1. In class, I defined the matrix-valued covariant derivative operator in the adjoint representation, $\mathscr{D}_\mu$, by

$$\mathscr{D}_\mu V \equiv (D_\mu V)_a T^a = \partial_\mu V + ig[A_\mu \, , \, V] \, , \tag{1}$$

where $V \equiv V^a T^a$ is a matrix-valued adjoint field and $(D_\mu)_{ab} \equiv \delta_{ab}\partial_\mu + gf_{cab}A^c_\mu$ is the covariant derivative acting on a field in the adjoint representation. The commutation relations satisfied by the generators of the Lie group G are given by $[T_a \, , \, T_b] = if_{abc}T_c$, and the indices a, b and c take on d_G possible values, where d_G is the dimension of G.

(a) Prove that for any pair of matrix-valued adjoint fields V and W,

$$[\mathscr{D}_\mu \, , \, V]\, W = (\mathscr{D}_\mu V)W \, ,$$

where $[\quad , \quad]$ is the usual matrix commutator. This means that $\mathscr{D}_\mu V = [\mathscr{D}_\mu \, , \, V]$ holds as an operator equation.

By definition of the commutator, for adjoint fields V and W,

$$\begin{aligned}
[\mathscr{D}_\mu \, , \, V]\, W \equiv (\mathscr{D}_\mu V - V\mathscr{D}_\mu)W &= \mathscr{D}_\mu(VW) - V\mathscr{D}_\mu W \\
&= \partial_\mu(VW) + ig[A_\mu \, , \, VW] - V\partial_\mu W - igV[A_\mu \, , \, W] \\
&= (\partial_\mu V)W + ig[A_\mu \, , \, V]W = \big\{\partial_\mu V + ig[A_\mu \, , \, V]\big\}W \\
&= (\mathscr{D}_\mu V)W \, .
\end{aligned}$$

This is true for an arbitrary adjoint field W. Hence,

$$\mathscr{D}_\mu V = [\mathscr{D}_\mu \, , \, V] \, , \tag{2}$$

holds as an operator identity.

(b) Prove that for any matrix-valued adjoint field V,

$$[\mathscr{D}_\mu \, , \, \mathscr{D}_\nu]V = ig[F_{\mu\nu} \, , \, V] \, ,$$

where $F_{\mu\nu} \equiv F^a_{\mu\nu}T^a$ is the matrix-value field strength tensor of the non-abelian gauge theory.

Using the definition of $\mathscr{D}_\mu$ given in eq. (1),

$$\begin{aligned}
[\mathscr{D}_\mu \, , \, \mathscr{D}_\nu]V &= \mathscr{D}_\mu\big(\partial_\nu V + ig[A_\nu \, , \, V]\big) - \mathscr{D}_\nu\big(\partial_\mu V + ig[A_\mu \, , \, V]\big) \\
&= \partial_\mu\big(\partial_\nu V + ig[A_\nu \, , \, V]\big) + ig\big[A_\mu \, , \, \partial_\nu V + ig[A_\nu \, , \, V]\big] \\
&\qquad -\partial_\nu\big(\partial_\mu V + ig[A_\mu \, , \, V]\big) - ig\big[A_\nu \, , \, \partial_\mu V + ig[A_\mu \, , \, V]\big] \\
&= ig\bigg\{[\partial_\mu A_\nu - \partial_\nu A_\mu \, , \, V] + ig\big[A_\mu \, , \, [A_\nu \, , \, V]\big] - ig\big[A_\nu \, , \, [A_\mu \, , \, V]\big]\bigg\} \, .
\end{aligned}$$

Using the Jacobi identity,

$$\big[A_\mu\,,\,[A_\nu\,,\,V]\big]+\big[V\,,\,[A_\mu\,,\,A_\nu]\big]+\big[A_\nu\,,\,[V\,,\,A_\mu]\big]=0\,.$$

and the antisymmetry of the commutator, e.g. $[V\,,\,A_\mu]=-[A_\mu\,,\,V]$, it follows that,

$$[\mathscr{D}_\mu\,,\,\mathscr{D}_\nu]V=ig\big[\partial_\mu A_\nu-\partial_\nu A_\mu+ig[A_\mu\,,\,A_\nu]\,,\,V\big]\,.$$

using the definition of the matrix field-strength tensor,

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+ig[A_\mu\,,\,A_\nu]\,,$$

we end up with

$$[\mathscr{D}_\mu\,,\,\mathscr{D}_\nu]V=ig\big[F_{\mu\nu}\,,\,V\big]\,. \tag{3}$$

An alternative derivation:

Note that by using the definition of the commutator, for adjoint fields V and W,

$$\begin{aligned}\big([\mathscr{D}_\mu\,,\,\mathscr{D}_\nu]V\big)W&=(\mathscr{D}_\mu\mathscr{D}_\nu-\mathscr{D}_\nu\mathscr{D}_\mu)W\\&=\big(\mathscr{D}_\mu[\mathscr{D}_\nu\,,\,V]-\mathscr{D}_\nu[\mathscr{D}_\mu\,,\,V]\big)W\\&=\big(\big[\mathscr{D}_\mu\,,\,[\mathscr{D}_\nu\,,\,V]\big]-\big[\mathscr{D}_\nu\,,\,[\mathscr{D}_\mu\,,\,V]\big]\big)W\,,\end{aligned} \tag{4}$$

after using eq. (2). The Jacobi identity implies that the following operator identity holds:

$$\big[\mathscr{D}_\mu\,,\,[\mathscr{D}_\nu\,,\,V]\big]+\big[V\,,\,[\mathscr{D}_\mu\,,\,\mathscr{D}_\nu\big]+\big[\mathscr{D}_\nu\,,\,[V\,,\,\mathscr{D}_\mu\big]=0\,.$$

Thus, using the Jacobi identity and the antisymmetry property of the commutator, eq. (4) yields

$$\big([\mathscr{D}_\mu\,,\,\mathscr{D}_\nu]V\big)W=\big[[\mathscr{D}_\mu\,,\,\mathscr{D}_\nu]\,,\,V\big]W\,.$$

This is true for an arbitrary adjoint field W. Hence,

$$[\mathscr{D}_\mu\,,\,\mathscr{D}_\nu]V=\big[[\mathscr{D}_\mu\,,\,\mathscr{D}_\nu]\,,\,V\big]\,, \tag{5}$$

holds as an operator identity.

The matrix field strength tensor was initially defined in class via the operator identity

$$[D_\mu\,,\,D_\nu]=igF_{\mu\nu}\,, \tag{6}$$

where D_μ is the covariant derivative defined by its action on a field that transforms according to an arbitrary representation of the Lie group G. In particular, eq. (6) must hold in the adjoint representation, which implies that

$$[\mathscr{D}_\mu\,,\,\mathscr{D}_\nu]=igF_{\mu\nu}$$

holds as an operator identity. Inserting this result into eq. (5) yields

$$[\mathscr{D}_\mu\,,\,\mathscr{D}_\nu]V=ig\big[F_{\mu\nu}\,,\,V\big]\,,$$

which again confirms eq. (3).

(c) Starting from the non-abelian Maxwell equation,

$$\mathscr{D}_\mu \, F^{\mu\nu} = j^\nu \,, \tag{7}$$

prove that the current j^ν is covariantly conserved. That is,

$$\mathscr{D}_\mu j^\mu = 0 \,.$$

Applying $\mathscr{D}_\nu$ to both sides of eq. (7),

$$\mathscr{D}_\nu j^\nu = \mathscr{D}_\nu \mathscr{D}_\mu F^{\mu\nu} = \tfrac{1}{2}\mathscr{D}_\nu \mathscr{D}_\mu (F^{\mu\nu} - F^{\nu\mu}) \,, \tag{8}$$

since $F^{\mu\nu} = -F^{\nu\mu}$. After relabeling $\mu \leftrightarrow \nu$ in the last term, we can rewrite eq. (8) as

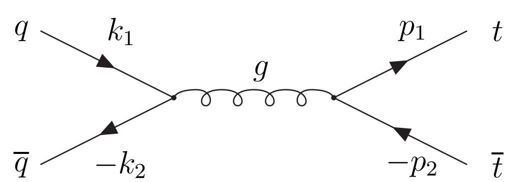
$$\mathscr{D}_\nu j^\nu = \tfrac{1}{2}(\mathscr{D}_\nu \mathscr{D}_\mu - \mathscr{D}_\mu \mathscr{D}_\nu) F^{\mu\nu} = -\tfrac{1}{2}\big[\mathscr{D}_\mu \,,\, \mathscr{D}_\nu\big] F^{\mu\nu} \,.$$

Since eq. (3) applies to any matrix-valued adjoint field V, we may use eq. (3) with $V = F^{\mu\nu}$ to obtain

$$\mathscr{D}_\nu j^\nu = -\tfrac{1}{2} i g \big[F_{\mu\nu} \,,\, F^{\mu\nu}\big] = 0 \,.$$

2. (a) Compute the differential cross section at $\mathcal{O}(\alpha_s^2)$ for $q\bar{q} \to t\bar{t}$ (where $q \neq t$ is any light quark and t is the top quark), in terms of the center-of-mass energy $\sqrt{s}$ and the squared four-momentum transfer t. Integrate your result over t to obtain the total cross section as a function of the squared center-of-mass energy s. In your calculation, average over initial colors and spins and sum over final colors and spins. You may assume that the initial quark and anti-quark are massless, but do *not* neglect the mass of the top-quark.

Only one Feynman diagram contributes,



where the direction of the four-momenta are indicated on the diagram (i.e. the *incoming* four-momenta are k_1 and k_2 and the *outgoing* four-momenta are p_1 and p_2).

The invariant matrix element for $q\bar{g} \to t\bar{t}$ is

$$i\mathcal{M} = \overline{u}(p_1)(-ig_s\gamma_\mu T^a)v(p_2)\left(\frac{-ig^{\mu\nu}\delta_{ab}}{s}\right)\overline{v}(k_2)(-ig_s\gamma_\nu T^b)v(k_1) \,, \tag{9}$$

where g_s is the strong coupling constant and the square of the center-of-mass energy,

$$s = (k_1 + k_2)^2 = (p_1 + p_2)^2 = 2k_1 \cdot k_2 = 2(M^2 + p_1 \cdot p_2) \,, \tag{10}$$

where we have neglected the masses of q and $\overline{q}$, and we have denoted the top quark mass by M. It is also convenient to introduce the kinematic invariants,

$$t = (k_1 - p_1)^2 = (p_2 - k_2)^2 = M^2 - 2p_1 \cdot k_1 = M^2 - 2p_2 \cdot k_2 \,, \tag{11}$$
$$u = (k_1 - p_2)^2 = (p_2 - k_1)^2 = M^2 - 2p_1 \cdot k_2 = M^2 - 2p_2 \cdot k_1 \,. \tag{12}$$

In particular, note the identity,

$$s + t + u = 4M^2 + 2p_1 \cdot (p_2 - k_1 - k_2) = 2M^2 \,, \tag{13}$$

after applying the conservation of momentum, $p_1 + p_2 = k_1 + k_2$, and using $p_1^2 = M^2$.

