(M^2 - t)} \left( \frac{N}{4N^2 - 1} \right) \tilde{C}_{\mu\nu} \text{Tr} \left[ \gamma_\mu (\not{p}_2 - M) \gamma^\nu (\not{p}_1 + M) \right],$$

$$2\text{Re}(M_b M_c^*)_{\text{ave}} = \frac{2g_s^4}{4s(M^2 - u)} \left( \frac{-N}{4N^2 - 1} \right) \tilde{C}_{\mu\nu} \text{Tr} \left[ \gamma_\mu (\not{p}_2 - M) \gamma^\nu (\not{p}_1 - k_2' + M) \gamma^\nu (\not{p}_1 + M) \right],$$

where $\tilde{C}_{\mu\nu\beta}$ is defined in eq. (77). The factor of 2 in the interference terms appear explicitly in the three initial numerator factors above. The color factors computed earlier are given inside parentheses in each expression. Note that for the case of $2\text{Re}(M_a M_b^*)_{\text{ave}}$, we have written

$$\frac{1}{4N(N^2 - 1)} = \frac{1}{4N} - \frac{N}{4N^2 - 1}.$$

The reason for doing this is that one can now see the existence of two independent color factors, $1/(4N)$ and $\frac{1}{2}N/(N^2 - 1)$. Indeed, $|M|^2_{\text{ave}}$ consists of two independent terms—one proportional to the first color factor and the other proportional to the second color factor. Although it may not be obvious, one can show that these two independent terms are separately gauge invariant. Indeed, this allows one to take the QED limit of the above result, simply by omitting all terms proportional to $\frac{1}{2}N/(N^2 - 1)$, since the latter involves the three gluon vertex, which is absent in QED. But, we also learn that part of $2\text{Re}(M_a M_b^*)_{\text{ave}}$ must also be omitted in the QED limit—this is the piece that is sensitive to the non-commuting nature of the generators $T^a$.

Identifying these two separate pieces is also useful computationally, as one might expect some significant cancellations within the two independent gauge invariant sets. It should be emphasized that this separation is only possible if one computes SU($N$) color factors rather
than SU(3) color factors, since the \( N \) dependence allows one to identify the two independent pieces of \( |\mathcal{M}|^2_{\text{ave}} \). Of course, at the end of the day, we will set \( N = 3 \) to get our final result.

The rest of the computation involves the calculation of traces. This is straightforward but tedious. After reducing the traces to products of dot products, one makes use of the kinematics \( \text{cf. eqs. (60)–(62)} \) to express the result in terms of the kinematic invariants. The end result is summarized below.

\[
|\mathcal{M}_a|_{\text{ave}}^2 = \frac{2g_s^4}{(M^2 - t)^2} \left( \frac{1}{4N} \right) \left[ (M^2 - t)(M^2 - u) - 2M^2(M^2 + t) \right],
\]

\[
|\mathcal{M}_b|_{\text{ave}}^2 = \frac{2g_s^4}{(M^2 - t)^2} \left( \frac{1}{4N} \right) \left[ (M^2 - t)(M^2 - u) - 2M^2(M^2 + u) \right],
\]

\[
2\text{Re}(\mathcal{M}_a\mathcal{M}_b^*)_{\text{ave}} = \frac{4g_s^4}{(M^2 - t)(M^2 - u)} \left( \frac{1}{4N} - \frac{N}{4(N^2 - 1)} \right) M^2(s - 4M^2),
\]

\[
|\mathcal{M}_c|_{\text{ave}}^2 = \frac{4g_s^4}{s^2} \left( \frac{N}{2(N^2 - 1)} \right) (M^2 - t)(M^2 - u),
\]

\[
2\text{Re}(\mathcal{M}_a\mathcal{M}_c^*)_{\text{ave}} = \frac{-4g_s^4}{s(M^2 - t)} \left( \frac{N}{4(N^2 - 1)} \right) \left[ (M^2 - t)(M^2 - u) + M^2(u - t) \right],
\]

\[
2\text{Re}(\mathcal{M}_b\mathcal{M}_c^*)_{\text{ave}} = \frac{4g_s^4}{s(M^2 - u)} \left( \frac{-N}{4(N^2 - 1)} \right) \left[ (M^2 - t)(M^2 - u) + M^2(t - u) \right].
\]

The invariant cross section is given by

\[
\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} |\mathcal{M}|_{\text{ave}}^2
\]

