1. By using the BRST-invariance of QED, one can derive the well known relation between the vertex function and the inverse propagator that is given in eq. (19.80) of Schwartz (on p. 352). Here is how to do it:

From part (d) of problem 1 of Problem Set 2, you know that:

\[ \langle \Omega | T (\psi(x) \overline{\psi}(y) \phi(z)) | \Omega \rangle = 0. \]  

(1)

Using the BRST-invariance of the theory, this Green function must remain zero under an (infinitesimal) BRST-transformation. Computing to first order, deduce an equation that relates one may eliminate it by explicitly evaluating the scalar field propagator [after invoking one of the relations proved in part (d) of problem 1 of Problem Set 2]. Then, one can derive the Ward Identity for QED that relates the vertex function and the inverse propagator. Transforming to momentum space, check that the final result coincides with eq. (19.80) of Schwartz.

Eq. (1) implies that under a BRST transformation, \( \delta \langle \Omega | T (\psi(x) \overline{\psi}(y) \phi(z)) | \Omega \rangle = 0. \) By definition, the variation of \( \langle \Omega | T (\psi(x) \overline{\psi}(y) \phi(z)) | \Omega \rangle \) is given by

\[
\langle \Omega | T ([\psi(x) + \delta \psi(x)] \overline{\psi}(y) + \delta \overline{\psi}(y))[\phi(z) + \delta \phi(z)] | \Omega \rangle - \langle \Omega | T (\psi(x) \overline{\psi}(y) \phi(z)) | \Omega \rangle.
\]

Working to first order in the field variations, we therefore conclude that

\[
\langle \Omega | T (\delta \psi(x) \overline{\psi}(y) \phi(z)) | \Omega \rangle + \langle \Omega | T (\psi(x) \delta \overline{\psi}(y) \phi(z)) | \Omega \rangle + \langle \Omega | T (\psi(x) \overline{\psi}(y) \delta \phi(z)) | \Omega \rangle = 0.
\]

(2)

Using eqs. (7) and (9) of Solution Set 2, the infinitesimal BRST transformation of the fermion and scalar fields are given by

\[
\delta \psi(x) = ie \epsilon \phi(x) \psi(x),
\]

(3)

\[
\delta \overline{\psi}(x) = -ie \epsilon \phi(x) \overline{\psi}(x),
\]

(4)

\[
\delta \phi(x) = -\frac{\epsilon}{a} \partial_\mu A_\mu(x).
\]

(5)

Inserting these in eq. (2) yields

\[
\frac{1}{a} \langle \Omega | T (\psi(x) \overline{\psi}(y) \phi(z)) | \Omega \rangle = 0.
\]

(6)

Using the path integral representation of the Green function,

\[
\langle \Omega | T (\psi(x) \overline{\psi}(y) \partial_\mu A_\mu(z)) | \Omega \rangle = N \int \mathcal{D}A_\mu \mathcal{D}\overline{\psi} \mathcal{D}\psi \psi(x) \overline{\psi}(y) \partial_\mu A_\mu(z) \exp \left[ i \int d^4x \mathcal{L}_{\text{QED}} \right]
\]

\[
= N \partial_\mu \int \mathcal{D}A_\mu \mathcal{D}\overline{\psi} \mathcal{D}\psi \psi(x) \overline{\psi}(y) A_\mu(z) \exp \left[ i \int d^4x \mathcal{L}_{\text{QED}} \right]
\]

\[
= \partial_\mu \langle \Omega | T (\psi(x) \overline{\psi}(y) A_\mu(z)) | \Omega \rangle.
\]

(7)
where
\[ N^{-1} = \int \mathcal{D}A^\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[ i \int \, d^4x \mathcal{L}_{\text{QED}} \right]. \]

Thus, eq. (6) can be written as
\[ ie \left[ \langle \Omega | T(\psi(x)\overline{\psi}(y)\phi(x)\phi(z))|\Omega \rangle - \langle \Omega | T(\psi(x)\overline{\psi}(y)\phi(z)\phi(z))|\Omega \rangle \right] - \frac{1}{a} \partial^\mu_z \langle \Omega | T(\psi(x)\overline{\psi}(y)A_\mu(z))|\Omega \rangle = 0. \]

Without loss of generality, we shall take all Green functions above to be connected. We can also use translational invariance of the Green functions to set \( y = 0 \). We define the connected two and three-point Green functions in momentum space as
\[ V_\mu(p, p + q) = \int d^4x d^4z e^{i(\mathbf{p} + \mathbf{q} + \mathbf{z})} \langle \Omega | \psi(x)\overline{\psi}(0) A_\mu(z)|\Omega \rangle. \]

From part (d) of problem 1 of Problem Set 2, we know that:
\[ \int d^4x d^4z e^{i(\mathbf{p} + \mathbf{q} + \mathbf{z})} \langle \Omega | \psi(x)\overline{\psi}(0)\phi(x)\phi(z)|\Omega \rangle = \int d^4x d^4z e^{i(\mathbf{p} + \mathbf{q} + \mathbf{z})} \langle \Omega | \psi(x)\overline{\psi}(0)|\Omega \rangle \langle \Omega | \phi(x)\phi(z)|\Omega \rangle, \]

Since \( \phi(x) \) is a free field, we have
\[ \langle \Omega | \phi(x)\phi(z)|\Omega \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{z})} \frac{i}{k^2 - i\epsilon}. \]

Inserting this in eq. (9),
\[ \int d^4x d^4z e^{i(\mathbf{p} + \mathbf{q} + \mathbf{z})} \langle \Omega | \psi(x)\overline{\psi}(0)\phi(x)\phi(z)|\Omega \rangle \]
\[ = \int d^4x d^4z \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - i\epsilon} e^{i(\mathbf{p} - \mathbf{k}) \cdot \mathbf{x}} e^{i(\mathbf{q} + \mathbf{k}) \cdot \mathbf{z}} \langle \Omega | \psi(x)\overline{\psi}(0)|\Omega \rangle \]
\[ = \int d^4k \frac{i}{k^2 - i\epsilon} S(p - k) \delta^4(q + k) = - \frac{1}{q^2 - i\epsilon} S(p + q). \]

Similarly,
\[ \int d^4x d^4z e^{i(\mathbf{p} + \mathbf{q} + \mathbf{z})} \langle \Omega | \psi(x)\overline{\psi}(0)\phi(0)\phi(z)|\Omega \rangle = - \frac{1}{q^2 - i\epsilon} S(p). \]

Finally, we observe that
\[ \int d^4x d^4z e^{i(\mathbf{p} + \mathbf{q} + \mathbf{z})} \partial^\mu_z \langle \Omega | \psi(x)\overline{\psi}(0) A_\mu(z)|\Omega \rangle = - i q^\mu V_\mu(p, p + q), \]after an integration by parts (making the usual assumption that the fields fall off fast enough at infinity so that there are no surface term contributions).
Thus, using eqs. (10), (11) and (12), we see that eq. (8) yields
\[
\frac{i}{a} q^\mu \mathcal{V}_\mu(p, p + q) = \frac{ie}{q^2 - i\epsilon} [S(p + q) - S(p)].
\]  
(13)

This result takes on a more familiar form when written in terms of the 1PI Green functions. Using eq. (18) of Solution Set 1,
\[
\mathcal{V}_\mu(x_1, x_2, x_3) = i \int d^4 y_1 d^4 y_2 d^4 y_3 \Gamma_\mu(y_1, y_2, y_3) G_2^{(2)}(x_1, y_1) G_2^{(2)}(x_2, y_2) G_2^{(2)}(x_3, y_3).
\]

In momentum space, it follows that
\[
\mathcal{V}_\mu(p, p') = \mathcal{D}_{\mu\nu}(q)i S(p')i \Gamma^\nu(p, p')i S(p),
\]
where \(p' = p + q\) and \(\mathcal{D}_{\mu\nu}(q)\) is the exact photon propagator. Consequently,
\[
\frac{i}{a} q^\mu \mathcal{V}_\mu(p, p') = \frac{1}{a} q^\mu \mathcal{D}_{\mu\nu}(q) S(p') \Gamma^\nu(p, p') S(p).
\]
(14)

In class, I proved that the exact photon propagator satisfies
\[
q^\nu \Gamma^\nu(p, p + q) = e[S^{-1}(p + q) - S^{-1}(p)].
\]

This result coincides with eq. (19.80) of Schwartz, and is known as the Ward-Takahashi identity.

**ADDITIONAL REMARKS:**

The time-ordered product defined via the path integral is slightly different from the original definition. Recall that the definition of the time-ordered product of two bosonic operators is
\[
T(\mathcal{O}_1(x)\mathcal{O}_2(y)) = \Theta(x^0 - y^0)\mathcal{O}_1(x)\mathcal{O}_2(y) + \Theta(y^0 - x^0)\mathcal{O}_2(y)\mathcal{O}_1(x),
\]
(15)

with a suitable generalization to the time-ordered product of three or more operators. In the case of fermionic operators, one must include a minus sign for any term in which the new ordering

\[\Theta(z) = \begin{cases} 1, & \text{if } z > 0, \\ 0, & \text{if } z < 0. \end{cases}\]

---

\[\text{The step function is defined by}\]

\[\Theta(z) = \begin{cases} 1, & \text{if } z > 0, \\ 0, & \text{if } z < 0. \end{cases}\]
of fermion fields differs from the original ordering by an odd number of interchanges. Taking a partial derivative of eq. (15) with respect to \( y \) yields,

\[
\partial_y T(\mathcal{O}_1(x)\mathcal{O}_2(y)) = T(\mathcal{O}_1(x)\partial^\mu \mathcal{O}_2(y)) - \delta(x^0 - y^0) [\mathcal{O}_1(x), \mathcal{O}_2(y)],
\]

which differs from the form of eq. (7) by the equal-time commutator term. (In the old literature, the time-ordered product that satisfies eq. (16) without the equal-time commutator term was called a \( T^\ast \)-product.) Nevertheless, in the problem, eq. (7) is also applicable for the time ordered product defined via eq. (15), since the equal time commutator of \( A_\mu(z) \) with a fermion field vanishes in the canonical approach to quantum field theory.

2. Consider the function of a real parameter \( z \)

\[
F(z) \equiv \int_0^1 dx \ln \left[ 1 - zx(1-x) - i\epsilon \right],
\]

which appeared in the computation of the one-loop correction to the four-point function in scalar field theory.

(a) Evaluate \( \text{Im} F(z) \). For what values of \( z \) does \( \text{Im} F \) vanish?

We shall denote the argument of the logarithm in eq. (17) by the function,

\[
f(x) \equiv zx^2 - zx + 1 \geq 0.
\]

First, we note that \( f(0) = f(1) = 1 \). Next, we compute the first and second derivatives,

\[
f'(x) = z(2x - 1), \quad f''(x) = 2z,
\]

Thus, \( f(x) \) has an extremum at \( x = \frac{1}{2} \). Since \( f''(\frac{1}{2}) = 2z \), it follows that \( x = \frac{1}{2} \) is a maximum if \( z < 0 \) and \( x = \frac{1}{2} \) is a minimum if \( z > 0 \). At \( z = 0 \), we have \( f(x) = 1 \) for all \( x \). Moreover, for \( z > 0 \), the minimum value of \( f(x) \) is equal to \( f(\frac{1}{2}) = 1 - \frac{1}{4}z \). Thus, for values of \( z \leq 0 \), we have \( f(x) \geq 1 \) in the region \( 0 \leq x \leq 1 \) and for values of \( 0 < z \leq 4 \), the minimum value of \( f(x) \) is non-negative (for all \( x \)).