Squaring the matrix element and performing an average over initial colors and spins and sum over final colors and spins, we first focus on the color sum and average. It is instructive to perform the color sum and average for an SU(N) gauge theory of strong interactions. (One can set $N = 3$ which is relevant for QCD at the end of the computation.) Consider the $N \times N$ matrix generators T^a in the fundamental representation of SU(N). The standard normalization for these generators are:

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta_{ab} \,. \tag{14}$$

We shall also employ the following identity [cf. eq. (10) of the class handout entitled, *Useful relations involving the generators of* SU*(N)*],

$$T^a_{ij} T^a_{k\ell} = \tfrac{1}{2}\left(\delta_{i\ell}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{k\ell}\right) \,, \tag{15}$$

where there is an implicit sum over $a = 1, 2, , \ldots, N^2 - 1$. Note that $i, j, k, \ell = 1, 2, \ldots, N$.

Focusing only on the color degrees of freedom and suppressing all other factors,

$$\begin{aligned} |\mathcal{M}|^2_{\rm ave} &= \frac{1}{N^2}\big[T^a_{ij}\delta_{ab}T^b_{k\ell}\big]\big[T^c_{ij}\delta_{cd}T^d_{k\ell}\big]^* = \frac{1}{N^2}\big[T^a_{ij}\delta_{ab}T^b_{k\ell}\big]\big[T^c_{ji}\delta_{cd}T^d_{\ell k}\big] \\ &= \frac{1}{N^2}\mathrm{Tr}(T^a T^c)\mathrm{Tr}(T^a T^c) = \frac{1}{4N^2}\delta_{ab}\delta_{ab} = \frac{N^2-1}{4N^2} \,, \end{aligned} \tag{16}$$

where we have used the fact that the generators are hermitian to write $(T^c_{ij})^* = T^c_{ji}$. Averaging over the colors of the incoming quark and antiquark yields two factors of $1/N$. Returning to the full expression given in eq. (9), the average over initial spins (which yields two factors of 1/2) and the sum over final spins yields

$$|\mathcal{M}|^2_{\rm ave} = \frac{1}{4}\,\frac{N^2-1}{4N^2}\,\frac{g_s^4}{s^2}\,\mathrm{Tr}\big[\gamma_\mu(\not{p}_2 - M)\gamma_\alpha(\not{p}_1 + M)\big]\,\mathrm{Tr}(\gamma^\mu \not{k}_1 \gamma^\alpha \not{k}_2) \,.$$

Working out the traces,

$$\begin{aligned} |\mathcal{M}|^2_{\rm ave} &= \frac{N^2-1}{4N^2}\,\frac{g_s^4}{4s^2}\cdot 16\big[p_{2\mu}p_{1\alpha} + p_{2\alpha}p_{1\mu} - g_{\mu\alpha}(p_1 \cdot p_2 + M^2)\big]\big[k^{1\mu}k^{2\alpha} + k^{1\alpha}k^{2\mu} - g^{\mu\alpha}k_1 \cdot k_2\big] \\ &= \frac{N^2-1}{4N^2}\,\frac{8g_s^4}{s^2}(p_1 \cdot k_1\, p_2 \cdot k_2 + p_1 \cdot k_2\, p_2 \cdot k_1 + M^2 k_1 \cdot k_2) \\ &= \frac{N^2-1}{4N^2}\,\frac{2g_s^4}{s^2}\left[(M^2 - t)^2 + (M^2 - u)^2 + 2M^2 s\right] \,, \end{aligned}$$

where we have used eqs. (10)–(12) to express the matrix element in terms of the invariants s, t and u. Defining $\alpha_s \equiv g_s^2/(4\pi)$ and setting $N=3$,

$$|\mathcal{M}|^2_{\rm ave} = \frac{64\pi^2\alpha_s^2}{9s^2}\left[(M^2-t)^2+(M^2-u)^2+2M^2s\right] .$$

The differential cross section for the scattering of two massless particles is given by

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}|\mathcal{M}|^2_{\rm ave} .$$

It then follows that

$$\frac{d\sigma}{dt} = \frac{4\pi\alpha_s^2}{9s^4}\left[(M^2-t)^2+(M^2-u)^2+2M^2s\right] .$$

We now integrate over t to get the total cross section. To obtain the limits of integration, we use eq. (11) to write

$$t = M^2 - 2p_1\!\cdot\! k_1 = M^2 - 2E_tE_q + 2\vec{\boldsymbol{p}}_1\!\cdot\!\vec{\boldsymbol{k}}_1$$

The corresponding energies are $E_t = E_q = \frac{1}{2}\sqrt{s}$, whereas the magnitudes of the corresponding three momenta are

$$|\vec{\boldsymbol{k}}_1| = \tfrac{1}{2}\sqrt{s}\,, \qquad\qquad |\vec{\boldsymbol{p}}_1| = \tfrac{1}{2}\sqrt{s}\beta\,,$$

where

$$\beta \equiv \sqrt{1-\frac{4M^2}{s}}\,. \tag{17}$$

Thus

$$t = M^2 - \tfrac{1}{2}s(1-\beta\cos\theta)\,, \tag{18}$$

where θ is the angle between the three-momenta $\vec{\boldsymbol{p}}_1$ and $\vec{\boldsymbol{k}}_1$. The minimum and maximum of t correspond to $\cos\theta = -1$ and $+1$, respectively. It is convenient to define $t_1 \equiv t - M^2$ and $u_1 \equiv u - M^2$. Then,

$$\sigma = \int_{-\frac{1}{2}s(1+\beta)}^{-\frac{1}{2}s(1-\beta)} \frac{d\sigma}{dt_1}\,dt_1 = \frac{4\pi\alpha_s^2}{9s^4}\int_{-\frac{1}{2}s(1+\beta)}^{-\frac{1}{2}s(1-\beta)} \left(t_1^2+u_1^2+2M^2s\right)dt_1$$

In light of eq. (13),

$$t_1^2+u_1^2+2M^2s = t_1^2+(s+t_1)^2+2M^2s = 2t_1^2+2t_1s+s(s+2M^2)\,.$$

Hence,

$$\begin{aligned}
\sigma &= \frac{4\pi\alpha_s^2}{9s^4}\int_{-\frac{1}{2}s(1+\beta)}^{-\frac{1}{2}s(1-\beta)}\left[2t_1^2+2t_1s+s(s+2M^2)\right]dt_1 \\
&= \frac{4\pi\alpha_s^4}{9s^4}\left[\tfrac{2}{3}t_1^3+t_1^2s+s(s+2M^2)t_1\right]\Bigg|_{-\frac{1}{2}s(1+\beta)}^{-\frac{1}{2}s(1-\beta)} \\
&= \frac{4\pi\alpha_s^4}{9s^4}\left\{-\tfrac{1}{12}s^3\left[(1-\beta)^3-(1+\beta)^3\right]+\tfrac{1}{4}s^3\left[(1-\beta)^2-(1+\beta)^2\right]+\beta s^2(s+2M^2)\right\}. \\
&= \frac{4\pi\alpha_s^4}{9s}\beta\left\{\tfrac{1}{6}\left(3+\beta^2\right)+\frac{2M^2}{s}\right\}
\end{aligned}$$

Using eq. (17), we end up with

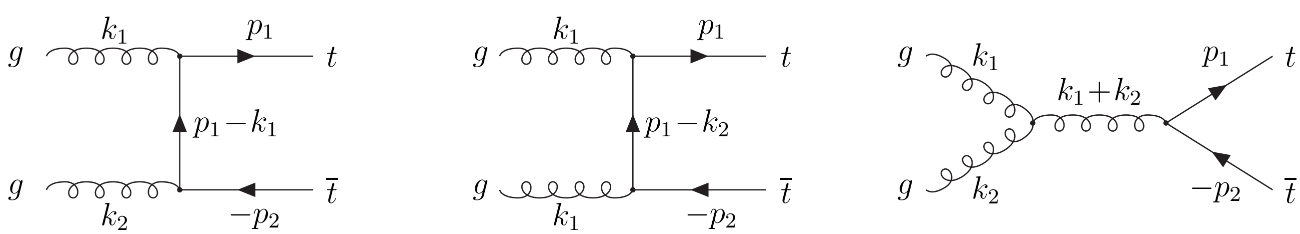
$$\sigma(q\overline{q} \to t\overline{t}) = \frac{8\pi\alpha_s^4}{27s}\left(1+\frac{2M^2}{s}\right)\left(1-\frac{4M^2}{s}\right)^{1/2} .$$

(b) Compute the differential cross section at $\mathcal{O}(\alpha_s^2)$ for $gg \to t\bar{t}$, where g is a gluon, in terms of the squared center-of-mass energy $\sqrt{s}$ and the squared four-momentum transfer t. Integrate your result over t to obtain the total cross section as a function of s. In your calculation, average over initial colors and spins and sum over final colors and spins.

Consider the process $gg \to t\bar{t}$. The incoming gluon momenta are denoted by k_1 and k_2, respectively, and the outgoing momenta of the t and $\bar{t}$ are denoted [as in part (a) of this problem] by p_1 and p_2, respectively. Once again we introduce the three kinematic invariants,

$$s = (k_1+k_2)^2 = (p_1+p_2)^2 = 2k_1\cdot k_2 = 2(M^2 + p_1\cdot p_2)\,, \tag{19}$$
$$t = (k_1-p_1)^2 = (p_2-k_2)^2 = M^2 - 2p_1\cdot k_1 = M^2 - 2p_2\cdot k_2\,, \tag{20}$$
$$u = (k_1-p_2)^2 = (p_2-k_1)^2 = M^2 - 2p_1\cdot k_2 = M^2 - 2p_2\cdot k_1\,, \tag{21}$$

where M is the top quark mass. The identity given in eq. (13) still holds, since the gluon is massless. Three Feynman diagrams contribute at tree-level to $gg \to t\bar{t}$,



(a) (b) (c)

and the corresponding invariant amplitudes will be denoted by $\mathcal{M}_a$, $\mathcal{M}_b$ and $\mathcal{M}_c$.

