1. Consider the spontaneous breaking of a gauge group G down to U(1). The unbroken generator $Q = c_a T^a$ is some real linear combination of the generators of G.

Before we solve parts (a)–(d) of this problem, we review the general structure of spontaneously gauge theories that is relevant for the analysis that follows. The Lagrangian for the gauge theory based on a simple compact gauge group G is

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \tfrac{1}{2}\big(D_\mu\Phi(x)\big)^{\mathsf{T}}\big(D_\mu\Phi(x)\big) - V(\Phi)\,, \tag{1}$$

where we are employing a real basis for the scalar fields, $\Phi(x) = \big(\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\big)$, and the covariant derivative is defined by

$$D_\mu \equiv \partial_\mu + igT^a A^a_\mu\,. \tag{2}$$

Since the $\phi_i(x)$ are real scalar fields, the generators iT^a are real antisymmetric matrices.[1] At the minimum of the scalar potential $V(\Phi)$, the scalar field vacuum expectation values are identified,

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\phi_i = v_i} = 0\,.$$

We then define shifted scalar fields,

$$\widetilde{\phi}_i \equiv \phi_i - v\,, \tag{3}$$

In more detail, the kinetic energy term of the scalar fields is given by

$$\mathscr{L} \ni \tfrac{1}{2}(\partial_\mu\phi_i + igT^a_{ij}A^a_\mu\phi_j)(\partial_\mu\phi_i + igT^b_{ik}A^b_\mu\phi_k)\,.$$

Writing $\mathscr{L}$ in terms of the shifted fields defined in eq. (3), we see that terms quadratic in the gauge fields arise corresponding to replacing $\phi_i \to v_i$,

$$\mathscr{L}_{\rm mass} = -\tfrac{1}{2}g^2 A^a_\mu A^{\mu b}(T^a_{ij}v_j)(T^b_{ik}v_k)\,. \tag{4}$$

Note that because T^a is antisymmetric, we can write $T^a_{ij}T^b_{ik} = -(T^aT^b)_{jk}$. We can identify the squared-mass matrix of the gauge bosons,

$$\mathscr{L}_{\rm mass} = \tfrac{1}{2}M_{ab}A^a_\mu A^{\mu b}\,, \tag{5}$$

where

$$M^2_{ab} \equiv g^2 v_j (T^aT^b)_{jk} v_k\,. \tag{6}$$

Suppose that the T^a are represented by $n \times n$ antisymmetric matrices, which act on a multiplet of n scalar fields. The group G is spontaneously broken down to U(1) when the

[1] Given a compact group G, the generators T^a are necessarily hermitian. In addition, the iT^a are real, since for a real representation of scalar fields $D_\mu\Phi$ must also be real. Hence, it follows that the T^a are antisymmetric.

vacuum expectation values of the scalar fields is given by v, which can be represented by a column vector of n rows. The unbroken generator,

$$Q = c_a T^a \,, \tag{7}$$

is some real linear combination of the generators of G that satisfies

$$Q_{jk} v_k = 0 \,, \tag{8}$$

where there is an implicit sum over $k = 1, 2, \ldots, n$.

If the gauge group G is a direct product of the form $\mathrm{G} = \mathrm{G}_1 \times \mathrm{G}_2 \cdots \mathrm{G}_r$, where the G_i are either simple compact Lie groups or U(1), then we associate an independent gauge coupling $g_1, g_2, \ldots, g_r$ with each factor. In this case, we generalize eq. (6) slightly,

$$M^2_{ab} = g_a g_b v_j (T^a T^b)_{jk} v_k \,, \tag{9}$$

where there is no sum over the repeated indices a and b. Here, we associate g_a with the generators that belong to the appropriate subgroup G_i of the direct product. In particular, if T^a and T^b are generators of G_i, then $g_a = g_b = g_i$ and

$$[T^a \,,\, T^b] = i f_{abc} T^c \,,$$

where the f_{abc} are the structure constants of G_i if T^c is a generator of G_i, and $f_{abc} = 0$ if T^c is a generator of G_j with $j \neq i$. Likewise, if T^a is a generator of G_i and T^b is a generator of G_j with $i \neq j$, then $g_a = g_i$, $g_b = g_j$ and $[T^a \,,\, T^b] = 0$ (or equivalently, $f_{abc} = 0$) as a consequence of the direct product structure of G. One consequence of these observations is that

$$(g_a - g_b) f_{abc} = 0 \,, \tag{10}$$

where there is no implicit sum over a and b.

(a) Prove that $x_b \equiv c_b / g_b$ is an (unnormalized) eigenvector of the vector boson squared-mass matrix, M^2_{ab}, with zero eigenvalue.