Observe that \( \text{Im} F(z) = 0 \) if \( f(z) \geq 0 \) for \( 0 \leq x \leq 1 \), which implies that \( \text{Im} F(z) = 0 \) if \( z \leq 4 \). When \( z > 4 \), the minimum value of \( f(x) \) at \( x = \frac{1}{2} \) is negative. Since \( f(0) = f(1) = 1 \), it follows that \( f(x) < 0 \) for values of \( x_+ < x < x_- \), where \( x_\pm \) are the roots of \( f(x) \),

\[
x_\pm = \frac{1}{2} \left[ 1 \pm \sqrt{1 - \frac{4}{z}} \right].
\]

Thus,

\[
\text{Im} F(z) = \Theta(z - 4) \int_{x_-}^{x_+} dx \text{Im} \ln \left[ 1 - zx(1-x) - i\epsilon \right],
\]

where we have explicitly included the step function to enforce the condition that \( \text{Im} F(z) = 0 \) if \( z \leq 4 \). To evaluate the imaginary part of the logarithm, we employ the principal value of the complex-valued logarithm, with the branch cut taken along the negative real axis. In particular,
assuming that \( y \) is a non-zero real number and \( \epsilon \) is a positive infinitesimal,
\[ \ln(y - i\epsilon) = \ln|y| - i\pi \Theta(-y). \]
It follows that \( \text{Im} \ln(y - i\epsilon) = -\pi \Theta(-y) \). Employing this result in eq. (19),
\[ \text{Im} F(z) = -\Theta(z - 4)\pi \int_{x_-}^{x_+} dx = -\Theta(z - 4)\pi(x_+ - x_-) = -\Theta(z - 4)\pi \sqrt{1 - \frac{4}{z}}, \]
after using the explicit form for \( x_\pm \) given in eq. (18).

(b) Let \( \Gamma^{(4)} \) be the 1PI four-point function in a field theory of a real scalar field (with an interaction Lagrangian given by \( \mathcal{L}_{\text{int}} = -\lambda \phi^4/4! \)). Using the cutting rules given in Section 24.1.2 [pp. 456–459] of Schwartz, evaluate \( \text{Im} \Gamma^{(4)} \) up to order \( \lambda^2 \). Check your result by starting with the full \( \mathcal{O}(\lambda^2) \) expression for \( \Gamma^{(4)} \) obtained in class, and implementing the results of part (a).

In class, we computed the 1PI 4-point Green function in scalar field theory to one-loop order. The result of this computation was
\[ \Gamma^{(4)} = -\lambda - \frac{1}{2} i\lambda^2 \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{1}{q^2 - m^2 + i\epsilon} \frac{1}{(q - p_1 - p_2)^2 - m^2 + i\epsilon} + (p_2 \to p_3) + (p_2 \to p_4) \right\}, \]
where the second and third terms above in the integrand are given by the first term with the momentum substitutions indicated. The three terms shown correspond to the \( s \)-channel, \( t \)-channel and \( u \)-channel diagrams.

We shall focus first on the \( s \) channel diagram. We expect that the singularity structure in the complex \( s \) plane to have a branch point at the threshold for the \( 2 \to 2 \) scattering process at threshold, \( s = 4m^2 \), and a branch cut extending to \( \infty \) along the positive real axis.\(^2\)

By definition, the discontinuity of \( \Gamma^{(4)}(s) \) across the branch cut is
\[ \text{Disc} \Gamma^{(4)}(s) \equiv \Gamma^{(4)}(s + i\epsilon) - \Gamma^{(4)}(s - i\epsilon), \]
where \( \epsilon \) is a positive infinitesimal. The cutting rules state that \( \text{Disc} \Gamma^{(4)}(s) \) is obtained by cutting the Feynman diagram

\[ \begin{array}{c}
  \text{cut} \rightarrow 2 \pi i \delta(q^2 - m^2) \Theta(q_0).
\end{array} \]

\(^2\)Note that \( s = (p_1 + p_2)^2 = 2(m^2 + p_1 \cdot p_2) = 2(m^2 + E_1E_2 - \vec{p}_1 \cdot \vec{p}_2). \) At threshold, \( \vec{p}_1 = \vec{p}_2 = 0 \) and \( E_1 = E_2 = m \), which implies that \( s = 4m^2 \) at threshold.
The discontinuity $\text{Disc} \Gamma^{(4)}(s)$ is related to $\text{Im} \Gamma^{(4)}(s)$ as follows. First, we observe that

$$
\Gamma^{(4)}(s - i\epsilon) = \Gamma^{(4)}(s + i\epsilon)^*.
$$

(21)

This result follows from the principle of analytic continuation. In particular, if $f(z)$ is an analytic function in some region of the complex plane, then so is $f^*(z^*)$. If $f(z)$ is a real valued function in a region of the complex plane that includes part of the real axis, then $f(z) = f^*(z^*)$ along that part of the real axis (since $z = z^*$ on the real axis). Consequently, $f^*(z^*)$ and $f(z)$ are analytic continuations of one another. As long as no singularities are encountered, it follows that $f(z) = f^*(z^*)$, which implies that $f^*(z) = f(z^*)$. That is, we have proven the reflection principle of complex analysis.

**Theorem** (Reflection principle): If $f(z)$ is real and analytic on part of the real axis, then $f^*(z^*)$ at all points in the complex plane where $f(z)$ is analytic.

Applying the reflection principle to $\Gamma^{(4)}(s + i\epsilon)$ yields eq. (21). We can therefore conclude that

$$
\text{Disc} \Gamma^{(4)}(s) \equiv \Gamma^{(4)}(s + i\epsilon) - \Gamma^{(4)}(s + i\epsilon)^* = 2i \text{Im} \Gamma^{(4)}(s),
$$

where we have defined

$$
\Gamma^{(4)}(s) \equiv \lim_{\epsilon \to 0} \Gamma^{(4)}(s + i\epsilon) .
$$

The upshot of this discussion is that the cutting rules provide a method for computing $\text{Im} \Gamma^{(4)}(s)$.

Applying the cutting rules to the $s$-channel one-loop diagram (shown above),

$$
2i \text{Im} \Gamma^{(4)}(s) = -\frac{1}{2}i\lambda^2(-2\pi i)^2 \int \frac{d^4q}{(2\pi)^4} \delta(q^2 - m^2)\Theta(q_0)\delta\left((q - p_1 - p_2)^2 - m^2\right)\Theta(p_{10} + p_{20} - q_0).
$$

(22)

It should be noted that the form of the $\Theta$-function corresponds to placing a cut propagator line on mass shell. To evaluate the integral in eq. (22), note that

$$
\int d^4q \delta(q^2 - m^2)\Theta(q_0) = \int d^3q \, dq_0 \delta(q_0^2 - |\vec{q}|^2 - m^2)\Theta(q_0)
$$

$$
= \int d^3q \, dq_0 \frac{1}{2\sqrt{|\vec{q}|^2 + m^2}} \left[\delta(q_0 - \sqrt{|\vec{q}|^2 + m^2}) + \delta(q_0 + \sqrt{|\vec{q}|^2 + m^2})\right]\Theta(q_0)
$$

$$
= \int \frac{d^3q}{2\sqrt{|\vec{q}|^2 + m^2}}.
$$

It follows that

$$
\int \frac{d^4q}{(2\pi)^4} \delta(q^2 - m^2)\Theta(q_0)\delta\left((q - p_1 - p_2)^2 - m^2\right)\Theta(p_{10} + p_{20} - q_0)
$$

$$
= \frac{1}{(2\pi)^4} \int \frac{d^3q}{2\sqrt{|\vec{q}|^2 + m^2}} \delta\left((q - p_1 - p_2)^2 - m^2\right)\Theta(p_{10} + p_{20} - q_0) \bigg|_{q_0=\sqrt{|\vec{q}|^2 + m^2}}
$$

\footnote{A very nice discussion of this point can be found in Paul Roman, *Introduction to Quantum Field Theory* (John Wiley & Sons, Inc., New York, NY, 1969) pp. 440–441.}
which can be rewritten as

\[
\int \frac{d^4q}{(2\pi)^4} \delta(q^2 - m^2) \Theta(q_0) \delta((q - p_1 - p_2)^2 - m^2) \Theta(p_10 + p_20 - q_0)
\]

\[
= \frac{1}{(2\pi)^4} \int \frac{d^3q}{2\sqrt{|\vec{q}|^2 + m^2}} \delta(s - 2q \cdot (p_1 + p_2)) \Theta(p_10 + p_20 - q_0) \bigg|_{q_0 = \sqrt{|\vec{q}|^2 + m^2}},
\] (23)

after using \( s \equiv (p_1 + p_2)^2 \) and noting that \( q^2 - m^2 = 0 \) is equivalent to \( q_0 = \sqrt{|\vec{q}|^2 + m^2} \).

The simplest way to evaluate the integral above is to work in the center-of-mass frame of the system, where

\[
p_1 + p_2 = (\sqrt{s}; \vec{0}).
\]

In this case,

\[
2q \cdot (p_1 + p_2) \bigg|_{q_0 = \sqrt{|\vec{q}|^2 + m^2}} = 2\sqrt{s} \sqrt{|\vec{q}|^2 + m^2},
\]

and eq. (23) reduces to

\[
\int \frac{d^4q}{(2\pi)^4} \delta(q^2 - m^2) \Theta(q_0) \delta((q - p_1 - p_2)^2 - m^2) \Theta(p_10 + p_20 - q_0)
\]

\[
= \frac{1}{(2\pi)^4} \int \frac{d^3q}{2\sqrt{|\vec{q}|^2 + m^2}} \delta(s - 2\sqrt{s} \sqrt{|\vec{q}|^2 + m^2}) \Theta(s - \sqrt{|\vec{q}|^2 + m^2}).
\] (24)

The delta function enforces \( \sqrt{s} = 2\sqrt{|\vec{q}|^2 + m^2} \), which means that the argument of the \( \Theta \) function is positive so that \( \Theta(s - \sqrt{|\vec{q}|^2 + m^2}) = 1 \). Hence,

\[
\int \frac{d^4q}{(2\pi)^4} \delta(q^2 - m^2) \Theta(q_0) \delta((q - p_1 - p_2)^2 - m^2) \Theta(p_10 + p_20 - q_0)
\]

\[
= \frac{1}{(2\pi)^4 \sqrt{s}} \int d^3q \delta(s - 2\sqrt{s} \sqrt{|\vec{q}|^2 + m^2}).
\] (25)

To evaluate the above integral, use spherical coordinates, \( d^3q = |\vec{q}|^2 d|\vec{q}| d\Omega = 4\pi |\vec{q}|^2 d|\vec{q}| \), since there is no dependence on the direction of \( \vec{q} \) in the integrand above. It is convenient to change the integration variable to \( E \equiv \sqrt{|\vec{q}|^2 + m^2} \), in which case \( |\vec{q}| d|\vec{q}| = EdE \). It follows that

\[
\int d^3q \delta(s - 2\sqrt{s} \sqrt{|\vec{q}|^2 + m^2}) = 4\pi \int_m^\infty |\vec{q}| E dE \delta(s - 2\sqrt{s}E)
\]

\[
= \frac{2\pi}{\sqrt{s}} \int_m^\infty E(E^2 - m^2)^{1/2} \delta(E - \frac{1}{2}\sqrt{s}) dE
\]

\[
= \frac{\pi \sqrt{s}}{2} \left( 1 - \frac{4m^2}{s} \right)^{1/2} \Theta(\sqrt{s} - 2m).
\]

Note that the \( \Theta \)-function appears, since if \( \sqrt{s} < 2m \), then the argument of the delta function is never zero over the range of integration \( m \leq E < \infty \), in which case the delta function must be set to zero.
\[ \int \frac{d^4q}{(2\pi)^4} \delta(q^2-m^2)\Theta(q_0)\delta((q-p_1-p_2)^2-m^2)\Theta(p_{10}+p_{20}-q_0) = \frac{1}{32\pi^3} \left( 1 - \frac{4m^2}{s} \right)^{1/2} \Theta(\sqrt{s}-2m). \]  