Employing the QCD Feynman rules,

$$i\mathcal{M}_a = \overline{u}(p_1)(-ig_s\gamma_\mu T^a)\frac{i(\not{p}_1 - \not{k}_1 + M)}{t-M^2}(-ig_s\gamma_\nu T^b)v(p_2)\epsilon_a^\mu(k_1,\lambda_1)\epsilon_b^\nu(k_2,\lambda_2)\,,$$

where $\epsilon_a^\mu(k,\lambda)$ is the polarization vector for a gluon of color a, helicity λ and four-momentum k. Slightly simplifying the above expression yields,

$$\mathcal{M}_a = \frac{g_s^2}{M^2-t}\,\epsilon_a^\mu(k_1,\lambda_1)\epsilon_b^\nu(k_2,\lambda_2)T^aT^b\,\overline{u}(p_1)\gamma_\mu(\not{p}_1 - \not{k}_1 + M)\gamma_\nu v(p_2)\,. \tag{22}$$

Next, $\mathcal{M}_b$ is obtained from $\mathcal{M}_a$ by exchanging the two initial gluons. Hence,

$$\mathcal{M}_b = \frac{g_s^2}{M^2-u}\,\epsilon_a^\mu(k_1,\lambda_1)\epsilon_b^\nu(k_2,\lambda_2)T^bT^a\,\overline{u}(p_1)\gamma_\nu(\not{p}_1 - \not{k}_2 + M)\gamma_\mu v(p_2)\,. \tag{23}$$

Finally, $\mathcal{M}_c$ involves the three-gluon vertex. After some minor simplification,

$$\mathcal{M}_c = \frac{g_s^2}{s}\,\epsilon_a^\mu(k_1,\lambda_1)\epsilon_b^\nu(k_2,\lambda_2)(if_{abc}T^c)\overline{u}(p_1)\gamma^\beta v(p_2)C_{\mu\nu\beta}\,, \tag{24}$$

where

$$C_{\mu\nu\beta} = g_{\mu\nu}(k_2-k_1)_\beta + g_{\beta\mu}(2k_1+k_2)_\nu + g_{\nu\beta}(-2k_2-k_1)_\mu\,. \tag{25}$$

Since these are complex calculations, it is always a good idea to check your results by some independent method. Here, I shall check gauge invariance. In class, I showed that given an invariant matrix element involving two external gluons, which is of the form

$$\mathcal{M}_{\mu\nu}\epsilon^\mu(k_1,\lambda_1)\epsilon^\nu(k_2,\lambda_2)\,,$$

where the color labels have been suppressed, then one must obtain zero if either $\epsilon^\mu(k_1,\lambda_1)$ is replaced by k_1^μ or if $\epsilon^\nu(k_2,\lambda_2)$ is replaced by k_2^μ. For example,

$$k_1^\mu\mathcal{M}_{\mu\nu}\epsilon^\nu(k_2,\lambda_2) = 0\,. \tag{26}$$

Let us now verify eq. (26). We consider the effect of replacing $\epsilon^\mu(k_1,\lambda_1)$ with k_1^μ in $\mathcal{M}_a$, $\mathcal{M}_b$ and $\mathcal{M}_c$, respectively. In the case of $\mathcal{M}_a$, we must evaluate

$$\begin{aligned}\overline{u}(p_1)\not{k}_1(\not{p}_1-\not{k}_1+M)\gamma_\nu v(p_2) &= \overline{u}(p_1)\not{k}_1(\not{p}_1+M)\gamma_\nu v(p_2)\\ &= 2k_1\!\cdot\! p_1\overline{u}(p_1)\gamma_\nu v(p_2) - \overline{u}(p_1)(\not{p}_1-M)\not{k}_1 v(p_2)\\ &= (M^2-t)u(p_1)\gamma_\nu v(p_2)\,.\end{aligned} \tag{27}$$

In obtaining eq. (27), we first used $\not{k}_1\not{k}_1 = k_1^2 = 0$ (since the gluon is massless). Next, we used the anticommutation relations of the gamma matrices, $\{\gamma_\mu\,,\,\gamma_\nu\} = 2g_{\mu\nu}$, to write

$$\not{k}_1\not{p}_1 = 2k_1\!\cdot\! p_1 - \not{p}_1\not{k}_1 = M^2 - t - \not{p}_1\not{k}_1\,.$$

Finally, we made use of the Dirac equation to obtain $\overline{u}(p_1)(\not{p}_1 - M) = 0$.

A similar computation arises when considering $\mathcal{M}_b$.

$$\begin{aligned}\overline{u}(p_1)\gamma_\nu(\not{p}_1-\not{k}_2+M)\not{k}_1 v(p_2) &= \overline{u}(p_1)\gamma_\nu(\not{k}_1-\not{p}_2+M)\not{k}_1 v(p_2)\\ &= \overline{u}(p_1)\gamma_\nu\not{k}_1(\not{p}_2+M)v(p_2) - 2k_1\!\cdot\! p_2\overline{u}(p_1)\gamma_\nu v(p_2)\\ &= (u-M^2)u(p_1)\gamma_\nu v(p_2)\,.\end{aligned} \tag{28}$$

Note that in this calculation, it was useful to use momentum conservation, $p_1 - k_2 = k_1 - p_2$. At the penultimate step, we invoked the Dirac equation, $(\not{p}_2 + M)v(p_2) = 0$.

When eq. (27) is employed in $\mathcal{M}_a$ and eq. (28) is employed in $\mathcal{M}_b$, we note that the denominators are conveniently canceled. Thus, we conclude that

$$(\mathcal{M}_a+\mathcal{M}_b)\Big|_{\epsilon_1\to k_1} = g_s^2\epsilon_b^\nu(k_2,\lambda_2)(T^aT^b - T^bT^a)\overline{u}(p_1)\gamma_\nu v(p_2)\,. \tag{29}$$

Finally, we examine $\mathcal{M}_c$ with $\epsilon^\mu(k_1,\lambda_1)$ replaced by k_1^μ. Then, using eq. (25),

$$k_1^\mu C_{\mu\nu\beta} = k_{1\nu}(k_2-k_1)_\beta + k_{1\beta}(2k_1+k_2)_\nu - 2g_{\nu\beta}k_1\!\cdot\! k_2\,,$$

after noting that $k_1^2=0$. In what follows, we shall denote $\epsilon_2^\nu \equiv \epsilon^\nu(k_2,\lambda_2)$. Then,

$$k_1^\mu \epsilon_2^\nu C_{\mu\nu\beta} = k_1\!\cdot\!\epsilon_2(k_1+k_2)_\beta - 2k_1\!\cdot\! k_2\,\epsilon_{2\beta}\,, \tag{30}$$

after using $k_2\!\cdot\!\epsilon_2=0$. The factor of $(k_1+k_2)_\beta$ then ends up in the expression,

$$\overline{u}(p_1)(\not{k}_1+\not{k}_2)v(p_2) = \overline{u}(p_1)(\not{p}_1+\not{p}_2)v(p_2) = 0\,,$$

after using four-momentum conservation and the Dirac equation. After noting that $s=2k_1\!\cdot\!k_2$, which conveniently cancels the denominator factor in $\mathcal{M}_c$, we are then left with

$$\mathcal{M}_c\Bigg|_{\epsilon_1\to k_1} = g_s^2\epsilon_b^\nu(k_2,\lambda_2)(-if^{abc}T^c)\overline{u}(p_1)\gamma_\nu v(p_2)\,.$$

Combining this with eq. (29),

$$(\mathcal{M}_a+\mathcal{M}_b+\mathcal{M}_c)\Bigg|_{\epsilon_1\to k_1} = g_s^2\epsilon_b^\nu(k_2,\lambda_2)\big(T^aT^b - T^bT^a - if^{abc}T^c\big)\overline{u}(p_1)\gamma_\nu v(p_2) = 0\,,$$

after employing the commutation relations of the generators, $[T^a\,,\,T^b] = if_{abc}T^c$. Thus, the gauge invariance check is successful.

It should be noted that an essential aspect of the gauge invariance check was keeping the factor of $\epsilon_b^\nu(k_2,\lambda_2)$, since eq. (30) was obtained only after using $k_2\!\cdot\!\epsilon_b^\nu(k_2,\lambda_2)=0$. This implies that although eq. (26) is satisfied, the following relations (which do hold in QED),

$$k_1^\mu\mathcal{M}_{\mu\nu} = 0\,, \qquad\qquad k_2^\nu\mathcal{M}_{\mu\nu} = 0 \tag{31}$$

do *not* hold in QCD. In For example, one can easily modify the above computation to obtain

$$k_1^\mu\mathcal{M}_{\mu\nu} = \frac{g_s^2}{s}\,if_{abc}T^c k_{2\mu}\overline{u}(p_1)\not{k}_1 v(p_2)\,. \tag{32}$$

Of course, eq. (26) does hold as expected, since if one multiplies the above equation by $\epsilon_b^\nu(k_2,\lambda_2)$ and employs $k_2\!\cdot\!\epsilon_b^\nu(k_2,\lambda_2)=0$, we do get zero as expected.

Looking ahead to the calculation of the spin and color summed and averaged squared matrix element, the gluon spin sum is given by

$$\sum_\lambda \epsilon_a^\mu(k,\lambda)\,\epsilon_b^\nu(k,\lambda)^* = \delta_{ab}\left(-g^{\mu\nu} + \frac{k^\mu\overline{k}^\nu + k^\nu\overline{k}^\mu}{k\!\cdot\!\overline{k}}\right)\,, \tag{33}$$

where $k^\mu = (k^0\,;\,\vec{\boldsymbol{k}})$ and $\overline{k}^\mu = (k^0\,;\,-\vec{\boldsymbol{k}})$. The consequence of eqs. (26) and (32) is that in a QCD scattering process, one is free to drop the $(k^\mu\overline{k}^{nu} + k^\nu\overline{k}^\mu)/k\!\cdot\!\overline{k}$ term in the spin sum corresponding to one (and only one) external gluon line. Thus, in the present calculation of $gg\to t\bar{t}$, one would have to employ the full expression given by eq. (33) for one of the two gluon spin sums. In contrast, if eq. (31) were to hold (as it does in QED), then one would be justified in making the replacement,

$$\sum_{\lambda} \epsilon_a^\mu(k,\lambda)\,\epsilon_b^\nu(k,\lambda)^* \longrightarrow -\delta_{ab}\, g^{\mu\nu}\,. \tag{34}$$

Let us now return to eqs. (24) and (25) and note that we can slightly simplify this result by writing

$$\mathcal{M}_c = \frac{g_s^2}{s}\,\epsilon_a^\mu(k_1,\lambda_1)\epsilon_b^\nu(k_2,\lambda_2)(if_{abc}T^c)\overline{u}(p_1)\gamma^\beta v(p_2)\widetilde{C}_{\mu\nu\beta}\,, \tag{35}$$

where

$$\widetilde{C}_{\mu\nu\beta} = g_{\mu\nu}(k_2-k_1)_\beta + 2g_{\beta\mu}k_{1\nu} - 2g_{\nu\beta}k_{2\mu}\,. \tag{36}$$

Here we have eliminated two of the terms in $C_{\mu\nu\beta}$ by invoking the properties of the gluon polarization vectors, $k_1\cdot\epsilon_a^\mu(k_1,\lambda_1) = k_2\cdot\epsilon_b^\nu(k_2,\lambda_2) = 0$. Using the version of $\mathcal{M}_c$ given in eq. (35) in the total amplitude $\mathcal{M}_{\mu\nu}\,\epsilon_a^\mu(k_1,\lambda_1)\epsilon_b^\nu(k_2,\lambda_2) \equiv \mathcal{M}_a + \mathcal{M}_b + \mathcal{M}_c$, it is a simple matter to verify that eq. (26) holds! Note that we have not actually changed the value of $\mathcal{M}_c$, but we have changed the value of $\mathcal{M}_{\mu\nu}$. Thus, by employing the form of $\mathcal{M}_c$ as given by eq. (35), we can now safely use eq. (34) in the gluon polarization sums for *both* gluons!