Adding up the six expressions above, putting \( N = 3 \) and defining \( \alpha_s^2 \equiv g_s^2/(4\pi) \), we end up with

\[
\frac{d\sigma}{dt} = \frac{2\pi\alpha_s^2}{s^2} \left\{ \left( \frac{1}{12} \right) \frac{(M^2 - t)(M^2 - u) - 2M^2(M^2 + t)}{(M^2 - t)^2} + \left( \frac{1}{12} \right) \frac{(M^2 - t)(M^2 - u) - 2M^2(M^2 + u)}{(M^2 - u)^2} 
\]

\[
+ \left( -\frac{1}{96} \right) \frac{2M^2(s - 4M^2)}{(M^2 - t)(M^2 - u)} + \left( \frac{3}{16} \right) \frac{2(M^2 - t)(M^2 - u)}{s^2} 
\]

\[
- \left( \frac{3}{32} \right) \frac{2[(M^2 - t)(M^2 - u) + M^2(u - t)]}{s(M^2 - t)} + \left( -\frac{3}{32} \right) \frac{2[(M^2 - t)(M^2 - u) + M^2(t - u)]}{s(M^2 - u)} \right\},
\]

where the color factors for \( N = 3 \) are specified inside the parentheses above.

In light of eq. (54), \( s + t + u = 2M^2 \), so that at fixed center-of-mass energy \( \sqrt{s} \), the variable \( u = 2M^2 - s - t \) is also a function of \( t \). Note that \( t \) is related to the scattering angle as indicated in eq. (59), which we repeat below:

\[
t = M^2 - \frac{1}{2}s(1 - \beta \cos \theta),
\]

24
where
\[ \beta \equiv \left( 1 - \frac{4M^2}{s} \right)^{1/2}. \] (84)

The kinematical limits on \( t \) are easily obtained by imposing the condition, \( |\cos \theta| \leq 1 \). It follows that
\[ M^2 - \frac{1}{2}s(1 + \beta) \leq t \leq M^2 - \frac{1}{2}s(1 - \beta). \] (85)

The final step is to integrate over \( t \) to obtain the total cross section. As in part (a), it is useful to define \( t_1 = t - M^2 \) and \( u_1 = u - M^2 \). Then,
\[
\frac{d\sigma}{dt_1} = \frac{2\pi\alpha_s^2}{s^2} \left\{ \frac{t_1u_1 - 2M^2(M^2 + t_1)}{12t_1^2} + \frac{t_1u_1 - 2M^2(M^2 + u_1)}{12u_1^2} - \frac{M^2(s - 4M^2)}{48t_1u_1} \right. \\
+ \left. \frac{3t_1u_1}{8s^2} + \frac{3[t_1u_1 + M^2(u_1 - t_1)]}{16st_1} + \frac{3[t_1u_1 + M^2(t_1 - u_1)]}{16su_1} \right\}, \] (86)

Using eq. (54), it follows that
\[ s + t_1 + u_1 = 0. \]

Due to the symmetry under the interchange of \( t_1 \) and \( u_1 \), it follows that \( d\sigma/dt_1 = d\sigma/du_1 \). This means that in computing the total cross section, the integral of the first two terms in eq. (86) yields the same result, as does the integral of the last two terms in eq. (86).

In addition, eq. (85) implies that
\[ -\frac{1}{2}s(1 + \beta) \leq t_1 \leq \frac{1}{2}s(1 - \beta), \]
which provides the limits for the integral over \( t_1 \). Hence, utilizing the symmetry noted above,
\[
\sigma = \frac{2\pi\alpha_s^2}{s^2} \int_{-\frac{1}{2}s(1-\beta)}^{\frac{1}{2}s(1+\beta)} dt_1 \left\{ -\frac{t_1(s + t_1) + 2M^2(2M^2 + t_1)}{6t_1^2} + \frac{M^2(s - 4M^2)}{48t_1(s + t_1)} - \frac{3t_1(s + t_1)}{8s^2} \\
- \frac{3[t_1(s + t_1) + M^2(s + 2t_1)]}{8st_1} \right\}. 
\]

The integration is straightforward. It is simple enough to perform by hand, although a computer algebra system such as Mathematica is also suitable. The end result is
\[
\sigma(gg \to t\bar{t}) = \frac{\pi\alpha_s^2}{3s} \left\{ \left( 1 + \frac{4M^2}{s} + \frac{M^4}{s^2} \right) \ln \left( \frac{1 + \sqrt{1 - \frac{4M^2}{s}}}{1 - \sqrt{1 - \frac{4M^2}{s}}} \right) - \frac{1}{4} \left( 1 - \frac{4M^2}{s} \right)^{1/2} \left( 7 + 31M^2 \right) \right\}, 
\]
after using eq. (84) to express the final result as a function of \( M^2 \) and \( s \).