We noted earlier that the \( \delta \)-function and \( \Theta \)-function conditions are satisfied only when \( s \geq 4m^2 \). This is true because the cut propagator lines are both on-shell, which means that to conserve both energy and three-momentum requires that \( s \geq 4m^2 \). This means that we can replace \( \Theta(\sqrt{s}-2m) \) above with \( \Theta(s-4m^2) \). In fact, our analysis in the center-of-mass frame implicitly assumed that \( s \) was positive since \( \sqrt{s} = p_{10} + p_{20} \) is real. Thus, our analysis above does not apply to the case of \( s < 0 \); in this latter case a second computation would be required. However, in practice we do not have to perform this second computation since the physical argument based on the mass-shell conditions imply that one cannot satisfy the \( \delta \)-function and \( \Theta \)-function conditions if \( s < 0 \). Hence, we can rewrite eq. (26) as

\[ \int \frac{d^4q}{(2\pi)^4} \delta(q^2-m^2)\Theta(q_0)\delta((q-p_1-p_2)^2-m^2)\Theta(p_{10}+p_{20}-q_0) = \frac{1}{32\pi^3} \left( 1 - \frac{4m^2}{s} \right)^{1/2} \Theta(s-4m^2). \]

Inserting this expression into eq. (22) yields our final result,

\[ \text{Im} \Gamma^{(4)}(s) = \frac{\lambda^2}{32\pi} \left( 1 - \frac{4m^2}{s} \right)^{1/2} \Theta(s-4m^2). \]  

It is instructive to compare eq. (27) with the explicit expression for \( \Gamma^{(4)} \) obtained in class,

\[ \Gamma^{(4)}(p_1, p_2, p_3, p_4) = -\lambda + \frac{\lambda^2}{32\pi} \left[ 3 \ln \left( \frac{m^2}{m^2} \right) - F \left( \frac{s}{m^2} \right) - F \left( \frac{t}{m^2} \right) - F \left( \frac{u}{m^2} \right) \right], \]

where the divergence has been absorbed by the counterterm. So far, we have only examined the \( s \)-channel piece of the above expression. Thus, we must compare eq. (27) with

\[ -\frac{\lambda^2}{32\pi} \text{Im} F \left( \frac{s}{m^2} \right) = \frac{\lambda^2}{32\pi^2} \left( 1 - \frac{4m^2}{s} \right)^{1/2} \Theta(s-4m^2), \]

after using eq. (20) from part (a). Indeed, we have reproduced the result of the cutting rules!

If we now include the \( t \)-channel and \( u \)-channel diagrams, it is clear that the only change in our analysis is to replace \( s \) with \( t \) and \( u \), respectively. Thus, applying the cutting rules or using the exact result, we conclude that

\[ \text{Im} \Gamma^{(4)}(p_1, p_2, p_3, p_4) = \frac{\lambda^2}{32\pi} \left[ \left( 1 - \frac{4m^2}{s} \right)^{1/2} \Theta(s-4m^2) \right. \]

\[ + \left. \left( 1 - \frac{4m^2}{t} \right)^{1/2} \Theta(t-4m^2) + \left( 1 - \frac{4m^2}{u} \right)^{1/2} \Theta(u-4m^2) \right]. \]

The physical region of scattering corresponds to \( s \geq 4m^2, t < 0 \) and \( u < 0 \). Thus, the last two terms on the right hand side do not survive in the physical scattering amplitude, in which case

\[ \text{Im} \Gamma^{(4)}(p_1, p_2, p_3, p_4) = \frac{\lambda^2}{32\pi} \left( 1 - \frac{4m^2}{s} \right)^{1/2}. \]
(c) Explain briefly when you expect the evaluation of a Feynman diagram to yield non-zero imaginary part.

The lesson of this problem is that Feynman diagrams exhibit non-zero imaginary parts when there exists at least one way to cut the diagram (i.e., put internal propagator lines on-shell) in which the cut lines represent a real physical process. As an example, when \( s \geq 4m^2 \), the diagram

\[
\begin{array}{c}
\text{p}_1 \\
\text{p}_2 \\
\text{p}_3 \\
\text{p}_4 \\
\end{array}
\begin{array}{c}
\text{q} \\
\text{p}_1 + \text{p}_2 - \text{q} \\
\end{array}
\]

represents a physical scattering of two on-shell scalar fields with four-momenta \( p_1 \) and \( p_2 \) into two on-shell scalar fields with momenta \( q \) and \( p_1 + p_2 - q \) followed by a physical scattering of two on-shell scalar fields with momenta \( q \) and \( p_1 + p_2 - q \) into on-shell two scalar fields with momenta \( p_3 \) and \( p_4 \). In these physical processes, the conservation of four-momentum and the mass-shell conditions are satisfied at each step.

3. The photon vacuum polarization function is defined to be:
\[
\Pi^{\mu\nu}(q) = (q^\mu q^\nu - g^{\mu\nu}q^2)\Pi(q^2).
\]
In class, we evaluated this function at one-loop in the \( \overline{\text{MS}} \) scheme. Consider a second scheme, called the on-shell scheme, in which we define \( \Pi(q^2 = 0) \equiv 0 \).

(a) Evaluate \( Z_3 \) in this scheme.

In class, we used the method of counterterms to derive the renormalized vacuum polarization in terms of the renormalized coupling \( e_R \) and mass \( m_R \). To simplify the typography, we shall henceforth drop the subscripts \( R \). The end result obtained in class was,
\[
\Pi(q^2) = \frac{2\alpha}{\pi}(4\pi)^\epsilon \Gamma(\epsilon) \int_0^1 dx \, x(1-x) \left[ \frac{m^2 - q^2 x(1-x)}{\mu^2} \right]^{-\epsilon} + Z_3 - 1
\]
\[
= \frac{\alpha}{3\pi} \left( \frac{1}{\epsilon} - \gamma + \ln 4\pi \right) - \frac{2\alpha}{\pi} \int_0^1 dx \, x(1-x) \ln \left[ \frac{m^2 - q^2 x(1-x)}{\mu^2} \right] + Z_3 - 1,
\]
after dropping terms of \( O(\epsilon) \) and higher. In the on-shell scheme, \( \Pi(q^2 = 0) = 0 \). That is
\[
\frac{\alpha}{3\pi} \left[ \frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln \left( \frac{m^2}{\mu^2} \right) \right] + Z_3 - 1 = 0.
\]
Solving for \( Z_3 \), we find
\[
Z_3 = 1 - \frac{\alpha}{3\pi} \left[ \frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln \left( \frac{m^2}{\mu^2} \right) \right].
\]
(b) Obtain asymptotic forms for $\Pi(q^2)$ in two limiting cases: (i) $q^2 \to 0$, and (ii) $q^2 \to \infty$.

Inserting the expression for $Z_3$ given in eq. (29) back into eq. (28) yields

$$\Pi(q^2) = -\frac{2\alpha}{\pi} \int_0^1 dx \frac{1}{x(1-x)} \ln \left[ \frac{m^2 - q^2 x(1-x)}{m^2} \right],$$

in the on-shell scheme.

Consider first the $q^2 \to 0$ limit. Expanding the logarithm,

$$\ln \left[ \frac{m^2 - q^2 x(1-x)}{m^2} \right] \simeq -\frac{q^2}{m^2} x(1-x).$$

Thus,

$$\Pi(q^2) \bigg|_{q^2 \to 0} \simeq \frac{2\alpha q^2}{\pi m^2} \int_0^1 x^2 (1-x)^2 \, dx = \frac{\alpha q^2}{15\pi m^2}. \quad (30)$$

Next, we consider the $q^2 \to \infty$ limit. In this case, we need to restore the positive infinitesimal $\epsilon$ (not to be confused with $\epsilon = 2 - \frac{1}{2}n$ of dimensional regularization) back into the argument of the logarithm using $m^2 \to m^2 - i\epsilon$. In the $q^2 \to \infty$ limit, we can drop the $m^2$ in the numerator of the argument of the logarithm, in which case,

$$\Pi(q^2) \bigg|_{q^2 \to \infty} = -\frac{2\alpha}{\pi} \int_0^1 dx \frac{1}{x(1-x)} \left\{ \ln \left( -\frac{q^2}{m^2} - i\epsilon \right) + \ln [x(1-x)] \right\}. $$

Performing the elementary integrations yields,

$$\Pi(q^2) \bigg|_{q^2 \to \infty} = -\frac{\alpha}{\pi} \ln \left( -\frac{q^2}{m^2} - i\epsilon \right) + \frac{5\alpha}{9\pi}. $$

(c) Using the $q^2 \to 0$ limit of part (b), compute the $O(\alpha)$ correction to the Coulomb potential. OPTIONAL: Compute the $O(\alpha)$ correction to the Coulomb potential without making the approximation of small $q^2$. Examine explicitly the limiting cases $m_e r \gg 1$ and $m_e r \ll 1$.

To find the correction to the Coulomb potential, consider the potential felt by an electron due to an infinitely heavy source of charge $Ze$. In this limit, if $p$ is the initial four-momentum and $p'$ is the final four-momentum, then the three-momentum is conserved but there is no energy transfer. Diagrammatically, we can represent this process by the interaction of an electron with a classical external source,
The kinematics of this process are:

\[ q = p' - p, \quad q_0 = 0, \quad q^2 = (q_0)^2 - |\vec{q}|^2 = -|\vec{q}|^2. \]

At tree-level, the matrix element is proportional to the propagator,

\[ \mathcal{M} \sim -\frac{Ze^2}{|\vec{q}|^2}. \]

Recalling the Born approximation of non-relativistic quantum mechanics,

\[ V(r) = -Ze^2 \int \frac{d^3q}{(2\pi^3)} \frac{e^{i\vec{q} \cdot \vec{r}}}{|\vec{q}|^2}. \]  

(31)

Here is a quick and dirty way to evaluate the above integral. Employing the identity,

\[ \nabla^2 \frac{1}{r} = -4\pi \delta^3(\vec{r}), \]

it follows that

\[ \frac{1}{r} = -4\pi \frac{1}{\nabla^2} \int \frac{d^3q}{(2\pi^3)} e^{i\vec{q} \cdot \vec{r}} = 4\pi \int \frac{d^3q}{(2\pi^3)} \frac{e^{i\vec{q} \cdot \vec{r}}}{|\vec{q}|^2}. \]

We conclude that

\[ V(r) = \frac{-Ze^2}{4\pi r}, \]

which is the well-known Coulomb potential.