We are now ready to perform the calculation of the cross section for $gg\to t\bar{t}$. In computing the absolute square of the invariant amplitude, there will be six terms,

$$|\mathcal{M}|^2 = |\mathcal{M}_a|^2| + \mathcal{M}_b|^2 + |\mathcal{M}_c|^2 + 2\mathrm{Re}(\mathcal{M}_a\mathcal{M}_b^*) + 2\mathrm{Re}(\mathcal{M}_a\mathcal{M}_c^*) + 2\mathrm{Re}(\mathcal{M}_b\mathcal{M}_c^*)\,. \tag{37}$$

We first focus on the color factors, suppressing all other factors. As in part (a), it is instructive to perform all color computations using an SU(N) color group, and only set $N=3$ at the end of the calculation.

In addition to eqs. (14) and (15), the following results, taken from the class handout entitled *Useful relations involving the generators of* SU*(N)*, will also prove useful. First, by setting $j=k$ in eq. (15) and summing over N colors yields the quadratic Casimir operator in the fundamental representation of SU(N),

$$(T_aT_a)_{i\ell} = \frac{N^2-1}{2N}\,\delta_{i\ell}\,. \tag{38}$$

Also useful is the Casimir operator in the adjoint representation of SU(N), which is given by

$$f_{abc}f_{abd} = N\delta_{cd}\,, \tag{39}$$

where there is an implicit sum over the two pairs of repeated indices. Two other trace relations obtained in eqs. (14) and (24) of the class handout cited above will be used in the computations that follow,

$$f_{abd}\,\mathrm{Tr}(T^aT^bT^c) = \tfrac{1}{4}iN\delta_{cd}\,, \tag{40}$$

$$\mathrm{Tr}(T^aT^bT^aT^c) = -\frac{1}{4N}\,\delta_{bc}\,. \tag{41}$$

The color factors for each of the six terms in eq. (37) can now be obtained. We sum over the final state colors of the t and $\bar{t}$ and average over the initial gluon colors. Since the gluon field transforms under the adjoint representation, it possesses N^2-1 colors in an SU(N) gauge theory. Thus, we must divide by $(N^2-1)^2$ to perform the color averaging of the initial gluons.

$$
\begin{aligned}
|\mathcal{M}_a|^2: &\quad \frac{(T^aT^b)_{ij}(T^bT^a)_{ji}}{(N^2-1)^2} = \frac{\mathrm{Tr}(T^aT^bT^bT^a)}{(N^2-1)^2} = \frac{1}{(N^2-1)^2}\left(\frac{N^2-1}{2N}\right)^2 N = \frac{1}{4N}\,, \\
|\mathcal{M}_b|^2: &\quad \frac{(T^bT^a)_{ij}(T^aT^b)_{ji}}{(N^2-1)^2} = \frac{\mathrm{Tr}(T^bT^aT^aT^b)}{(N^2-1)^2} = \frac{1}{4N}\,, \\
|\mathcal{M}_c|^2: &\quad \frac{(if_{abc}T^c_{ij})(-if_{abe}T^e_{ji})}{(N^2-1)^2} = \frac{f_{abc}f_{abe}\,\mathrm{Tr}(T^cT^e)}{(N^2-1)^2} = \frac{N\delta_{ce}\delta_{ce}}{2(N^2-1)^2} = \frac{N}{2(N^2-1)}\,, \\
2\mathrm{Re}(\mathcal{M}_a\mathcal{M}_b^*): &\quad \frac{(T^aT^b)_{ij}(T^aT^b)_{ji}}{(N^2-1)^2} = \frac{\mathrm{Tr}(T^aT^bT^aT^b)}{(N^2-1)^2} = -\frac{1}{4N(N^2-1)^2}\delta_{bb} = -\frac{1}{4N(N^2-1)}\,, \\
2\mathrm{Re}(\mathcal{M}_a\mathcal{M}_c^*): &\quad \frac{(T^aT^b)_{ij}(-if_{abe}T^e_{ji})}{(N^2-1)^2} = -\frac{if_{abe}\,\mathrm{Tr}(T^aT^bT^e)}{(N^2-1)^2} = \frac{N}{4(N^2-1)^2}\delta_{ee} = \frac{N}{4(N^2-1)}\,, \\
2\mathrm{Re}(\mathcal{M}_b\mathcal{M}_c^*): &\quad \frac{(T^bT^a)_{ij}(-if_{abe}T^e_{ji})}{(N^2-1)^2} = -\frac{if_{abe}\,\mathrm{Tr}(T^bT^aT^e)}{(N^2-1)^2} = -\frac{N}{4(N^2-1)^2}\delta_{ee} = -\frac{N}{4(N^2-1)}\,.
\end{aligned}
$$

Note that the factors of 2 appearing in the interference terms are *not* incorporated in the color factors obtained above. These factors of 2 will be included separately below.

We now proceed to write out the spin and color averaged matrix element, considering separately the six terms in eq. (35). The average over gluon helicities is performed using eq. (34) [although the δ_{ab} factor has already been employed in obtaining the color factors above]. The sums over the final state t and $\bar{t}$ spins are computed in the usual fashion, resulting in traces of gamma matrices. In the average over initial gluon helicities, we include a factor of 1/2 for each gluon since we are averaging over two polarization states. Thus, we obtain

$$
\begin{aligned}
|\mathcal{M}_a|^2_{\text{ave}} &= \frac{g_s^4}{4(M^2-t)^2}\left(\frac{1}{4N}\right)\mathrm{Tr}\big[\gamma_\mu(\not p_1-\not k_1+M)\gamma_\nu(\not p_2-M)\gamma^\nu(\not p_1-\not k_1+M)\gamma^\mu(\not p_1+M)\big]\,, \\
|\mathcal{M}_b|^2_{\text{ave}} &= \frac{g_s^4}{4(M^2-u)^2}\left(\frac{1}{4N}\right)\mathrm{Tr}\big[\gamma_\nu(\not p_1-\not k_2+M)\gamma_\mu(\not p_2-M)\gamma^\mu(\not p_1-\not k_2+M)\gamma^\nu(\not p_1+M)\big]\,, \\
2\mathrm{Re}(\mathcal{M}_a\mathcal{M}_b^*)_{\text{ave}} &= \frac{2g_s^4}{4(M^2-t)(M^2-u)}\left(\frac{1}{4N}-\frac{N}{4(N^2-1)}\right)\mathrm{Tr}\big[\gamma_\mu(\not p_1-\not k_1+M)\gamma_\nu(\not p_2-M)\gamma^\mu \\
&\qquad\qquad \times(\not p_1-\not k_2+M)\gamma^\nu(\not p_1+M)\big]\,, \\
|\mathcal{M}_c|^2_{\text{ave}} &= \frac{g_s^4}{4s^2}\left(\frac{N}{2(N^2-1)}\right)\widetilde{C}_{\mu\nu\alpha}\widetilde{C}^{\mu\nu\beta}\,\mathrm{Tr}\big[\gamma_\beta(\not p_2-M)\gamma^\alpha(\not p_1+M)\big]\,, \\
2\mathrm{Re}(\mathcal{M}_a\mathcal{M}_c^*)_{\text{ave}} &= \frac{2g_s^4}{4s(M^2-t)}\left(\frac{N}{4(N^2-1)}\right)\widetilde{C}_{\mu\nu\beta}\,\mathrm{Tr}\big[\gamma^\beta(\not p_2-M)\gamma^\nu(\not p_1-\not k_1+M)\gamma^\mu(\not p_1+M)\big]\,, \\
2\mathrm{Re}(\mathcal{M}_b\mathcal{M}_c^*)_{\text{ave}} &= \frac{2g_s^4}{4s(M^2-u)}\left(\frac{-N}{4(N^2-1)}\right)\widetilde{C}_{\mu\nu\beta}\,\mathrm{Tr}\big[\gamma^\beta(\not p_2-M)\gamma^\mu(\not p_1-\not k_2+M)\gamma^\nu(\not p_1+M)\big]\,,
\end{aligned}
$$

where $\widetilde{C}_{\mu\nu\beta}$ is defined in eq. (36). The factor of 2 in the interference terms appear explicitly in the three initial numerator factors above. The color factors computed earlier are given inside parentheses in each expression. Note that for the case of $2\mathrm{Re}(\mathcal{M}_a\mathcal{M}_b^*)_{\rm ave}$, we have written

$$-\frac{1}{4N(N^2-1)} = \frac{1}{4N} - \frac{N}{4(N^2-1)}\,.$$

The reason for doing this is that one can now see the existence of two independent color factors, $1/(4N)$ and $\frac{1}{2}N/(N^2-1)$. Indeed, $|\mathcal{M}|^2_{ave}$ consists of two independent terms—one proportional to the first color factor and the other proportional to the second color factor. Although it may not be obvious, one can show that these two independent terms are separately gauge invariant. Indeed, this allows one to take the QED limit of the above result, simply by omitting all terms proportional to $\frac{1}{2}N/(N^2-1)$, since the latter involves the three gluon vertex, which is absent in QED. But, we also learn that part of $2\mathrm{Re}(\mathcal{M}_a\mathcal{M}_b^*)_{\rm ave}$ must also be omitted in the QED limit—this is the piece that is sensitive to the non-commuting nature of the generators T^a.

Identifying these two separate pieces is also useful computationally, as one might expect some significant cancellations within the two independent gauge invariant sets. It should be emphasized that this separation is only possible if one computes SU(N) color factors rather than SU(3) color factors, since the N dependence allows one to identify the two independent pieces of $|\mathcal{M}|^2_{ave}$. Of course, at the end of the day, we will set $N=3$ to get our final result.