We now consider he spontaneous breaking of a gauge group G down to U(1). The unbroken generator $Q = c_a T^a$ is some real linear combination of the generators of G. Then,

$$\sum_b M^2_{ab} \frac{c_b}{g_b} = \sum_b g_a g_b v_j (T^a T^b)_{jk} v_k \,\frac{c_b}{g_b} = g_a v_j T^a_{j\ell} \left(\sum_b c_b T^b_{\ell k} \right) v_k = g_a v_j T^a_{j\ell}\, Q_{\ell k} v_k = 0 \,,$$

after using eqs. (8) and (9). That is, c_b/g_b is an eigenvector of M^2_{ab} with eigenvalue zero. This corresponds to the massless U(1) gauge boson which remains massless due to the residual unbroken U(1) gauge symmetry.

(b) Suppose that A_μ is the massless gauge field that corresponds to the generator Q. Show that the covariant derivative can be expressed in the following form:

$$D_\mu = \partial_\mu + ieQA_\mu + \ldots \,, \tag{11}$$

where we have omitted terms in eq. (11) corresponding to all the other gauge bosons and

$$e = \left[\sum_a \left(\frac{c_a}{g_a}\right)^2\right]^{-1/2} . \tag{12}$$

Employing a real basis for the scalar fields, we define real antisymmetric generators via

$$L^a \equiv i g_a T^a \,, \tag{13}$$

where there is no implicit sum over a [cf. the comment following eq. (9)]. Using eq. (4), we can rewrite the gauge boson squared-mass matrix [cf. eq. (9)] as

$$M^2_{ab} = (L_a \vec{\boldsymbol{v}}, L_b \vec{\boldsymbol{v}}) \,. \tag{14}$$

Here, we have employed a convenient notation where the components of $\vec{\boldsymbol{v}}$ are v_i and

$$(x, y) \equiv \sum_i x_i y_i \,. \tag{15}$$

The gauge boson squared-mass matrix is real symmetric, so it can be diagonalized with an orthogonal similarity transformation:

$$\mathcal{O} M^2 \mathcal{O}^{\mathsf{T}} = \text{diag}\,(0, 0, \ldots, 0,\; m_1^2, m_2^2, \ldots) \,. \tag{16}$$

The corresponding gauge boson mass-eigenstates are:[2]

$$\widetilde{A}^a_\mu \equiv \mathcal{O}_{ab} A^b_\mu \,. \tag{17}$$

Likewise, we may define a new basis for the Lie algebra:

$$\widetilde{L}_a \equiv \mathcal{O}_{ab} L_b \,. \tag{18}$$

It then follows that:

$$(\mathcal{O} M^2 \mathcal{O}^{\mathsf{T}})_{ab} = (\widetilde{L}_a \vec{\boldsymbol{v}}, \widetilde{L}_b \vec{\boldsymbol{v}}) = m_a^2 \delta_{ab} \,, \tag{19}$$

is the diagonalized vector boson squared-mass matrix, and the covariant derivative is given by

$$D_\mu = \partial_\mu + L_a A^a_\mu = \partial_\mu + \widetilde{L}_a \widetilde{A}^a_\mu = \partial_\mu + ieQA_\mu + \cdots \,,$$

where A_μ is the gauge boson corresponding to the unbroken U(1) generator.

Let us choose $\mathcal{O}$ such that $m_1 = 0$ is the mass of the gauge boson that corresponds to the unbroken U(1). Then, we can identify the unbroken generator as

$$\widetilde{L}_1 = ieQ = \mathcal{O}_{1b} L_b \,, \tag{20}$$

where there is an implicit sum over b. Moreover, eq. (19) yields $\mathcal{O}_{1a} M^2_{ab} \mathcal{O}_{1b} = 0$. The rows of the diagonalizing matrix $\mathcal{O}$ correspond to the normalized eigenvectors of M^2. Thus,

$$\mathcal{O}_{1b} = \frac{1}{N} \frac{c_b}{g_b} \tag{21}$$

is the eigenvector corresponding to the zero eigenvalue found in part (a), and N is a constant.

[2]Indeed, one can easily check that $M^2_{ab} A^a_\mu A^{\mu b} = \sum_a m_a^2 \widetilde{A}^a_\mu \widetilde{A}^{\mu a}$.