We next examine the effects of vacuum polarization at one-loop. As shown in class, the photon propagator is modified as follows

\[ D_{\mu\nu}(q^2) = \frac{-i}{q^2[1 + \Pi(q^2)]} \left( g_{\mu\nu} - \frac{g_{\mu}q_{\nu}}{q^2} \right) - i\alpha \frac{q_{\mu}q_{\nu}}{q^4}. \]  

(32)

It is convenient to work in the Feynman gauge with \( a = 1 \), in which case,

\[ D_{\mu\nu}(q^2) = \frac{-ig_{\mu\nu}}{q^2[1 + \Pi(q^2)]}. \]

In the static limit (corresponding to the \( q^2 \to 0 \) limit of part (b), we make use of eq. (30) to obtain

\[ \mathcal{M} \sim -\frac{Ze^2}{|\vec{q}|^2} \left( 1 + \frac{\alpha|\vec{q}|^2}{15\pi m^2} \right)^{-1}, \]

after putting \( q^2 = -|\vec{q}|^2 \). In the static approximation, \( |\vec{q}| \to 0 \), and we can expand to first order in \( \vec{q} \),

\[ \mathcal{M} \sim -\frac{Ze^2}{|\vec{q}|^2} \left( 1 - \frac{\alpha|\vec{q}|^2}{15\pi m^2} \right). \]

Thus, eq. (31) is modified,

\[ V(r) = -Ze^2 \int \frac{d^3q}{(2\pi^3)^3} \frac{e^{i\vec{q} \cdot \vec{r}}}{|\vec{q}|^2} \left( 1 - \frac{\alpha|\vec{q}|^2}{15\pi m^2} \right) = -\frac{Ze^2}{4\pi r} - \frac{Ze^2\alpha}{15\pi m^2} \delta^3(\vec{r}). \]

This is the famous Uehling potential—the correction to the Coulomb potential of a heavy nucleus due to vacuum polarization.
Suppose we do not use the $|\vec{q}| \to 0$ limit of $\Pi(q^2)$. Then, in the static approximation,

$$M \sim -\frac{Ze^2}{|\vec{q}|^2} \left\{ 1 + \frac{2\alpha}{\pi} \int_0^1 dx \, x(1-x) \ln \left[ \frac{m^2 + |\vec{q}|^2 x(1-x)}{m^2} \right] \right\}.$$ 

To simplify the notation, for the rest of this calculation I shall denote $q \equiv |\vec{q}|$. Following our previous analysis,

$$V(r) = -Ze^2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{q} \cdot \vec{r}}}{q^2} \left\{ 1 + \frac{2\alpha}{\pi} \int_0^1 dx \, x(1-x) \ln \left[ \frac{m^2 + q^2 x(1-x)}{m^2} \right] \right\}.$$ 

Thus, we need to examine,

$$\int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{q} \cdot \vec{r}}}{q^2} \ln \left[ 1 + \frac{q^2}{m^2} x(1-x) \right]\ln \left[ 1 + \frac{q^2}{m^2} x(1-x) \right] \right\}.$$ 

Inserting this result into eq. (33),

$$V(r) = -\frac{Ze^2}{2\pi^2 r} \int_0^1 dq \frac{q \sin qr}{q} \left\{ 1 + \frac{2\alpha}{\pi} \int_0^1 dx \, x(1-x) \ln \left[ 1 + \frac{q^2}{m^2} x(1-x) \right] \right\}.$$ 

It will prove useful to change of variables, $x = \frac{1}{2}(1-y)$. Then, the resulting $y$-integration, which now goes from $y = -1$ to 1, is over an even function of $y$. Hence, we can take the limits of integration to go from $y = 0$ to 1 and multiply by 2. Thus,

$$\int_0^1 dx \, x(1-x) \ln \left[ 1 + \frac{q^2}{m^2} x(1-x) \right] = \frac{1}{2} \int_0^1 dy \, (1-y^2) \ln \left[ 1 + \frac{q^2(1-y^2)}{4m^2} \right]$$

$$= \frac{1}{2}q^2 \int_0^1 \frac{y^2(1-\frac{1}{3}y^2)dy}{4m^2 + q^2(1-y^2)}.$$ 

To achieve the last step, we integrated by parts by taking $u = \ln \left[ 1 + q^2(1-y^2)/(4m^2) \right]$ and $dv = (1-y^2)dy$. Using the above result in eq. (34),

$$V(r) = -\frac{Ze^2}{2\pi^2 r} \int_0^1 dq \frac{q \sin qr}{q} \left\{ 1 + \frac{aq^2}{\pi} \int_0^1 \frac{y^2(1-\frac{1}{3}y^2)dy}{4m^2 + q^2(1-y^2)} \right\}.$$ 

We can perform the integration over $q$ using

$$\int_0^\infty \frac{q \sin qr}{q^2 + a^2} dq = \frac{1}{2} \pi e^{-ar},$$

either using the calculus of residues or by consulting a good table of integrals.
We end up with
\[ V(r) = -\frac{Ze^2}{4\pi r} \left\{ 1 + \frac{\alpha}{\pi} \int_0^1 dy \frac{y^2(1 - \frac{1}{2}y^3)}{1 - y^2} \exp \left[ -\frac{2mr}{\sqrt{1 - y^2}} \right] \right\}. \]

We can rewrite the integral over \( y \) with another change of variables:
\[ u = \frac{1}{\sqrt{1 - y^2}}, \quad du = \frac{ydy}{(1 - y^2)^{3/2}} = u^2 \sqrt{u^2 - 1} \, du. \]

Then \( y = \sqrt{u^2 - 1}/u \) and we obtain
\[ V(r) = -\frac{Ze^2}{4\pi r} \left\{ 1 + \frac{2\alpha}{3\pi} \int_1^\infty du \frac{e^{-2mr}}{u^2} \left[ 1 + \frac{1}{2u^2} \frac{\sqrt{u^2 - 1}}{u^2} \right] \right\}. \]

The analysis of the limiting cases for \( mr \ll 1 \) and \( mr \gg 1 \) is described in V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii, *Quantum Electrodynamics* (Pergamon Press, Oxford, UK, 1980) pp. 504–508. In the two limiting cases,
\[ V(r) = -\frac{Ze^2}{4\pi r} \times \begin{cases} 
1 - \frac{2\alpha}{3\pi} \ln(mr) + \gamma + \frac{5}{6} + \cdots, & \text{for } mr \ll 1, \\
1 + \frac{\alpha}{4\sqrt{\pi}(mr)^{3/2}} e^{-2mr} + \cdots, & \text{for } mr \gg 1.
\end{cases} \]

(d) Show that the quantity:
\[ \alpha_{\text{eff}}(q^2) \equiv \frac{\alpha}{1 + \Pi(q^2)} \tag{35} \]

is independent of whether you evaluate this expression using bare or renormalized quantities. As a result, argue that \( \alpha_{\text{eff}}(q^2) \) is independent of renormalization scheme. Outline how you would relate the coupling constants defined in the \( \overline{\text{MS}} \) and on-shell schemes. Sketch a graph of \( \alpha_{\text{eff}}(-q^2) \) at one-loop, in the on-shell scheme, i.e. for negative values of the argument.

**NOTE:** In the on-shell scheme, \( \alpha_{\text{eff}}(0) \) is the fine structure constant, which is approximately equal to 1/137.

Consider the quantity defined in eq. (35), where \( \alpha \) and \( \Pi(q^2) \) are the bare coupling and vacuum polarization, respectively. Note that in part (a), the method of counterterms was used to obtain the renormalized vacuum polarization in terms of the renormalized coupling. For typographical simplicity, we omitted the subscript \( R \) on all relevant quantities. To distinguish between bare and renormalized quantities, we shall instead include a subscript \( B \) on bare quantities. Thus, \( \alpha_B \) will denote the bare coupling, \( \alpha_B = e^2_B/(4\pi) \), and \( \Pi_B(q^2) \) will be the vacuum polarization as computed with the original Lagrangian expressed in terms of bare fields, couplings and masses.

We therefore define
\[ \alpha_{\text{eff}}(q^2) \equiv \frac{\alpha_B}{1 + \Pi_B(q^2)}. \]
The relation between bare and renormalized quantities were obtained in class. In particular,

\[ A_B^\mu = Z_3^{1/2} A^\mu, \quad e_B = Z_1 Z_2^{-1} Z_3^{-1/2} \mu^\epsilon e, \quad a_B = Z_a a. \]  

(36)

Using of the Ward-Takahashi identity for gauge invariance to deduce that \( Z_1 = Z_2 \), it follows that \( e_B = Z_3^{-1/2} \mu^\epsilon e \), or in terms of \( \alpha \equiv \epsilon^2/(4\pi) \),

\[ \alpha_B = Z_3^{-1} \mu^{2\epsilon} \alpha. \]  

(37)

The connected 2-point Green function is

\[ \mathcal{D}_B^{\mu\nu}(x,y) = \langle \Omega | A_B^\mu(x) A_B^\nu(y) | \Omega \rangle_{\text{conn}} = Z_3 \langle \Omega | A^\mu(x) A^\nu(y) | \Omega \rangle_{\text{conn}} = Z_3 \mathcal{D}^{\mu\nu}(x,y). \]  

(38)

Eq. (32) applies to both the bare and the renormalized connected 2-point Green functions. Hence, it follows that

\[ \mathcal{D}_B^{\mu\nu}(q^2) = -i q^2 \left[ 1 + \Pi_B(q^2) \right] \left( g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) - i a_B \frac{q^\mu q^\nu}{q^4}, \]

\[ \mathcal{D}^{\mu\nu}(q^2) = -i q^2 \left[ 1 + \Pi(q^2) \right] \left( g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) - i a \frac{q^\mu q^\nu}{q^4}, \]

In light of eq. (38), it follows that \( Z_3 = Z_a \), which was a result previously noted in class. In addition, we conclude that

\[ \frac{1}{1 + \Pi_B(q^2)} = \frac{Z_3}{1 + \Pi(q^2)}. \]  

(39)

In the limit of \( \epsilon \to 0 \), eq. (37) yields \( Z_3 = \alpha/\alpha_B \), from which it follows that

\[ \frac{\alpha_B}{1 + \Pi_B(q^2)} = \frac{\alpha}{1 + \Pi(q^2)}. \]  

(40)

That is, the definition of \( \alpha_{\text{eff}} \) in eq. (35) does not depend on whether it is computed using bare quantities or renormalized quantities. Moreover, in deriving eq. (40), no specific renormalization scheme was imposed. Thus, \( \alpha_{\text{eff}} \) is renormalization scheme independent!

Thus, if we denote the minimal subtraction scheme by \( \overline{\text{MS}} \) and the on-shell scheme by \( \text{OS} \), then

\[ \alpha_{\text{eff}}(q^2) = \frac{\alpha_{\overline{\text{MS}}}}{1 + \Pi_{\overline{\text{MS}}}(q^2)} = \frac{\alpha_{\text{OS}}}{1 + \Pi_{\text{OS}}(q^2)}. \]  

(41)

In class, we derived

\[ \Pi(q^2) = -\frac{2\alpha_{\overline{\text{MS}}}}{\pi} \int_0^1 dx x(1-x) \ln \left[ \frac{m^2 - q^2 x(1-x)}{\mu^2} \right], \]

which should be compared with the one-loop vacuum polarization obtained in part (b),

\[ \Pi(q^2) = -\frac{2\alpha_{\text{OS}}}{\pi} \int_0^1 dx x(1-x) \ln \left[ \frac{m^2 - q^2 x(1-x)}{m^2} \right], \]
Employing eq. (41) to one-loop order, we can expand the denominators,

\[ \alpha_{\text{MS}}[1 - \Pi_{\text{MS}}(q^2)] = \alpha_{\text{OS}}[1 - \Pi_{\text{OS}}(q^2)] , \]

which then yields

\[
\alpha_{\text{MS}}\left\{ 1 + \frac{2\alpha_{\text{MS}}}{\pi} \int_0^1 dx \, x(1-x) \ln \left( \frac{m^2 - q^2 x(1-x)}{\mu^2} \right) \right\} = \alpha_{\text{OS}}\left\{ 1 + \frac{2\alpha_{\text{OS}}}{\pi} \int_0^1 dx \, x(1-x) \ln \left( \frac{m^2 - q^2 x(1-x)}{m^2} \right) \right\} . \tag{42}
\]

We can express \( \alpha_{\text{OS}} \) as a power series in \( \alpha_{\text{MS}} \),

\[ \alpha_{\text{OS}} = \alpha_{\text{MS}} + O(\alpha_{\text{MS}}^2) . \]

Then, eq. (42) yields

\[
\alpha_{\text{OS}} = \alpha_{\text{MS}}\left\{ 1 + \frac{2\alpha_{\text{MS}}}{\pi} \int_0^1 dx \, x(1-x) \ln \left( \frac{m^2 - q^2 x(1-x)}{\mu^2} \right) \right\} + O(\alpha_{\text{MS}}^3) \\
= \alpha_{\text{MS}}\left\{ 1 + \frac{\alpha_{\text{OS}}}{3\pi} \ln \left( \frac{m^2}{\mu^2} \right) \right\} + O(\alpha_{\text{MS}}^3) .
\]