The rest of the computation involves the calculation of traces. This is straightforward but tedious. After reducing the traces to products of dot products, one makes use of the kinematics [cf. eqs. (19)–(21)] to express the result in terms of the kinematic invariants. The end result is summarized below.

$$|\mathcal{M}_a|^2_{\rm ave} = \frac{2g_s^4}{(M^2-t)^2}\left(\frac{1}{4N}\right)\left[(M^2-t)(M^2-u) - 2M^2(M^2+t)\right],$$

$$|\mathcal{M}_b|^2_{\rm ave} = \frac{2g_s^4}{(M^2-u)^2}\left(\frac{1}{4N}\right)\left[(M^2-t)(M^2-u) - 2M^2(M^2+u)\right],$$

$$2\mathrm{Re}(\mathcal{M}_a\mathcal{M}_b^*)_{\rm ave} = \frac{4g_s^4}{(M^2-t)(M^2-u)}\left(\frac{1}{4N} - \frac{N}{4(N^2-1)}\right)M^2(s-4M^2)\,,$$

$$|\mathcal{M}_c|^2_{\rm ave} = \frac{4g_s^4}{s^2}\left(\frac{N}{2(N^2-1)}\right)(M^2-t)(M^2-u)\,,$$

$$2\mathrm{Re}(\mathcal{M}_a\mathcal{M}_c^*)_{\rm ave} = \frac{-4g_s^4}{s(M^2-t)}\left(\frac{N}{4(N^2-1)}\right)\left[(M^2-t)(M^2-u) + M^2(u-t)\right],$$

$$2\mathrm{Re}(\mathcal{M}_b\mathcal{M}_c^*)_{\rm ave} = \frac{4g_s^4}{s(M^2-u)}\left(\frac{-N}{4(N^2-1)}\right)\left[(M^2-t)(M^2-u) + M^2(t-u)\right].$$

The invariant cross section is given by

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}|\mathcal{M}|^2_{\rm ave}$$

Adding up the six expressions above, putting $N=3$ and defining $\alpha_s^2 \equiv g_s^2/(4\pi)$, we end up with

$$\frac{d\sigma}{dt} = \frac{2\pi\alpha_s^2}{s^2}\Biggl\{\left(\frac{1}{12}\right)\frac{(M^2-t)(M^2-u)-2M^2(M^2+t)}{(M^2-t)^2} + \left(\frac{1}{12}\right)\frac{(M^2-t)(M^2-u)-2M^2(M^2+u)}{(M^2-u)^2}$$
$$+\left(-\frac{1}{96}\right)\frac{2M^2(s-4M^2)}{(M^2-t)(M^2-u)} + \left(\frac{3}{16}\right)\frac{2(M^2-t)(M^2-u)}{s^2}$$
$$-\left(\frac{3}{32}\right)\frac{2\bigl[(M^2-t)(M^2-u)+M^2(u-t)\bigr]}{s(M^2-t)} + \left(-\frac{3}{32}\right)\frac{2\bigl[(M^2-t)(M^2-u)+M^2(t-u)\bigr]}{s(M^2-u)}\Biggr\},$$

where the color factors for $N=3$ are specified inside the parentheses above.

In light of eq. (13), $s+t+u=2M^2$, so that at fixed center-of-mass energy $\sqrt{s}$, the variable $u=2M^2-s-t$ is also a function of t. Note that t is related to the scattering angle as indicated in eq. (18), which we repeat below:

$$t = M^2 - \tfrac{1}{2}s(1-\beta\cos\theta)\,,$$

where

$$\beta \equiv \left(1-\frac{4M^2}{s}\right)^{1/2}. \tag{42}$$

The kinematical limits on t are easily obtained by imposing the condition, $|\cos\theta| \leq 1$. It follows that

$$M^2 - \tfrac{1}{2}s(1+\beta) \leq t \leq M^2 - \tfrac{1}{2}s(1-\beta)\,. \tag{43}$$

The final step is to integrate over t to obtain the total cross section. As in part (a), it is useful to define $t_1 = t - M^2$ and $u_1 = u - M^2$. Then,

$$\frac{d\sigma}{dt_1} = \frac{2\pi\alpha_s^2}{s^2}\Biggl\{\frac{t_1u_1-2M^2(M^2+t_1)}{12t_1^2} + \frac{t_1u_1-2M^2(M^2+u_1)}{12u_1^2} - \frac{M^2(s-4M^2)}{48t_1u_1}$$
$$+\frac{3t_1u_1}{8s^2} + \frac{3\bigl[t_1u_1+M^2(u_1-t_1)\bigr]}{16st_1} + \frac{3\bigl[t_1u_1+M^2(t_1-u_1)\bigr]}{16su_1}\Biggr\}, \tag{44}$$

Using eq. (13), it follows that

$$s+t_1+u_1=0\,.$$

Due to the symmetry under the interchange of t_1 and u_1, it follows that $d\sigma/dt_1 = d\sigma/du_1$. This means that in computing the total cross section, the integral of the first two terms in eq. (44) yields the same result, as does the integral of the last two terms in eq. (44).

In addition, eq. (43) implies that

$$-\tfrac{1}{2}s(1+\beta) \leq t_1 \leq \tfrac{1}{2}s(1-\beta)\,,$$

which provides the limits for the integral over t_1. Hence, utilizing the symmetry noted above,

$$\sigma = \frac{2\pi\alpha_s^2}{s^2}\int_{-\frac{1}{2}s(1+\beta)}^{-\frac{1}{2}s(1-\beta)} dt_1 \Biggl\{ -\frac{t_1(s+t_1)+2M^2(2M^2+t_1)}{6t_1^2} + \frac{M^2(s-4M^2)}{48t_1(s+t_1)} - \frac{3t_1(s+t_1)}{8s^2} - \frac{3\bigl[t_1(s+t_1)+M^2(s+2t_1)\bigr]}{8st_1}\Biggr\}.$$

The integration is straightforward. It is simple enough to perform by hand, although a computer algebra system such as Mathematica is also suitable. The end result is

$$\sigma(gg\to t\bar{t}) = \frac{\pi\alpha_s^2}{3s}\Biggl\{\left(1+\frac{4M^2}{s}+\frac{M^4}{s^2}\right)\ln\left(\frac{1+\sqrt{1-\frac{4M^2}{s}}}{1-\sqrt{1-\frac{4M^2}{s}}}\right)-\frac{1}{4}\left(1-\frac{4M^2}{s}\right)^{1/2}\left(7+\frac{31M^2}{s}\right)\Biggr\},$$

after using eq. (42) to express the final result as a function of M^2 and s.

3. Consider the following model Lagrangian density for a theory with two real scalar fields ϕ_1 and ϕ_2 and a Dirac fermion field ψ,

$$\mathscr{L} = \tfrac{1}{2}\bigl[(\partial_\mu\phi_1)^2+(\partial_\mu\phi_2)^2\bigr]+\tfrac{1}{2}\mu^2\bigl(\phi_1^2+\phi_2^2\bigr)-\tfrac{1}{4}\lambda\bigl(\phi_1^2+\phi_2^2\bigr)^2+i\overline{\psi}\partial\!\!\!/\psi-g\overline{\psi}(\phi_1+i\gamma_5\phi_2)\psi\,, \quad (45)$$

where the parameters μ^2 and λ are assumed to be positive.

(a) Show that this theory possesses the following global symmetry,

$$\phi_1 \to \phi_1\cos\alpha - \phi_2\sin\alpha \quad (46)$$

$$\phi_2 \to \phi_1\sin\alpha + \phi_2\cos\alpha \quad (47)$$

$$\psi \to \exp\bigl\{-\tfrac{1}{2}i\alpha\gamma_5\bigr\}\,. \quad (48)$$

Show that the solution to the classical field equations with the minimum energy breaks this symmetry spontaneously.

Define the transformed fields,

$$\phi_1' = \phi_1\cos\alpha - \phi_2\sin\alpha$$

$$\phi_2' = \phi_1\sin\alpha + \phi_2\cos\alpha$$

$$\psi' = \exp\bigl\{-\tfrac{1}{2}i\alpha\gamma_5\bigr\}\psi\,.$$

Then, it is straightforward to check that

$$\tfrac{1}{2}\bigl[(\partial_\mu\phi_1')^2+(\partial_\mu\phi_2')^2\bigr]+\tfrac{1}{2}\mu^2\bigl[(\phi_1')^2+(\phi_2')^2\bigr]-\tfrac{1}{4}\lambda\bigl[(\phi_1')^2+(\phi_2')^2\bigr]^2$$
$$=\tfrac{1}{2}\bigl[(\partial_\mu\phi_1)^2+(\partial_\mu\phi_2)^2\bigr]+\tfrac{1}{2}\mu^2\bigl(\phi_1^2+\phi_2^2\bigr)-\tfrac{1}{4}\lambda\bigl(\phi_1^2+\phi_2^2\bigr)^2\,.$$

This is most easily verified using matrix notation by writing

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \mathcal{O} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad \text{where } \mathcal{O} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

with $\mathcal{O}^{\mathsf{T}}\mathcal{O} = \mathbb{1}$ (i.e. $\mathcal{O}$ is a real orthogonal matrix).

Next,

$$\begin{aligned}
\overline{\psi'}\gamma^\mu\psi' = \psi'^{\dagger}\gamma^0\gamma^\mu\psi' &= \psi'^{\dagger}\exp\{\tfrac{1}{2}i\alpha\gamma_5\}\gamma^0\gamma^\mu\exp\{-\tfrac{1}{2}i\alpha\gamma_5\}\psi \\
&= \psi'^{\dagger}\gamma^0\gamma^\mu\exp\{\tfrac{1}{2}i\alpha\gamma_5\}\,\exp\{-\tfrac{1}{2}i\alpha\gamma_5\}\psi = \overline{\psi}\gamma^\mu\psi\,,
\end{aligned}$$

after using the anticommutativity property, $\{\gamma_5, \gamma^\mu\} = 0$, to write

$$\exp\{\tfrac{1}{2}i\alpha\gamma_5\}\gamma^0\gamma^\mu = \gamma^0\gamma^\mu\exp\{\tfrac{1}{2}i\alpha\gamma_5\}\,.$$

Finally, using $(\gamma_5)^2 = 1$, one can use the series expansion of the exponential to compute

$$\begin{aligned}
\exp\{i\alpha\gamma_5\} &= \mathbb{1} + i\alpha\gamma_5 + \frac{(i\alpha)^2}{2!}(\gamma_5)^2 + \frac{(i\alpha)^3}{3!}(\gamma_5)^3 + \cdots \\
&= \mathbb{1}\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots\right) + i\gamma_5\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \cdots\right), \qquad (49)
\end{aligned}$$

which we recognize as

$$\exp\{i\alpha\gamma_5\} = \mathbb{1}\cos\alpha + i\gamma_5\sin\alpha\,. \qquad (50)$$

Multiplying eq. (50) by $i\gamma_5$ yields

$$i\gamma_5\exp\{i\alpha\gamma_5\} = -\mathbb{1}\sin\alpha + i\gamma_5\cos\alpha\,. \qquad (51)$$

Hence, it follows that

$$\begin{aligned}
\overline{\psi'}(\phi_1' + i\gamma_5\phi_2')\psi' &= \overline{\psi}\exp\{-\tfrac{1}{2}i\alpha\gamma_5\}\big[\mathbb{1}(\phi_1\cos\alpha - \phi_2\sin\alpha) + i\gamma_5(\phi_1\sin\alpha + \phi_2\cos\alpha)\big]\exp\{-\tfrac{1}{2}i\alpha\gamma_5\}\psi \\
&= \overline{\psi}\exp\{-i\alpha\gamma_5\}\big[(\mathbb{1}\cos\alpha + i\gamma_5\sin\alpha)\phi_1 + (-\mathbb{1}\sin\alpha + i\gamma_5\cos\alpha)\phi_2\big]\psi \\
&= \overline{\psi}\exp\{-i\alpha\gamma_5\}\,\exp\{i\alpha\gamma_5\}(\mathbb{1}\phi_1 + i\gamma_5\phi_2)\psi \\
&= \overline{\psi}(\phi_1 + i\gamma_5\phi_2)\psi\,,
\end{aligned}$$

after employing eqs. (50) and (51) at the penultimate step. That is, $\mathscr{L}$ given by eq. (45) is invariant under the transformation of fields given in eqs. (46)–(48).