In particular, in the one-loop approximation, we have

\[ \alpha_{\text{OS}} = \alpha_{\text{MS}}(\mu = m) . \tag{43} \]

We can sketch a graph of \( \alpha_{\text{eff}}(q^2) \). Note that for \( q^2 > 4m^2 \), \( \Pi(q^2) \) acquires an imaginary part due to the on-shell production of intermediate state \( e^+e^- \) pairs. Thus, to avoid this region, we shall examine \( \Pi(-q^2) \) which is a real function for positive values of \( q^2 \). Using the results of part (b),

\[ \text{As } q^2 \to 0, \quad \Pi_{\text{OS}}(-q^2) \to 0, \]
\[ \text{As } q^2 \to \infty, \quad \Pi_{\text{OS}}(-q^2) \to -\frac{\alpha_{\text{OS}}}{3\pi} \ln \left( \frac{q^2}{m^2} \right) . \tag{44} \]

Thus, \( \alpha_{\text{eff}}(-q^2) \) blows up when

\[ 1 - \frac{\alpha_{\text{OS}}}{3\pi} \ln \left( \frac{q^2}{m^2} \right) = 0 , \]

that is, when \( q^2 = -m^2 \exp(3\pi/\alpha_{\text{OS}}) \). A sketch of \( \alpha_{\text{eff}}/\alpha \) vs. \( -q^2/\Lambda^2 \) is shown at the top of the next page, where \( \alpha \equiv \alpha_{\text{eff}} \equiv \alpha_{\text{eff}}(0) \) and \( \Lambda \equiv -m^2 \exp(3\pi/\alpha_{\text{OS}}) \). Note that

\[ \alpha_{\text{eff}}(0) = \alpha_{\text{OS}} \approx 1/137 \tag{45} \]

is the standard definition of the QED coupling constant based on the Thomson limit.
(e) Calculate the numerical value of the momentum scale (in GeV units) where $\alpha_{\text{eff}}(-q^2)$ blows up.

As is evident from the above plot, $\alpha_{\text{eff}}(-q^2)$ blows up at $q^2 = -\Lambda^2 = m^2 \exp\left(3\pi/\alpha_{\text{OS}}\right)$. Using $\alpha_{\text{OS}} = 1/137$ and $m = m_e = 0.511 \times 10^{-3}$ GeV, we obtain

$$\Lambda = (0.511 \times 10^{-3} \text{ GeV}) e^{3\pi/137/2}.$$

Using

$$e^{3\pi/137/2} = 10^{[3\pi/137/2]/\ln 10} = 2.4 \times 10^{280}.$$

Hence,

$$\Lambda \approx 10^{277} \text{ GeV},$$

which is an incredibly large number (well beyond the Planck scale, $M_{\text{Pl}} = 10^{19}$ GeV, at which quantum gravitational effects become significant and QED surely must break down).

4. Consider QED coupled to a neutral scalar field:

$$\mathcal{L} = \mathcal{L}_{\text{QED}} + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 - g \bar{\psi} \gamma^{\mu} \gamma_{\nu} \psi \psi \phi. \quad (46)$$

(a) Compute the amplitude for the decay $\phi \rightarrow \gamma\gamma$, as a function of $m_e$, $m$, $g$, and $\alpha \equiv e^2/(4\pi)$, using perturbation theory at one-loop. Simplify your answer by invoking the kinematics of the problem, i.e. momentum conservation and the on-shell conditions for the external particles. Take care to consider two diagrams which differ only in the direction of flow of electric charge in the loop. Do you need to add a counterterm in order to remove an infinity? Explain.
Because there is no coupling of $\phi$ to two photons in the bare Lagrangian [cf. eq. (46)], there is no counterterm for the $\phi\gamma\gamma$ vertex. Thus, the renormalizability of the theory implies that the sum of all loop diagrams that contribute to $\phi \to \gamma\gamma$ must be finite.\footnote{Indeed, if a term $\phi F_{\mu\nu} F^{\mu\nu}$, which has mass-dimension 5, did appear in eq. (46) [such a term would then contribute at tree-level to $\phi \to \gamma\gamma$], the resulting theory would be non-renormalizable.}

There are two Feynman diagrams contributing to $\phi\gamma\gamma$ at one loop:

Diagrams (a) and (b) differ in that the outgoing photons are interchanged. Equivalently, one can say that in diagram (b) the flow of electric charge is opposite to that of diagram (a) [by rotating the triangle by 180° out of the plane at the scalar–fermion vertex].

Applying the Feynman rules, and recalling the minus sign for the closed fermion loop,

$$i\mathcal{M}_a = -\int \frac{d^4 q}{(2\pi)^4} \left[ \epsilon^\nu(k_1, \lambda_1) \epsilon^\mu(k_2, \lambda_2) \right]$$

where the factors of $i$ arise from the three fermion propagators. Next, $\mathcal{M}_b$ is obtained from $\mathcal{M}_a$ by interchanging $k_1 \to k_2$ and $\mu \to \nu$,

$$i\mathcal{M}_b = -\int \frac{d^4 q}{(2\pi)^4} \left[ \epsilon^\nu(k_1, \lambda_1) \epsilon^\mu(k_2, \lambda_2) \right]$$

We now evaluate the trace that appears in the numerator in eq. (47),

$$\text{Tr}[(\not{q} - \not{p} + m_e)\gamma^\nu(\not{q} - \not{k}_1 + m_e)\gamma^\mu(\not{q} + m_e)]$$

$$= m_e^3 \text{Tr}(\gamma^{\nu}\gamma^{\mu}) + m_e \left\{ \text{Tr}[(\not{q} - \not{p})\gamma^\nu(\not{q} - \not{k}_1)\gamma^\mu] + \text{Tr}[\gamma^\nu(\not{q} - \not{k}_1)\gamma^\mu] + \text{Tr}[(\not{q} - \not{p})\gamma^\nu(\not{q} - \not{k}_1)\gamma^\mu] \right\}$$

$$= 4m_e^3 g^{\mu\nu} + 4m_e \left\{ (q-p)^\mu(q-k_1)^\nu + (q-p)^\nu(q-k_1)^\mu - g^{\mu\nu}(q-p)(q-k_1) 

+ (q-p)^\nu q^\mu + g^{\mu\nu} q^\nu(q-p) - (q-p)^\mu q^\nu + (q-k_1)^\mu q^\nu - g^{\mu\nu} q^\nu(q-k_1) \right\}$$

$$= 4m_e \left\{ g^{\mu\nu} \left[ m_e^2 q^2 - 2q\cdot k_1 - p\cdot k_1 \right] + 4q^\mu q^\nu - 2q^\mu (k_1+p)^\nu - 2q^\nu k_1^\mu + p^\mu k_1^\nu + p^\nu k_1^\mu \right\}.$$
For the physical $\phi \to \gamma\gamma$ decay, we have $p^2 = m^2$ and $k_1^2 = 0$, where $m$ is the mass of the scalar particle. Then,

$$D = q^2 = 2q \cdot (px + k_1y) + m^2 x - m_e^2.$$ 

Hence,

$$\mathcal{M}_a = 8ie^2 g_{\mu\nu} \epsilon^*_\mu(k_1, \lambda_1) \epsilon^*_\nu(k_2, \lambda_2) \int_0^1 dx \int_0^{1-x} dy \int \frac{d^nx q}{(2\pi)^n} \frac{4g^\mu q^\nu - g^\mu q^2}{[q^2 - 2q \cdot (px + k_1y) + m^2 x - m_e^2]^3}.$$

It is convenient to isolate the numerator term that is quadratic in $q$, since this term yields a potential divergence. Let us write

$$\mathcal{M}_a = \mathcal{M}_a^{(1)\mu\nu} + \mathcal{M}_a^{(2)\mu\nu} \epsilon^*_\mu(k_1, \lambda_1) \epsilon^*_\nu(k_2, \lambda_2),$$

where

$$\mathcal{M}_a^{(1)\mu\nu} = 8ie^2 g_{\mu\nu} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^nx q}{(2\pi)^n} \frac{4g^\mu q^\nu - g^\mu q^2}{[q^2 - 2q \cdot (px + k_1y) + m^2 x - m_e^2]^3}.$$

$$\mathcal{M}_a^{(2)\mu\nu} = 8ie^2 g_{\mu\nu} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^nx q}{(2\pi)^n} \frac{4g^\mu(q^\nu - 2q^\nu q^\mu) + 2g^\mu k^\mu - 2q^\nu k^\nu + q^\nu k^\mu + q^\mu k^\nu}{[q^2 - 2q \cdot (px + k_1y) + m^2 x - m_e^2]^3}.$$

Using the formulae given in the dimensional regularization class handout,

$$\int \frac{d^nx q}{(2\pi)^n} \frac{4g^\mu q^\nu - g^\mu q^2}{[q^2 - 2q \cdot P - M^2 + i\epsilon]^3} = \frac{-i\Gamma(4\epsilon)(4\epsilon)^\epsilon}{32\pi^2} (P^2 + M^2)^{-1-\epsilon} [4\epsilon P^\mu P^\nu - \epsilon(2P^2 + M^2)g^\mu\nu],$$

where $\epsilon = 2 - \frac{1}{2} n$. Using $\epsilon\Gamma(\epsilon) = \Gamma(1 + \epsilon)$, we see that the integral is finite as $\epsilon \to 0$. Hence taking the $n \to 4$ limit,

$$\lim_{n \to 4} \int \frac{d^nx q}{(2\pi)^n} \frac{4g^\mu q^\nu - g^\mu q^2}{[q^2 - 2q \cdot P - M^2 + i\epsilon]^3} = \frac{-i}{32\pi^2 (P^2 + M^2)} [4P^\mu P^\nu - g^\mu\nu (2P^2 + M^2)].$$

In computing $\mathcal{M}_a^{(1)\mu\nu}$, we identify $P = px + k_1y$ and $M^2 = m^2 x - m_e^2$. Hence,

$$P^2 + M^2 = m_e^2 - m^2 x(1-x) + 2p \cdot k_1xy = m_e^2 - m^2 x(1-x - y).$$

At the final step above, we evaluated $p \cdot k_1$ using the kinematic constraints of the $\phi \to \gamma\gamma$ decay.\(^6\)

Hence,

$$\mathcal{M}_a^{(1)\mu\nu} = \frac{e^2 g_{\mu\nu}}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ -2g^\mu + \frac{4(px + k_1y)^\mu(px + k_1y)^\nu + g^\mu(m_e^2 - m^2 x)}{m_e^2 - m^2 x(1-x - y)} \right\}. \quad (49)$$

\(^6\)Since momentum conservation implies that $k_2 = p - k_1$, we have

$$0 = k_2^2 = (p - k_1)^2 = p^2 - 2p \cdot k_1 + k_1^2 = m^2 - 2p \cdot k_1,$$

after using $k_1^2 = k_2^2 = 0$ and $p^2 = m^2$. Hence, we conclude that $m^2 = 2p \cdot k_1$. 