The classical field equations are obtained from the Lagrange field equations (cf. Sections 3.1 and 3.2 of Schwartz), which yield

$$\begin{aligned}
\Box\phi_1 &= -\frac{\partial V}{\partial\phi_1} - g\overline{\psi}\,\psi\,, \\
\Box\phi_2 &= -\frac{\partial V}{\partial\phi_2} - ig\overline{\psi}\gamma_5\psi\,, \\
i\,\partial\!\!\!/\psi &= g(\phi_1 + i\gamma_5\phi_2)\psi\,,
\end{aligned}$$

where we have introduced the scalar potential,

$$V(\phi_1, \phi_2) = -\tfrac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) + \tfrac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2 \,. \tag{52}$$

The total energy density can either be identified with the Hamiltonian density (written in terms of the fields and their derivatives) or by the 00 component of the energy-momentum tensor. Given the Lagrangian density specified in eq. (45), the corresponding Hamiltonian is

$$\mathscr{H} = \frac{1}{2}\left[\left(\frac{\partial\phi_1}{\partial t}\right)^2 + \left(\frac{\partial\phi_2}{\partial t}\right)^2 + (\boldsymbol{\nabla}\phi_1)^2 + (\boldsymbol{\nabla}\phi_2)^2\right] - i\overline{\psi}\,\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\psi + V(\phi_1,\phi_2) + g\overline{\psi}(\phi_1 + i\gamma_5\phi_2)\psi \,.$$

The solution to the classical field equations with the minimum energy corresponds to choosing $\psi = \overline{\psi} = 0$, and $\phi_1 = v_1$ and $\phi_2 = v_2$, where v_1 and v_2 are constants chosen to minimize the function $V(v_1, v_2)$. This implies that

$$\left.\frac{\partial V}{\partial \phi_1}\right|_{\phi_1 = v_1\,,\,\phi_2 = v_2} = \left.\frac{\partial V}{\partial \phi_2}\right|_{\phi_1 = v_1\,,\,\phi_2 = v_2} = 0\,.$$

These conditions yield

$$\left[-\mu^2 + \lambda(v_1^2 + v_2^2)\right] v_1 = 0\,, \tag{53}$$

$$\left[-\mu^2 + \lambda(v_1^2 + v_2^2)\right] v_2 = 0\,. \tag{54}$$

Eqs. (53) and (54) have two possible solutions:

$$\text{(i)} \qquad v_1 = v_2 = 0\,, \quad \Longrightarrow \quad V(v_1, v_2) = 0\,,$$

$$\text{(ii)} \qquad v_1^2 + v_2^2 = \mu^2/\lambda \quad \Longrightarrow \quad V(v_1, v_2) = -\frac{\mu^2}{4\lambda}\,.$$

Since $\mu^2 > 0$ and $\lambda > 0$ by assumption, it follows that the minimum energy vacuum field configuration corresponds to $\psi = \overline{\psi} = 0$ and $\phi_1^2 + \phi_2^2 = \mu^2/\lambda$. The latter condition implies that at least one of the scalar vacuum expectation values is nonzero.

Without loss of generality, we can choose to expand around the vacuum state corresponding to vacuum expectation values, $\langle\Omega|\phi_1|\Omega\rangle = v$ and $\langle\Omega|\phi_2|\Omega\rangle = 0$, where

$$v = \frac{|\mu|}{\sqrt{\lambda}}\,. \tag{55}$$

However, this choice is not invariant under the transformation of fields given in eqs. (46) and (47). Indeed any choice of scalar field vacuum expectation values, subject to the requirement that $\langle\Omega|\,\phi_1^2 + \phi_2^2\,|\Omega\rangle = \mu^2/\lambda$, is not invariant under these transformations. Hence, the corresponding global symmetry is spontaneously broken.

(b) Without loss of generality, one can assume that ϕ_1 possesses a non-zero vacuum expectation value, $\langle\phi_1\rangle = v$, in the ground state, whereas $\langle\phi_2\rangle = 0$. Define new scalar fields $\sigma(x)$ and $\pi(x)$ such that,

$$\big(\phi_1(x)\,,\,\phi_2(x)\big) = \big(v + \sigma(x)\,,\,\pi(x)\big)\,. \tag{56}$$

Write out the Lagrangian in terms of the new scalar fields $\sigma(x)$ and $\pi(x)$, and show that the fermion acquires a mass. Evaluate the fermion mass in terms of g and v.

Inserting eq. (56) into eq. (45) yields

$$\begin{aligned}\mathscr{L} &= \tfrac{1}{2}\big[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2\big] + \tfrac{1}{2}\mu^2(v^2 + 2v\sigma + \sigma^2 + \pi^2) - \tfrac{1}{4}(v^2 + 2v\sigma + \sigma^2 + \pi^2)^2 \\ &\qquad\qquad + i\overline{\psi}\partial\!\!\!/\psi - g\overline{\psi}\big(v + \sigma + i\gamma_5\pi\big)\psi\,, \\ &= \tfrac{1}{2}\big[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2\big] + \tfrac{1}{2}\mu^2v^2 + \mu^2 v\sigma + \tfrac{1}{2}\mu^2\sigma^2 + \tfrac{1}{2}\mu^2\pi^2 - \tfrac{1}{4}\lambda v^2(v^2 + 4v\sigma + 4\sigma^2) \\ &\qquad\qquad - \tfrac{1}{2}\lambda v(v + 2\sigma)(\sigma^2 + \pi^2) - \tfrac{1}{4}\lambda(\sigma^2 + \pi^2)^2 + i\overline{\psi}\partial\!\!\!/\psi - g\overline{\psi}\big(v + \sigma + i\gamma_5\pi\big)\psi\,.\end{aligned}$$

using eq. (55), we can eliminate μ^2 in favor of v. The end result is:

$$\mathscr{L} = \tfrac{1}{4}\lambda v^4 + \tfrac{1}{2}\big[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2\big] - \lambda v^2\sigma^2 - \lambda v\sigma(\sigma^2 + \pi^2) - \tfrac{1}{4}\lambda(\sigma^2 + \pi^2)^2 + i\overline{\psi}\partial\!\!\!/\psi - g\overline{\psi}\big(v + \sigma + i\gamma_5\pi\big)\psi\,. \tag{57}$$

The constant term, $\frac{1}{4}\lambda v^2 = \mu^2/(4\lambda)$, corresponds to the negative of the vacuum energy, which can be dropped. The fermion mass would correspond to a term in the Lagrangian density, $\mathscr{L} \ni -m_\psi\overline{\psi}\psi$. Comparing with eq. (57), we conclude that

$$m_\psi = gv\,.$$

(c) What are the masses of the physical scalar bosons of this model?

To determine the masses of the physical scalars, we compared eq. (57) with

$$\mathscr{L} \ni -\tfrac{1}{2}(m_\sigma^2\sigma^2 + m_\pi^2\pi^2)\,.$$

It immediately follows that $m_\sigma^2 = 2\lambda v^2 = 2\mu^2$ and $m_\pi = 0$. We recognize π as the Goldstone boson field, which is massless as expected.

4. Consider the Abelian Higgs model (i.e., scalar electrodynamics where the U(1) gauge symmetry is spontaneously broken).

(a) Suppose you wish to do calculations in the R_ξ–gauge. Derive the Faddeev-Popov Lagrangian and the corresponding Feynman rules for the ghost propagator and vertices.

In class, the Abelian Higgs model Lagrangian (prior to gauge fixing) was given by

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi,\phi^*)\,,$$

where the covariant derivative acting on the complex scalar field is given by

$$D_\mu\phi = (\partial_\mu + ieA_\mu)\phi\,,$$

and the scalar potential is given by

$$V(\phi,\phi^*) = -\mu^2\phi^*\phi + \tfrac{1}{4}\lambda(\phi^*\phi)^2\,,$$

with μ^2 and λ assumed to be positive. It is convenient to express the complex field ϕ in terms of two real fields, $\phi \equiv (\Phi_1 + i\phi_2)/\sqrt{2}$. As in problem 3, the U(1) symmetry is spontaneously broken. The scalar potential is minimized for $v^2 \equiv \langle \phi_1^2 + \phi_2^2 \rangle = 4\mu^2/\lambda$. We therefore define shifted fields, $(\phi_1(x)\,,\,\phi_2(x) = \bigl(v + \widetilde{\phi}_1(x)\,,\,\widetilde{\phi}_2(x)\bigr)$. In terms of the shifted fields,

$$\begin{aligned}\mathscr{L} = {} & \tfrac{1}{2}\bigl[(\partial_\mu\widetilde{\phi}_1)^2 + (\partial_\mu\widetilde{\phi}_2)^2\bigr] - \tfrac{1}{4}\lambda v^2(\widetilde{\phi}_1)^2 - \tfrac{1}{4}\lambda\widetilde{\phi}_1\bigl[(\widetilde{\phi}_1)^2 + (\widetilde{\phi}_2)^2\bigr] - \tfrac{1}{16}\lambda\bigl[(\widetilde{\phi}_1)^2 + (\widetilde{\phi}_2)^2\bigr]^2 \\ & -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + evA_\mu\partial^\mu\widetilde{\phi}_2 + \tfrac{1}{2}e^2v^2A_\mu A^\mu + e^2v\widetilde{\phi}_1A_\mu A^\mu + eA^\mu\widetilde{\phi}_1\overleftrightarrow{\partial^\mu}\widetilde{\phi}_2 + \tfrac{1}{2}e^2A_\mu A^\mu\bigl[(\widetilde{\phi}_1)^2 + (\widetilde{\phi}_2)^2\bigr]\,. \end{aligned} \tag{58}$$

Note that $\widetilde{\phi}_1$ is the Higgs field, $\widetilde{\phi}_2$ is the Goldstone field, and a photon mass, $m_\gamma = ev$, has been generated.

The gauge fixing term for the R_ξ-gauge in the Abelian Higgs model is

$$\mathscr{L}_{\rm GF} = -\frac{1}{2\xi}\bigl(\partial_\mu A^\mu - \xi ev\widetilde{\phi}_2\bigr)^2\,.$$

Adding $\mathscr{L}_{\rm FP}$ to eq. (58) and dropping the total divergence, $ev(A_\mu\partial^\mu\widetilde{\phi}_2 + \widetilde{\phi}_2\partial_\mu A^\mu) = ev\partial_\mu(A^\mu\widetilde{\phi}_2)$, we end up with

$$\begin{aligned}\mathscr{L} = {} & \tfrac{1}{2}\bigl[(\partial_\mu\widetilde{\phi}_1)^2 + (\partial_\mu\widetilde{\phi}_2)^2\bigr] - \tfrac{1}{4}\lambda v^2(\widetilde{\phi}_1)^2 - \tfrac{1}{2}\xi m_\gamma^2(\widetilde{\phi}_2)^2 - \tfrac{1}{4}\lambda v\widetilde{\phi}_1\bigl[(\widetilde{\phi}_1)^2 + (\widetilde{\phi}_2)^2\bigr] - \tfrac{1}{16}\lambda\bigl[(\widetilde{\phi}_1)^2 + (\widetilde{\phi}_2)^2\bigr]^2 \\ & -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \tfrac{1}{2}m_\gamma^2A_\mu A^\mu + em_\gamma\widetilde{\phi}_1A_\mu A^\mu + eA^\mu\widetilde{\phi}_1\overleftrightarrow{\partial^\mu}\widetilde{\phi}_2 + \tfrac{1}{2}e^2A_\mu A^\mu\bigl[(\widetilde{\phi}_1)^2 + (\widetilde{\phi}_2)^2\bigr]\,. \end{aligned} \tag{59}$$

The Faddeev-Popov determinant is

$$\det\left(\frac{\delta F(x)}{\delta\Lambda(y)}\right)\,, \qquad \text{where} \quad F(x) \equiv \partial_\mu A^\mu(x) - \xi ev\widetilde{\phi}_2(x) - f(x),$$

and

$$\frac{\delta F(x)}{\delta\Lambda(y)} = \int d^4z\left(\frac{\delta F(x)}{\delta A_\mu(z)}\frac{\delta A_\mu(z)}{\delta\Lambda(y)} + \frac{\delta F(x)}{\delta\widetilde{\phi}_2(z)}\frac{\delta\widetilde{\phi}_2(z)}{\delta\Lambda(y)}\right)\,.$$

Under an infinitesimal gauge transformation,

$$\begin{aligned} A_\mu(x) &\to A_\mu(x) + \partial_\mu\Lambda(x)\,, \\ \phi(x) &\to \bigl[1 - ie\Lambda(x)\bigr]\phi(x)\,, \end{aligned} \tag{60}$$

where $\phi(x) = v + \widetilde{\phi}_1(x) + i\widetilde{\phi}_2(x)$. That is, $A_\mu \to A_\mu + \delta A_\mu$ and $\widetilde{\phi}_2 \to \widetilde{\phi}_2 + \delta\widetilde{\phi}_2$, where

$$\delta A_\mu = \partial_\mu\Lambda\,, \qquad \delta\widetilde{\phi}_1 = e\Lambda\widetilde{\phi}_2\,, \qquad \delta\widetilde{\phi}_2 = -e\Lambda(\widetilde{\phi}_1 + v)\,.$$

Hence, it follows that

$$\begin{aligned}\frac{\delta F(x)}{\delta\Lambda(y)} &= \int d^4z\,\bigl[\Box_x\delta^4(x-z)\delta^4(y-z) + e^2v\xi\bigl(\widetilde{\phi}_1(x) + v\bigr)\delta^4(x-z)\delta^4(y-z)\bigr] \\ &= \bigl[\Box_x + e^2v^2\xi + e^2v\,\xi\widetilde{\phi}_1(x)\bigr]\delta^4(x-y)\,. \end{aligned} \tag{61}$$

Therefore, following the steps given in class, the Faddeev-Popov Lagrangian is obtained via the path integral representation,

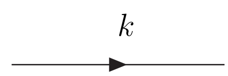
$$
\begin{aligned}
\det\left(\frac{\delta F(x)}{\delta\Lambda(y)}\right) &= \mathcal{N}\int \mathcal{D}\eta^*\mathcal{D}\eta\,\exp\left\{-i\int d^4x\,d^4y\,\eta^*(y)\frac{\delta F(x)}{\delta\Lambda(y)}\eta(x)\right\} \\
&= \mathcal{N}\int \mathcal{D}\eta^*\mathcal{D}\eta\,\exp\left\{-i\int d^4x\,\eta^*(x)\left[\Box_x + e^2v^2\xi + e^2v\,\xi\widetilde{\phi}_1(x)\right]\eta(x)\right\} \\
&= \mathcal{N}\int \mathcal{D}\eta^*\mathcal{D}\eta\,\exp\left\{i\int d^4x\,\mathscr{L}_{\rm FP}\right\},
\end{aligned}
$$

where

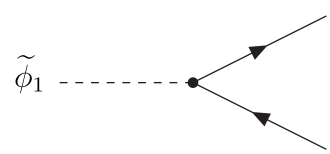
$$
\mathscr{L}_{\rm FP} = \partial_\mu\eta^*\partial^\mu\eta - \xi m_\gamma^2\eta^*\eta - em_\gamma\,\xi\widetilde{\phi}_1(x)\eta^*\eta\,, \tag{62}
$$

after performing an integration by parts (assuming that the fields fall off at infinity so that we can drop the surface term). Note that we have identified the photon mass, $m_\gamma = ev$. We can now read off the squared-mass of the Faddeev-Popov ghost fields, ξm_γ^2, and the strength of the coupling of the Faddeev-Popov ghosts to the Higgs field, $g_{\eta^*\eta\tilde{\phi}_1} = -e\xi m_\gamma$.

The Feynman rules for the Faddeev-Popov fields can be immediately read off from eq. (62). The rules for the Faddeev ghost propagator and interactions are given by:



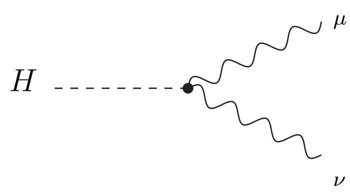
$$\frac{i}{k^2 - \xi m_\gamma^2}$$



$$-ie\xi m_\gamma$$

(b) Let m_H and m_V be the masses of the Higgs boson (H) and vector boson (V) respectively. Assuming that $m_H > 2m_V$, compute the tree-level rate for the decay $H \to VV$.

We identify $H = \widetilde{\phi}_1$ and $V = \gamma$. The Feynman rule for the $\widetilde{\phi}_1\gamma\gamma$ vertex can be read off from eq. (59) and is exhibited below. Note the factor of 2 due to identical photons.



$$2iem_\gamma g^{\mu\nu}$$

The invariant matrix element for $H \to VV$ is

$$
\mathcal{M} = 2em_\gamma\,\epsilon_1^*\!\cdot\epsilon_2^*\,, \tag{63}
$$

where the ϵ_i are the polarization vectors of the outgoing spin-1 particles. To compute the decay rate, we shall first square the amplitude and sum over outgoing polarizations. The spin sum for a massive spin-1 boson is

$$\sum_\lambda \epsilon^\mu(p,\lambda)\epsilon^\nu(p,\lambda)^* = -g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}\,,$$

where the sum is taken over the three physical polarization states of the massive photon, V. Thus, denoting the four-momenta of the outgoing massive photons by p_1 and p_2, it follows that

$$\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 = 4e^2 m_\gamma^2 \left(-g^{\mu\nu} + \frac{p_1^\mu p_1^\nu}{m_\gamma^2}\right)\left(-g_{\mu\nu} + \frac{p_{2\mu}p_{2\nu}}{m_\gamma^2}\right) = 4e^2 m_\gamma^2 \left[2 + \frac{(p_1\!\cdot\! p_2)^2}{m_\gamma^4}\right]\,, \tag{64}$$

where we have employed the mass-shell conditions, $p_1^2 = p_2^2 = m_\gamma^2$.

In light of the kinematics of the decay, we have

$$m_H^2 = (p_1+p_2)^2 = 2m_\gamma^2 + 2p_1\!\cdot\! p_2\,,$$

from which it follows that $p_1\!\cdot\! p_2 = \frac{1}{2}m_H^2 - m_\gamma^2$. Inserting this result in eq. (64), we end up with

$$\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 = \frac{e^2(m_H^4 - 4m_H^2 m_\gamma^2 + 12 m_\gamma^4)}{m_\gamma^2}\,. \tag{65}$$

In the rest frame of H, the Higgs boson four-momentum is The total decay rate is $p = (m_H\,;\,\vec{\boldsymbol{0}})$, and the rate of Higgs boson decay into a pair of massive photons is given by

$$\Gamma = \frac{1}{2m_H(2\pi)^2}\int \frac{d^3p_1}{2E_1}\,\frac{d^3p_2}{2E_2}\,\delta^4(p-p_1-p_2)\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2\,, \tag{66}$$

To evaluate the phase space integral above, we employ the following identity,

$$\begin{aligned}\int \frac{d^3p_1}{2E_1}\,\frac{d^3p_2}{2E_2}\,\delta^4(p-p_1-p_2) &= \int d^4p_1\,d^4p_2\,\delta(p^1-m_\gamma^2)\delta(p^2-m_\gamma^2)\delta^4(p-p_1-p_2)\theta(p_{10})\theta(p_{20}) \\ &= \int d^4p_1\,\delta(p_1^2-m_\gamma^2)\delta\big((p-p_1)^2-m_\gamma^2\big)\theta(p_{10})\,,\end{aligned} \tag{67}$$

after using the delta function to integrate over p_2. Next, we make use of a well known delta function identity and write[1]

$$\theta(p_{10})\delta(p_1^2-m_\gamma^2) = \theta(p_{10})\delta(p_{10}^2 - \vec{\boldsymbol{p}}_1^{\,2} - m_\gamma^2) = \frac{1}{2p_{10}}\delta\left(p_{10} - \sqrt{\vec{\boldsymbol{p}}_1^{\,2} + m_\gamma^2}\right)\,.$$