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Further simplification can be achieved by using the properties of the photon polarization vectors, 

\[ k_1^\mu \epsilon_\mu(k_1, \lambda_1) = k_2^\nu \epsilon^\nu(k_2, \lambda_2) = 0. \]  

(50)

By writing \( p = k_1 + k_2 \) in the numerator of the integrand in eq. (49), we can then omit any terms proportional to \( k_1^\mu \) and/or \( k_2^\nu \). The end result is,

\[ M^{(1)\mu\nu}_a = \frac{e^2 g m_e}{4 \pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{g^{\mu\nu}[\frac{m_e^2 + m^2 x(1 - 2x - 2y)}{m_e^2 - m^2 x(1 - x - y)} + 4x(x + y)k_2^\mu k_1^\nu]}{m_e^2 - m^2 x(1 - x - y)}. \]  

(51)

To evaluate \( M^{(2)\mu\nu}_a \) we can set \( \epsilon \to 0 \) immediately, since the loop integral is manifestly finite. Using the formulae given in the dimensional regularization class handout,

\[ M^{(2)\mu\nu}_a = \frac{e^2 g m_e}{4 \pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{g^{\mu\nu}[m_e^2 - p \cdot k_1 + 2k_1 \cdot (px + k_1 y)] + p^\mu k_1^\nu + p^\nu k_1^\mu - 2(px + k_1 y)^\mu(k_1 + p)^\nu - 2(px + k_1 y)^\nu(k_1 + p)^\mu}{m_e^2 - m^2 x(1 - x - y)}. \]

We can simplify this result by imposing the kinematical constraints [cf. footnote 5],

\[ k_1^2 = k_2^2 = 0, \quad p^2 = m^2 = 2p \cdot k_1. \]

in addition, we put \( p + k_1 + k_2 \) and drop terms proportional to \( k_1^\mu \) and/or \( k_2^\nu \), as noted below eq. (50). The end result is,

\[ M^{(2)\mu\nu}_a = \frac{e^2 g m_e}{4 \pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{g^{\mu\nu}[\frac{m_e^2 + m^2 x(1 - 1/2)}{m_e^2 - m^2 x(1 - x - y)} + (1 - 4x)k_2^\mu k_1^\nu]}{m_e^2 - m^2 x(1 - x - y)}. \]  

(52)

Adding up eqs. (51) and (52) yields,

\[ M^{\mu\nu}_a = \frac{e^2 g m_e}{8 \pi^2} (m^g g^{\mu\nu} - 2k_1^\mu k_1^\nu) \int_0^1 dx \int_0^{1-x} dy \frac{4x(1 - x - y) - 1}{m_e^2 - m^2 x(1 - x - y)}. \]

We can immediately write down the result for \( M^{\mu\nu}_a \) by interchanging \( k_1 \leftrightarrow k_2 \) and \( \mu \leftrightarrow \nu \). It immediately follows that \( M^{\mu\nu}_a = M^{\nu\mu}_b \). Hence, the sum of the amplitudes resulting from the two contributing one-loop Feynman diagrams is

\[ M = \frac{\alpha g m_e}{\pi} (m^g g^{\mu\nu} - 2k_1^\mu k_1^\nu) \epsilon^*_\mu(k_1, \lambda_1) \epsilon^*_\nu(k_2, \lambda_2) \int_0^1 dx \int_0^{1-x} dy \frac{4x(1 - x - y) - 1}{m_e^2 - m^2 x(1 - x - y)}, \]

after writing \( \alpha \equiv e^2/(4\pi) \). As advertised, the amplitude is manifestly finite, and no counterterm is required.

(b) Denote the amplitude for the scalar decay by \( M_{\mu\nu} \), where \( \mu \) and \( \nu \) are the photon Lorentz indices. Gauge invariance implies that \( k_1^\mu M_{\mu\nu} = k_2^\nu M_{\mu\nu} = 0 \), where \( k_1 \) and \( k_2 \) are the respective photon momenta. Does your amplitude of part (a) respect this requirement?
The result from part (a) yields
\[ M_{\mu\nu} = \frac{\alpha g_m}{\pi} (m^2 g_{\mu\nu} - 2k_2 \cdot k_{1\nu}) \int_0^1 dx \int_0^{1-x} dy \frac{4x(1-x-y) - 1}{m_e^2 - m^2 x(1-x-y)}, \]
(53)
It is straightforward to verify that \( k_1^\mu M_{\mu\nu} = k_2^\nu M_{\mu\nu} = 0 \). For example,
\[ k_1^\mu (m^2 g_{\mu\nu} - 2k_2 \cdot k_{1\nu}) = (m^2 - 2k_1 \cdot k_2) k_{1\nu} = 0, \]
after noting that
\[ 2k_1 \cdot k_2 = (k_1 + k_2)^2 - k_1^2 - k_2^2 = p^2 = m^2, \]
where we have used \( p = k_1 + k_2 \) and \( k_1^2 = k_2^2 = 0 \). Likewise,
\[ k_2^\nu (m^2 g_{\mu\nu} - 2k_2 \cdot k_{1\nu}) = (m^2 - 2k_1 \cdot k_2) k_{2\mu} = 0, \]
(c) Work out all integrals explicitly and evaluate the imaginary part of \( M_{\mu\nu} \). For what range of \( m_e/m \) is the amplitude purely real? Explain the physical significance of the non-zero imaginary part.

**HINT:** You may find the following integral useful:
\[ \int_0^1 \frac{dy}{y} \log [1 - 4Ay(1 - y)] = -2 \left( \sin^{-1} \sqrt{A} \right)^2, \]
for \( 0 \leq A \leq 1 \). For values of \( A \) outside this region, you may analytically continue the above result. The imaginary part of this integral is easily computed once the \( i\epsilon \) factor is restored in the argument of the logarithm.

We examine the integral,
\[ I = \int_0^1 dx \int_0^{1-x} dy \frac{4x(1-x-y) - 1}{1 - Rx(1-x-y)}, \]
(54)
where \( R \equiv m^2/m_e^2 \). Rewrite the numerator as
\[ 4x(1-x-y) - 1 = \frac{4[Rx(1-x-y) - 1] + 4 - R}{R}. \]
Then,
\[ I = -\frac{4}{R} \int_0^1 dx \int_0^{1-x} dy + \frac{4 - R}{R} \int_0^1 dx \int_0^{1-x} dy \frac{1}{1 - Rx(1-x-y)} \]
\[ = -\frac{2}{R} + \frac{R - 4}{R^2} \int_0^1 \frac{dx}{x} \ln [1 - Rx(1-x)]. \]
(55)
Thus, we must now evaluate

\[ J \equiv \int_0^1 \frac{dx}{x} \ln[1 - Rx(1 - x)]. \]

Using the hint provided,

\[ J = -2 \left[ \sin^{-1} \left( \frac{1}{2} \sqrt{R} \right) \right]^2, \quad \text{for } 0 \leq R \leq 4. \]

To analytically continue beyond \( R = 4 \), we make use of

\[ \sin^{-1} z = -i \ln \left[ iz + \sqrt{1 - z^2} \right] \]

For \( z > 1 \), we have \( \sqrt{1 - z^2} = \pm i \sqrt{z^2 - 1} \), where the sign ambiguity will be addressed shortly. Then,

\[
\begin{align*}
\sin^{-1} z &= -i \ln \left[ i(z \pm \sqrt{z^2 - 1}) \right] \\
&= -i \left[ \frac{1}{2} i \pi + \ln(z \pm \sqrt{z^2 - 1}) \right] = \frac{1}{2} \pi \mp i \ln(z \pm \sqrt{z^2 - 1}).
\end{align*}
\]

To obtain the final result above, we used the fact that

\[ z - \sqrt{z^2 - 1} = \frac{1}{z + \sqrt{z^2 - 1}}, \]

which implies that

\[ \ln(z - \sqrt{z^2 - 1}) = -\ln(z + \sqrt{z^2 - 1}). \]

To resolve the sign ambiguity, we shall compute \( \text{Im} J \) directly following the procedure of Problem 2. Here, we will need to put back the factor of \( i \epsilon \) by replacing \( m_\epsilon^2 \to m_\epsilon^2 - i \epsilon \). Since \( R \equiv m^2/m_\epsilon^2 \), this means that we should replace \( R \to R + i \epsilon \). Noting that \( x(1-x) > 0 \) for \( 0 < x < 1 \), we examine,

\[ J \equiv \int_0^1 \frac{dx}{x} \ln[1 - Rx(1 - x) - i \epsilon]. \]

Following eqs. (18) and (20), the roots of the argument of the logarithm are given by

\[ x_\pm = \frac{1}{2} \left[ 1 \pm \sqrt{1 - \frac{4}{R}} \right]. \quad (56) \]

Thus,

\[
\begin{align*}
\text{Im } J &= \Theta(R - 4) \int_{x_-}^{x_+} \frac{dx}{x} \text{Im} \ln[1 - Rx(1 - x) - i \epsilon] \\
&= -\Theta(R - 4) \pi \int_{x_-}^{x_+} \frac{dx}{x} \\
&= -\Theta(R - 4) \pi \ln \left( \frac{x_+}{x_-} \right) = -\Theta(R - 4) \pi \ln \left( \frac{1 + \sqrt{1 - \frac{4}{R}}}{1 - \sqrt{1 - \frac{4}{R}}} \right). \quad (57)
\end{align*}
\]
It follows that for $R > 4$, the correct analytic continuation is

$$J = -2 \left[ \frac{1}{2} \pi + i \ln \left( \frac{\sqrt{R}}{2} + \sqrt{\frac{R}{4} - 1} \right) \right]^2.$$  

We check this by computing $\text{Im} J$,

$$\text{Im} J = -2 \pi \ln \left( \frac{\sqrt{R}}{2} + \sqrt{\frac{R}{4} - 1} \right) = -\pi \ln \left( \frac{\sqrt{R}}{2} + \sqrt{\frac{R}{4} - 1} \right)^2 = -\pi \ln \left( \frac{1 + \sqrt{1 - \frac{4}{R}}}{1 - \sqrt{1 - \frac{4}{R}}} \right)$$

in agreement with eq. (57). We can thus rewrite $J$ in the following form,

$$J = \begin{cases} 
-2 \left[ \sin^{-1} \left( \frac{1}{2} \sqrt{R} \right) \right]^2, & \text{for } 0 \leq R \leq 4 \\
-\frac{1}{2} \left[ \pi + i \ln \left( \frac{1 + \sqrt{1 - \frac{4}{R}}}{1 - \sqrt{1 - \frac{4}{R}}} \right) \right]^2, & \text{for } R > 4.
\end{cases}$$

It is convenient to introduce a function $f(R)$ defined by

$$f(R) = \begin{cases} 
\sin^{-1} \left( \frac{1}{2} \sqrt{R} \right), & \text{for } 0 \leq R \leq 4 \\
\frac{1}{2} \left[ \pi + i \ln \left( \frac{1 + \sqrt{1 - \frac{4}{R}}}{1 - \sqrt{1 - \frac{4}{R}}} \right) \right], & \text{for } R > 4.
\end{cases}$$

Then $J = -2[F(R)]^2$, and eq. (55) yields,

$$I = -\frac{2}{R} \left\{ 1 + \left( 1 - \frac{4}{R} \right) [f(R)]^2 \right\}.$$  

In light of eq. (54), we see that eq. (53) yields

$$\mathcal{M}_{\mu \nu} = -\frac{2\alpha m_e}{\pi m^2} \left( m^2 g_{\mu \nu} - 2k_2 \epsilon_{1 \nu} \right) \left\{ 1 + \left( 1 - \frac{4}{R} \right) [f(R)]^2 \right\}.$$  

In particular,

$$\text{Im} \mathcal{M}_{\mu \nu} = -\frac{\alpha m_e}{\pi m^2} \left( m^2 g_{\mu \nu} - 2k_2 \epsilon_{1 \nu} \right) \left( 1 - \frac{4}{R} \right) \ln \left( \frac{1 + \sqrt{1 - \frac{4}{R}}}{1 - \sqrt{1 - \frac{4}{R}}} \right) \Theta(R - 4).$$

Thus, $\text{Im} \mathcal{M}_{\mu \nu} \neq 0$ when $R = m^2/m^2_e > 4$, which corresponds to $m > 2m_e$. In this case, the kinematics allows the scalar particle to decay into an $e^+e^-$ pair. Thus, we can cut the triangle diagrams to reveal the on-shell electron and positron. By the cutting rules, $\text{Disc} \text{Im} \mathcal{M}_{\mu \nu} \neq 0$, and we expect a non-zero imaginary part.
(d) Evaluate the leading behavior of $\mathcal{M}_{\mu\nu}$ in the limit of $m_e \to \infty$.