Inserting this result in eq. (67), we can immediately perform the integration over p_{10}. Using $p = (m_H\,;\,\vec{\boldsymbol{0}})$, we obtain $(p-p_1)^2 - m_\gamma^2 = p^2 - 2p\!\cdot\! p_1 = m_H^2 - 2m_H p_{10}$. It follows that

[1]Note that due to the presence of the step function, $\theta(p_{10})$, the second term of the delta function identity involving $\delta\left(p_{10} + \sqrt{\vec{\boldsymbol{p}}_1^{\,2} + m_\gamma^2}\right)$ does not contribute.

$$\int \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\,\delta^4(p-p_1-p_2) = \int \frac{d^3p_1}{2\sqrt{\vec{\boldsymbol{p}}_1^{\,2}+m_\gamma^2}}\,\delta(m_H^2-2m_H p_{10})\bigg|_{p_{10}=\sqrt{\vec{\boldsymbol{p}}_1^{\,2}+m_\gamma^2}}\,. \tag{68}$$

We now use spherical coordinates to carry out the integration over $\vec{\boldsymbol{p}}$. Note that

$$d^3p_1 = |\vec{\boldsymbol{p}}_1|^2 d|\vec{\boldsymbol{p}}_1| d\Omega = |\vec{\boldsymbol{p}}_1| E_1 dE_1 d\Omega\,,$$

where $E_1 \equiv \sqrt{\vec{\boldsymbol{p}}_1^{\,2}+m_\gamma^2}$. Hence, eq. (68) can be evaluated as follows:

$$\begin{aligned}
\int \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\,\delta^4(p-p_1-p_2) &= \frac{1}{2}\int dE_1\,d\Omega\,\sqrt{E_1^2-mk_\gamma^2}\,\delta(m_H^2-2m_H E_1) \\
&= \frac{1}{4m_H}\int dE_1\,d\Omega\,\sqrt{E_1^2-mk_\gamma^2}\,\delta(E_1-\tfrac{1}{2}m_H) \\
&= \frac{\sqrt{m_H^2-4m_\gamma^2}}{8m_H}\int d\Omega\,.
\end{aligned} \tag{69}$$

Thus, eqs. (66) and (69) yield

$$\Gamma = \frac{\sqrt{m_H^2-4m_\gamma^2}}{64\pi^2 m_H^2}\int d\Omega \sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2\,.$$

The integrand is independent of angles, so we can perform the integration over solid angles immediately. But, we must be careful since the two outgoing massive photons are identical. Thus, integrating over 4π steradians double counts, since the two outgoing massive photons are indistinguishable. Thus, we should only integrate over 2π steradians (or equivalently, integrate over the full 4π steradians and then include an extra factor of $1/2$ due to two identical particles in the final state). Hence,

$$\Gamma = \frac{\sqrt{m_H^2-4m_\gamma^2}}{32\pi m_H^2}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2\,.$$

Employing eq. (65) then yields our final result

$$\Gamma(H\to VV) = \frac{e^2(m_H^4-4m_H^2m_\gamma^2+12m_\gamma^4)}{32\pi m_\gamma^2 m_H}\left(1-\frac{4m_\gamma^2}{m_H^2}\right)^{1/2}\,.$$

(c) The Equivalence Theorem states that the S-matrix amplitude involving external longitudinally polarized gauge bosons may be evaluated in the R_ξ gauge by substituting the corresponding Goldstone bosons as external particles. This equality holds up to corrections of order m_V/E_V, where E_V is the vector boson energy. Verify this theorem by applying it to the Higgs boson decay of part (b).

The amplitude for $H \to VV$ was obtained in eq. (63), which we reproduce here:

$$\mathcal{M} = 2em_\gamma\, \epsilon_1^* \!\cdot\! \epsilon_2^* \,. \tag{70}$$

We are interested in the limit where $E_\gamma \gg m_H$, where E_γ is the energy of either photon in the rest frame of the Higgs boson. Indeed, energy conservation applied to the Higgs boson rest frame yields $E_\gamma = \frac{1}{2}m_H$. To evaluate the amplitude in this limit, we need to examine the explicit forms for the massive photon polarization vectors, $\epsilon^\mu(\vec{\boldsymbol{k}}_\gamma, \lambda)$, where $\vec{\boldsymbol{k}}_\gamma, \lambda$ are the photon three-momentum and helicity, respectively, which are given in eqs. (8) and (22) of the class handout entitled *Polarization Sum for Massless Spin-One Particles*.[2] The transverse and longitudinal polarization vectors of a massive spin-1 boson traveling in the direction $\vec{\boldsymbol{k}} = k_\gamma \hat{\boldsymbol{z}}$ are given by:

$$\epsilon^\mu(k_\gamma \hat{\boldsymbol{z}}, \pm 1) = \frac{1}{\sqrt{2}}\bigl(0\,;\, \mp 1\,,\, -i\,,\, 0\bigr) \tag{71}$$

$$\epsilon^\mu(k_\gamma \hat{\boldsymbol{z}}, 0) = \frac{1}{m_\gamma}\bigl(k_\gamma\,;\, 0\,,\, 0\,,\, E_\gamma\bigr)\,, \tag{72}$$

where $E_\gamma = \sqrt{k_\gamma^2 + m_\gamma^2}$. Note that the polarization vectors satisfy $k \!\cdot\! \epsilon(\vec{\boldsymbol{k}}_\gamma, \lambda) = 0$, where the four-vector of the massive photon is given by

$$k^\mu = (E_\gamma\,;\, 0\,,\, 0\,,\, k_\gamma)\,. \tag{73}$$

Comparing eqs. (72) and (73) and noting the identities,

$$\frac{E_\gamma}{m_\gamma} - \frac{m_\gamma}{k_\gamma + E_\gamma} = \frac{E_\gamma^2 - m_\gamma^2 + k_\gamma E_\gamma}{m_\gamma(k_\gamma + E_\gamma)} = \frac{k_\gamma}{m_\gamma}\,,$$

$$\frac{k_\gamma}{m_\gamma} + \frac{m_\gamma}{k_\gamma + E_\gamma} = \frac{k_\gamma^2 + m_\gamma^2 + k_\gamma E_\gamma}{m_\gamma(k_\gamma + E_\gamma)} = \frac{E_\gamma}{m_\gamma}\,,$$

after employing $E_\gamma^2 = k_\gamma^2 + m_\gamma^2$, it follows that

$$\epsilon^\mu(k_\gamma \hat{\boldsymbol{z}}, 0) = \frac{k^\mu}{m_\gamma} - \frac{m_\gamma}{k_\gamma + E_\gamma}\bigl(1\,;\, 0\,,\, 0\,,\, -1\bigr) = \frac{k^\mu}{m_\gamma} + \mathcal{O}\left(\frac{m_\gamma}{E_\gamma}\right)\,.$$

Indeed, the result

$$\epsilon^\mu(\vec{\boldsymbol{k}}_\gamma, 0) = \frac{k^\mu}{m_\gamma} + \mathcal{O}\left(\frac{m_\gamma}{E_\gamma}\right)\,, \tag{74}$$

is more general and applies to a massive photon traveling in an arbitrary direction $\vec{\boldsymbol{k}}_\gamma$. This can be easily demonstrated by an appropriate rotation of the coordinate system.

A comparison of eqs. (71) and (74) shows that in the limit of $E_\gamma \gg m_\gamma$, the contribution of the longitudinal (helicity-zero) polarizations dominates over the contribution of the transverse

[2] Alternatively, see e.g., Elliot Leader, *Spin in Particle Physics* (Cambridge University Press, Cambridge, UK, 2001) p. 71; Hartmut Pilkuhn, *The Interaction of Hadrons* (North-Holland Publishing Company, Amsterdam, 1967) p. 62.

(i.e., helicity ± 1) polarizations in eq. (70). Hence, employing eq. (72) for the polarization vectors in eq. (74),

$$\mathcal{M} \simeq 2em_\gamma \frac{p_1\cdot p_2}{m_\gamma^2}\,.$$

Using

$$2p_1\cdot p_2 = (p_1+p_2)^2 - p_1^2 - p_2^2 = m_H^2 - 2m_\gamma^2 \simeq 2m_H^2\,,$$

in the limit of $E_\gamma = \frac{1}{2}m_H \gg m_\gamma$, it follows that

$$\mathcal{M} \simeq \frac{em_H^2}{m_\gamma}\,. \tag{75}$$

Hence, summing over the polarizations, the contribution of the longitudinal polarizations dominates which means that

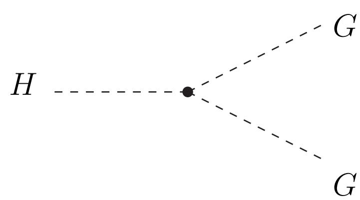
$$\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 = \frac{e^2 m_H^4}{m_\gamma^2}\left[1 + \mathcal{O}\left(\frac{m_\gamma^2}{m_H^2}\right)\right]\,,$$

which is consistent with the exact result obtained in eq. (65).

We shall now compare the amplitude for the decay of the Higgs boson to a pair of Goldstone bosons. We identify $H = \widetilde{\phi}_1$ and $G = \widetilde{\phi}_2$, so that the relevant interaction term is the $\widetilde{\phi}_1(\widetilde{\phi}_2)^2$ term in eq. (59). This equation also provides the mass of the Higgs boson $\widetilde{\phi}_1$,

$$m_H^2 = \tfrac{1}{2}\lambda v^2\,. \tag{76}$$

The Feynman rule for the $\widetilde{\phi}_1(\widetilde{\phi}_2)^2$ vertex is



$$-\tfrac{1}{2}i\lambda v = -i\frac{m_H^2}{v}$$

where we have included a factor of 2 for the identical Goldstone bosons and we have made use of eq. (76) to express the HGG coupling in terms of the Higgs boson mass. Thus, we identify the amplitude for $H \to GG$ as

$$\mathcal{M} = -\frac{m_H^2}{v} = -\frac{em_H^2}{m_\gamma}\,,$$

after using $m_\gamma = ev$. That is, we have verified the relation,

$$\mathcal{M}(H \to VV)\bigg|_{m_H \gg m_\gamma} = -\mathcal{M}(H \to GG)\,.$$

Apart from an unimportant overall minus sign, we have demonstrated that the high energy limit of the $H \to VV$ decay amplitude is given by the amplitude for $H \to GG$ computed in the R_ξ gauge. This is an explicit example of the Equivalence Theorem. It is a particularly useful observation since the computation of the amplitude for a process consisting entirely of spin-zero bosons is far easier as compared to the computation of the amplitude for a process that involves massive spin-1 bosons.