The limit of $m_e \to \infty$ corresponds to $R = m^2/m_e^2 \to 0$. Using eq. (58), in the limit of $R \to 0$,

$$1 + \left(1 - \frac{4}{R}\right) \left[f(R)\right]^2 = 1 + \left(1 - \frac{4}{R}\right) \left[\sin^{-1}\left(\frac{1}{2}\sqrt{R}\right)\right]^2$$

$$\approx 1 + \left(1 - \frac{4}{R}\right) \left[\frac{\sqrt{R}}{2} + \frac{1}{6} \left(\frac{\sqrt{R}}{2}\right)^3\right]^2$$

$$\approx 1 + \left(1 - \frac{4}{R}\right) \frac{R}{4} \left(1 + \frac{R}{24}\right)^2$$

$$\approx 1 + \left(\frac{R}{4} - 1\right) \left(1 + \frac{R}{12}\right)$$

$$\approx \frac{R}{6}.$$  

Hence, eq. (59) yields

$$\mathcal{M}_{\mu\nu}(\phi \to \gamma\gamma)\bigg|_{m_e \to \infty} = -\frac{\alpha g}{3\pi m_e} \left(m^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\right).$$

This is an example of the decoupling theorem, which states that the effects of very massive particles in internal loops of Feynman diagrams yield contributions to the corresponding amplitude that vanish in the infinite mass limit. The decoupling theorem relies on an assumption that it is possible to take the infinite mass limit while holding all coupling constants of the theory fixed. This assumption is valid in the present application, where it is possible to take $m_e \to \infty$ while holding $e$ and $g$ fixed.

5. In QED, the renormalization group functions are:

$$\beta(e) = \mu \frac{de_R}{d\mu},$$

$$\delta(e) = \mu \frac{da_R}{d\mu},$$

$$m_R \gamma_m(e) = \mu \frac{dm_R}{d\mu},$$

$$\gamma_i(e) = \frac{1}{2} \mu \frac{\partial}{\partial\mu} \ln Z_i \quad (i = 2, 3).$$

(a) Compute $\beta(e)$, $\delta(e)$, $\gamma_m(e)$, and $\gamma_i(e)$ in the one-loop approximation, using the MS-renormalization scheme.
In eq. (36), we noted that
\[ e = \mu e Z_1 Z_2^{-1} Z_3^{-1/2} e_R , \]
where in this problem we use the subscript \( R \) to denoted renormalized parameters, and quantities without subscripts to be bare parameters. Using of the Ward-Takahashi identity for gauge invariance to deduce that \( Z_1 = Z_2 \), it follows that
\[ e = Z_3^{-1/2} \mu e_R . \]
The bare parameters are independent of \( \mu \). Hence,
\[ 0 = \mu \frac{d e}{d \mu} = \mu \frac{d}{d \mu} \left( Z_3^{-1/2} \mu e_R \right) . \]
In the MS renormalization scheme,
\[ Z_3 = 1 + \sum_{k=1}^{\infty} a_k(e_R) \frac{1}{\epsilon^k} . \] (60)
Using the chain rule of differentiation,
\[ \epsilon e_R Z_3^{-1/2} + \beta(e_R, \epsilon) \left( \frac{d}{d \epsilon} \frac{1}{2} Z_3^{-1/2} \right) = 0 , \]
where
\[ \beta(e_R, \epsilon) \equiv \mu \frac{d e_R}{d \mu} . \] (61)
Noting that
\[ \frac{d Z_3^{-1/2}}{d e_R} = -\frac{1}{2} Z_3^{-3/2} \frac{d Z_3}{d e_R} , \]
it follows that
\[ \left[ \beta(e_R, \epsilon) + \epsilon e_R - \frac{1}{2} e_R \beta(e_R, \epsilon) Z_3^{-1} \frac{d}{d e_R} \right] Z_3 = 0 . \] (62)
Inserting the expansion of \( Z_3 \) given in eq. (60), it follows that a solution which is consistent with the \( 1/\epsilon \) expansion of \( Z_3 \) is
\[ \beta(e_R, \epsilon) = -\epsilon e_R + \beta(e_R) , \] (63)
where \( \beta(e_R) \) is independent of \( \epsilon \). Note that
\[ \beta(e_R) = \lim_{\epsilon \to 0} \beta(e_R, \epsilon) . \] (64)
Eq. (62) can therefore be written as
\[ \left[ \beta(e_R) - \frac{1}{2} e_R \beta(e_R) Z_3^{-1} \frac{d}{d e_R} + \frac{1}{2} \epsilon e_R^2 Z_3^{-1} \frac{d}{d e_R} \right] Z_3 = 0 . \]
Inserting eq. (60), and performing a formal expansion in $1/\epsilon$, we deduce that all coefficients of $1/\epsilon^k$ should vanish. Of particular interest to us here is the coefficient corresponding to $k = 1$. For this equation, we may take $Z_3^{-1} = 1$, in which case,

$$\beta(e_R) + \frac{1}{2}e_R^2\frac{da_1}{de_R} = 0. \quad (65)$$

Using eq. (29), it follows that in the one-loop approximation in the MS scheme,

$$Z_3 = 1 - \frac{\alpha}{3\pi \epsilon} = 1 - \frac{e_R^2}{12\pi^2 \epsilon}. \quad (66)$$

That is, $a_1 = -e_R^2/(12\pi^2)$. Then, eq. (65) yields

$$\beta(e_R) = \frac{e_R^3}{12\pi^2}. \quad (67)$$

Next we compute $\gamma_m$. The starting point is

$$m = Z_m m_R. \quad \text{(68)}$$

Again, we note that the bare mass is independent of $\mu$. Hence,

$$0 = \mu \frac{d m}{d \mu} = \mu \frac{d}{d \mu} (Z_m m_R) = m_R \mu \frac{d Z_m}{d \mu} + Z_m \mu \frac{d m_R}{d \mu}. \quad \text{(69)}$$

By definition,

$$m_R \gamma_m(e_R) = \mu \frac{d m_R}{d \mu}. \quad \text{(70)}$$

Thus, using the chain rule, we can write

$$\mu \frac{d e_R}{d \mu} \frac{d Z_m}{d e_R} + \gamma_m(e_R) Z_m = 0. \quad \text{(71)}$$

Using eqs. (61) and (63),

$$[\beta(e_R) - e \epsilon_R] \frac{d Z_m}{d e_R} + \gamma_m(e_R) Z_m = 0. \quad (72)$$

In the MS renormalization scheme,

$$Z_m = 1 + \sum_{k=1}^{\infty} \frac{b_k(e_R)}{e^k}. \quad (73)$$

Inserting this expansion into eq. (72), we can extract the equation corresponding to $k = 1$,

$$\gamma_m(e_R) - e_R \frac{d b_1}{d e_R} = 0. \quad (74)$$
In class, we computed $Z_m$ in the one-loop approximation in the MS scheme,

$$Z_m = 1 - \frac{3\alpha}{4\pi\epsilon} = 1 - \frac{3e_R^2}{16\pi^2\epsilon}.$$  

That is, $b_1 = -3e_R^2/(16\pi^2)$. Then, eq. (70) yields

$$\gamma_m(e_R) = -\frac{3e_R^2}{8\pi^2}.$$  

Next, we present the one-loop computation of

$$\gamma_i(e_R) \equiv \frac{1}{2} \mu \frac{d}{d\mu} \ln Z_i = \frac{1}{2} Z_i^{-1} \mu \frac{dZ_i}{d\mu}, \quad \text{for } i = 2, 3. \tag{71}$$

In the MS renormalization scheme,

$$Z_2 = 1 + \sum_{k=1}^{\infty} c_k(e_R) \frac{e^k}{e^k}. \tag{72}$$

Thus, using the chain rule,

$$\gamma_2(e_R) = \frac{1}{2} Z_2 \frac{d e_R}{d e_R} \frac{d Z_2}{d e_R}. \tag{73}$$

Using eqs. (61) and (63),

$$\gamma_2(e_R) = \frac{1}{2} Z_2 \left[ \beta(e_R) - \epsilon e_R \right] \frac{d Z_2}{d e_R}. \tag{74}$$

Inserting eq. (72) into eq. (73), we can extract the equation corresponding to $k = 1$ by setting $Z_2 = 1$,

$$\gamma_2(e_R) = -\frac{1}{2} e_R \frac{d c_1}{d e_R}. \tag{75}$$

A similar analysis yields

$$\gamma_3(e_R) = -\frac{1}{2} e_R \frac{d a_1}{d e_R}. \tag{76}$$

In class, we computed $Z_2$ in the one-loop approximation in the MS scheme,

$$Z_2 = 1 - \frac{\alpha}{4\pi\epsilon} = 1 - \frac{e_R^2}{16\pi^2\epsilon}.$$  

That is, $c_1 = -e_R^2/(16\pi^2)$. Then, eq. (70) yields

$$\gamma_2(e_R) = \frac{e_R^2}{16\pi^2}. \tag{77}$$

Likewise, using $a_1 = -e_R^2/(12\pi^2)$ [as noted below eq. (66)],

$$\gamma_3(e_R) = \frac{e_R^2}{12\pi^2}. \tag{78}$$

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Finally, we compute $\delta$. The starting point is

$$a = Z_a a_R = Z_3 a_R,$$

where we have employed the identity $Z_a = Z_3$ previously noted above eq. (39). Hence, following the well known procedure,

$$0 = \mu \frac{da}{d\mu} = \mu \frac{d}{d\mu}(Z_3 a_R) = \mu a_R \frac{dZ_3}{d\mu} + Z_3 \mu \frac{da_R}{d\mu}.$$

By definition,

$$\delta(e_R) = \mu \frac{da_R}{d\mu}.$$

Thus, it follows that

$$\mu a_R \frac{dZ_3}{d\mu} + \delta(e_R) Z_3 = 0.$$

Solving for $\delta(e_R)$,

$$\delta(e_R) = -\mu a_R \frac{d}{d\mu} \ln Z_3 = -2a_R \gamma_3(e_R),$$

after using eq. (71) for $\gamma_3(e_R)$. Using eq. (76), it follows that in the one-loop approximation in the MS renormalization scheme,

$$\delta(e_R) = -\frac{a_R e_R^2}{6\pi^2}.$$

This completes the one-loop calculation of the renormalization group functions of QED in the MS renormalization scheme.

(b) The running coupling constant in QED can be written as:

$$\bar{\alpha}(Q) = \frac{3\pi}{\ln(\Lambda^2/Q^2)}, \tag{77}$$

in the one loop approximation. Using the boundary condition $\bar{\alpha}(\mu) \equiv e_R^2/4\pi$, express $\Lambda$ in terms of $\mu$ and $e_R$. Show that $\Lambda$ is a renormalization group invariant, that is:

$$\mu \frac{d\Lambda}{d\mu} = 0.$$

Evaluate $\Lambda$ numerically.

In class, we showed that the running coupling constant of QED in the one-loop approximation was given by

$$\bar{\alpha}(Q) = \frac{\alpha_R}{1 - \frac{2\alpha_R}{3\pi} \ln \left( \frac{Q}{\mu} \right)},$$

where $\alpha_R \equiv \bar{\alpha}(\mu)$. Comparing this with eq. (77), it follows that

$$\frac{2}{3\pi} \ln \left( \frac{\Lambda}{Q} \right) = \frac{1}{\alpha_R} - \frac{2}{3\pi} \ln \left( \frac{Q}{\mu} \right).$$
Simplify this expression yields
\[ \frac{2}{3\pi} \ln \left( \frac{\Lambda}{\mu} \right) = \frac{1}{\alpha_R}. \]
Hence,
\[ \Lambda = \mu \exp \left( \frac{3\pi}{2\alpha_R} \right). \]
To show that \( \lambda \) is formally independent of \( \mu \), we evaluate,
\[ \mu \frac{d\Lambda}{d\mu} = \mu \exp \left( \frac{3\pi}{2\alpha_R} \right) \left[ 1 + \frac{3}{2\pi} \frac{d\alpha_R^{-1}}{d\mu} \right]. \]  
(78)
However, note that
\[ \mu \frac{d\alpha_R^{-1}}{d\mu} = 4\pi \mu \frac{d}{d\mu} \left( \frac{1}{e_R^3} \right) = -8\pi \mu \frac{de_R}{e_R^3} d\mu = -8\pi \beta(e_R) \frac{e_R^3}{e_R^3} = -\frac{2}{3\pi}, \]
where we have used eqs. (62) and (64), and have employed the one-loop approximation for the \( \beta \)-function given in eq. (67). Inserting this result back in eq. (78), we end up with
\[ \mu \frac{d\Lambda}{d\mu} = 0. \]
Since \( \Lambda \) is independent of \( \mu \), we conclude that it is a physically measurable observable of QED. To see what its numerical value, recall that eqs. (43) and (45) imply that \( \alpha(m_e) \simeq 1/137 \). It follows that
\[ \Lambda = m_e \exp \left( \frac{3}{2\pi} \cdot 137 \right) \simeq 10^{277} \text{ GeV}. \]
This is precisely the same \( \Lambda \) that we computed in part (e) of problem 3. This is called the Landau pole (in the one-loop approximation) where the QED running coupling constant blows up.

**ADDITIONAL REMARKS:**

The definition of \( \Lambda \) above is based on the one-loop approximation. In fact, it is not difficult to define a \( \mu \)-independent \( \Lambda \) to all orders in perturbation theory. We begin with the formal definition of the running coupling,
\[ s \frac{\partial \tau(s)}{\partial s} = \beta(\tau(s)), \quad \text{where } \tau(s = 1) = e_R. \]
Integrating this equation and putting \( s = Q/\mu \),
\[ \ln \left( \frac{Q}{\mu} \right) = \int_{e_R}^{\tau(Q)} \frac{de}{\beta(e)}. \]  
(79)
Let us define the indefinite integral
\[ G(e) \equiv \int_{e}^{e'} \frac{de'}{\beta(e')} . \]
Then, eq. (79) can be rewritten as

$$\ln \left( \frac{Q}{\mu} \right) = G(\overline{\rho}(Q)) - G(e_R) .$$

(80)

We now define $\Lambda$ via the equation

$$\ln \left( \frac{\Lambda}{\mu} \right) = -G(e_R) .$$

Inserting this result back into eq. (80) yields

$$\ln \left( \frac{Q}{\Lambda} \right) = G(\overline{\rho}(Q)) .$$

(81)

No perturbation approximation has been made here. Moreover, $\Lambda$ defined via eq. (81) is explicitly independent of $\mu$. Finally, it is straightforward to check that in the one-loop approximation, we recover our previous results.

(c) Find the relation between the $\overline{\text{MS}}$ mass parameter, $m_R$, and the physical electron mass $m_e$ (i.e., the pole mass) in the one-loop approximation.

In class, we derived

$$\Sigma(p) = -\frac{1}{p} \left\{ Z_2 - 1 + \frac{\alpha_R}{2\pi} (4\pi)^\epsilon \Gamma(\epsilon)(1 - \epsilon) \int_0^1 dx \,(1 - x)x^{-\epsilon} \left[ \frac{m_R^2 - p^2(1 - x)}{\mu^2} \right]^{-\epsilon} \right\}$$

$$+ m_R \left\{ Z_m Z_2 - 1 + \frac{\alpha_R}{2\pi} (4\pi)^\epsilon \Gamma(\epsilon)(2 - \epsilon) \int_0^1 dx \, x^{-\epsilon} \left[ \frac{m_R^2 - p^2(1 - x)}{\mu^2} \right]^{-\epsilon} \right\} .$$

(82)

In the $\overline{\text{MS}}$ renormalization scheme,

$$Z_2 = 1 - \frac{\alpha_R}{4\pi} (4\pi)^\epsilon \Gamma(\epsilon) ,$$

$$Z_2 Z_m = 1 - \frac{\alpha_R}{4\pi} (4\pi)^\epsilon \Gamma(\epsilon) .$$

Inserting these results back into eq. (82)

$$\Sigma(p) = -\frac{1}{p} \frac{\alpha_R}{4\pi} (4\pi)^\epsilon \Gamma(\epsilon)(A - 1) + \frac{m_R \alpha_R}{\pi} \frac{1}{(4\pi)^\epsilon} \Gamma(\epsilon)(B - 1) ,$$

(83)

where $A$ and $B$ are the following loop integrals,

$$A \equiv 2(1 - \epsilon) \int_0^1 dx \,(1 - x)x^{-\epsilon} \left[ \frac{m_R^2 - p^2(1 - x)}{\mu^2} \right]^{-\epsilon}$$

$$B \equiv (1 - \frac{1}{2\epsilon}) \int_0^1 dx \, x^{-\epsilon} \left[ \frac{m_R^2 - p^2(1 - x)}{\mu^2} \right]^{-\epsilon} .$$
Expanding about $\epsilon = 0$,

\[
A = 1 - \epsilon \left\{ 1 + 2 \int_0^1 (1-x) \ln x \, dx + 2 \int_0^1 (1-x) \ln \left[ \frac{m_R^2 - p^2(1-x)}{\mu^2} \right] \, dx \right\} + O(\epsilon^2),
\]

\[
B = 1 - \epsilon \left\{ \frac{1}{2} + \int_0^1 \ln x \, dx + \int_0^1 \ln \left[ \frac{m_R^2 - p^2(1-x)}{\mu^2} \right] \, dx \right\} + O(\epsilon^2).
\]

We record below the relevant integrals:

\[
\int_0^1 (1-x) \ln x = -\frac{3}{4},
\]

\[
\int_0^1 \ln x \, dx = -1,
\]

\[
\int_0^1 \ln \left[ \frac{m_R^2 - p^2(1-x)}{\mu^2} \right] \, dx = \frac{m_R^2}{p^2} \ln \left( \frac{m_R^2}{\mu^2} \right) + \left( 1 - \frac{m_R^2}{p^2} \right) \ln \left( \frac{m_R^2 - p^2}{\mu^2} \right) - 1.
\]

Inserting these results into the expressions for $A$ and $B$ and performing some simplification yields,

\[
A = 1 + \epsilon \left\{ 1 + \frac{m_R^2}{p^2} - \ln \left( \frac{m_R^2 - p^2}{\mu^2} \right) + \frac{m_R^4}{p^4} \ln \left( 1 - \frac{p^2}{m_R^2} \right) \right\} + O(\epsilon^2),
\]

\[
B = 1 + \epsilon \left\{ \frac{3}{2} - \ln \left( \frac{m_R^2 - p^2}{\mu^2} \right) + \frac{m_R^2}{p^2} \ln \left( 1 - \frac{p^2}{m_R^2} \right) \right\} + O(\epsilon^2).
\]

Using these explicit expressions for $A$ and $B$ in eq. (83),

\[
\Sigma(p) = -\frac{\alpha_R}{4\pi} \left\{ 1 + \frac{m_R^2}{p^2} - \ln \left( \frac{m_R^2 - p^2}{\mu^2} \right) + \frac{m_R^4}{p^4} \ln \left( 1 - \frac{p^2}{m_R^2} \right) \right\}
\]

\[
+ \frac{m_R \alpha_R}{\pi} \left\{ \frac{3}{2} - \ln \left( \frac{m_R^2 - p^2}{\mu^2} \right) + \frac{m_R^2}{p^2} \ln \left( 1 - \frac{p^2}{m_R^2} \right) \right\},
\]

after taking the $\epsilon \to 0$ limit.

The one-loop correction to the inverse propagator is

\[
\Gamma^{(2)}(p) = \frac{\pi}{\mu} - m_R - \Sigma(p)
\]

\[
= \frac{\alpha_R}{4\pi} \left\{ 1 + \frac{m_R^2}{p^2} - \ln \left( \frac{m_R^2 - p^2}{\mu^2} \right) + \frac{m_R^4}{p^4} \ln \left( 1 - \frac{p^2}{m_R^2} \right) \right\}
\]

\[
- m_R \left\{ \frac{3}{2} - \ln \left( \frac{m_R^2 - p^2}{\mu^2} \right) + \frac{m_R^2}{p^2} \ln \left( 1 - \frac{p^2}{m_R^2} \right) \right\}.
\]
In this expression \( m_R \equiv m_R(\mu) \) is the renormalized mass, which differs from the physical pole mass. The definition of the \( \overline{\text{MS}} \) mass is obtained by setting \( \mu = m_R \). That is, the \( \overline{\text{MS}} \) mass is defined as \( m_R(m_R) \). Thus, we set \( \mu = m_R \) in eq. (84) and obtain,

\[
\Gamma^{(2)}(p) = \hat{\rho} \left\{ 1 + \frac{\alpha_R}{4\pi} \left[ 1 + \frac{m_R^2}{p^2} - \left( 1 - \frac{m_R^4}{p^4} \right) \ln \left( 1 - \frac{p^2}{m_R^2} \right) \right] \right\} \\
- m_R \left\{ 1 + \frac{\alpha_R}{\pi} \left[ \frac{3}{2} - \left( 1 - \frac{m_R^2}{p^2} \right) \ln \left( 1 - \frac{p^2}{m_R^2} \right) \right] \right\},
\]

(85)

where \( m_R \equiv m_R(m_R) \).

The physical pole mass, denoted by \( m_e \), corresponds to a zero of the inverse propagator. That is, \( m_e \) is defined by the condition

\[
\Gamma^{(2)}(p) \bigg|_{p = m_e} = 0 .
\]

(86)

One can expand the \( \overline{\text{MS}} \) mass perturbatively in terms of the physical mass \( m_e \),

\[
m_R(m_R) = m_e \left[ 1 + \frac{\alpha_R}{\pi} \kappa + \mathcal{O}(\alpha_R^2) \right].
\]

(87)

Inserting this into eq. (85) and imposing the condition specified in eq. (86), we can solve for \( \kappa \). At one-loop accuracy,

\[
\Gamma^{(2)}(p) = \hat{\rho} \left\{ 1 + \frac{\alpha_R}{4\pi} \left[ 1 + \frac{m_e^2}{p^2} - \left( 1 - \frac{m_e^4}{p^4} \right) \ln \left( 1 - \frac{p^2}{m_e^2} \right) \right] \right\} \\
- m_e \left\{ 1 + \frac{\alpha_R}{\pi} \left[ \kappa + \frac{3}{2} - \left( 1 - \frac{m_e^2}{p^2} \right) \ln \left( 1 - \frac{p^2}{m_e^2} \right) \right] \right\},
\]

(88)

Setting \( \hat{\rho} = m \) and \( p^2 = \hat{\rho} \hat{\rho} = m^2 \), we end up with

\[
\frac{\alpha_R}{2\pi} - \frac{\alpha_R}{\pi} \left( \kappa + \frac{3}{2} \right) = 0 .
\]

It follows that \( \kappa = -1 \). Inserting this result back into eq. (87), we conclude that the relation between the \( \overline{\text{MS}} \) mass and the physical mass \( m_e \) is given to one-loop accuracy by

\[
m_R(m_R) = m_e \left( 1 - \frac{\alpha_R}{\pi} \right).
\]

The inverse relation can also be obtained to one-loop accuracy,

\[
m_e = m_R(m_R) \left( 1 + \frac{\alpha_R}{\pi} \right).
\]