The normalization constant N that appears in eq. (21) is chosen such that the eigenvector has unit length,

$$N^2 = \sum_b \left(\frac{c_b}{g_b}\right)^2 . \tag{22}$$

Eqs. (20) and (21) yield

$$ieQ = \frac{1}{N}\sum_b \frac{c_b}{g_b} L_b = \frac{i}{N}\sum_b c_b T^b = \frac{i}{N}\, Q\,,$$

after making use of eqs. (7) and (13). It immediately follows that $e = 1/N$. In light of eq. (22),

$$e = \left[\sum_b \left(\frac{c_b}{g_b}\right)^2\right]^{-1/2} . \tag{23}$$

By convention, we take $e > 0$.

(c) Evaluate Q in the adjoint representation (that is, $Q = c_a T^a$, where the $(T^a)_{bc} = -if_{abc}$ are the generators of the gauge group in the adjoint representation). Show that $Q_{bc}x_c = 0$, where x_c is defined in part (a). What is the physical interpretation of this result?

Using $x_c = c_c/g_c$ and eq. (7), we obtain

$$Q_{bc}x_c = c_a T^a_{bc} x_c = -if_{abc}\frac{c_a c_c}{g_c} = if_{acb}\frac{c_a c_c}{g_c}\,, \tag{24}$$

using the antisymmetry properties of the f_{abc}. Employing eq. (10),

$$\sum_{a,c} f_{acb}\frac{c_a c_c}{g_c} = \sum_{a,c} g_a f_{acb}\frac{c_a c_c}{g_a g_c} = \sum_{a,c} g_c f_{acb}\frac{c_a c_c}{g_a g_c} = \sum_{a,c} f_{acb}\frac{c_a c_c}{g_a} = \sum_{a,c} f_{cab}\frac{c_a c_c}{g_c} = -\sum_{a,c} f_{acb}\frac{c_a c_c}{g_c}\,,$$

where in the penultimate step we relabeled $a \to c$ and $c \to a$, and in the last step we used $f_{cab} = -f_{acb}$. Hence,

$$\sum_{a,c} f_{acb}\frac{c_a c_c}{g_c} = 0\,,$$

which means that $Q_{bc}x_c = 0$. The physical interpretation of this statement is that the U(1) gauge boson is neutral with respect to the unbroken generator Q.

(d) Prove that the commutator $[Q\,,\,M^2] = 0$, where Q is the unbroken U(1) generator in the adjoint representation and M^2 is the gauge boson squared-mass matrix. Conclude that one can always choose the eigenstates of the gauge boson squared-mass matrix to be states of definite unbroken U(1)-charge.

In the adjoint representation [cf. eq. (24)], $Q_{bc} = i\sum_e f_{ecb}c_e$, where there is an implicit sum over the repeated index a. Using eq. (9),

$$[Q\,,\,M^2]_{ac} = \sum_b \bigl(Q_{ab}M^2_{bc} - M^2_{ab}Q_{bc}\bigr) = iv^{\mathsf{T}}\sum_{b,e}\bigl(g_b g_c c_e f_{eba}T^b T^c - g_a g_b c_e f_{ecb} T^a T^b\bigr)v\,, \tag{25}$$

where $v^{\mathsf{T}} T^b T^c v \equiv v_j (T^b T^c)_{jk} v_k$, etc. Note that all sums are explicitly exhibited; there are no implicit sums over repeated indices in eq. (25). Employing eq. (10),

$$\sum_b g_b f_{eba} T^b = \sum_b g_b f_{bae} T^b = g_a \sum_b f_{bae} T^b \,,$$

$$\sum_b g_b f_{ecb} T^b = \sum_b g_b f_{cbe} T^b = g_c \sum_b f_{cbe} T^b \,.$$

Using the commutation relations of the generators,

$$\begin{aligned} i f_{bae} T^b = i f_{aeb} T^b = [T^a \,,\, T^e] = T^a T^e - T^e T^a \,, \\ i f_{cbe} T^b = i f_{ecb} T^b = [T^e \,,\, T^c] = T^e T^c - T^c T^e \,. \end{aligned}$$

Inserting these results back into eq. (25) yields

$$[Q \,,\, M^2]_{ac} = g_a g_c v^{\mathsf{T}} \sum_e \Big\{ c_e (T^a T^e - T^e T^a) T^c - c_e T^a (T^e T^c - T^c T^e) \Big\} v \,. \tag{26}$$

The T^a are the generators in the representation that acts on the scalar fields. In this representation the charge operator, which will be denoted by $\mathcal{Q}$ to distinguish it from the charge operator in the adjoint representation Q, is defined by

$$\mathcal{Q} = \sum_e c_e T^e \,.$$

Hence, eq. (26) yields

$$\begin{aligned} [Q \,,\, M^2]_{ac} &= g_a g_c v^{\mathsf{T}} \Big\{ (T^a \mathcal{Q} - \mathcal{Q} T^a) T^c - T^a (\mathcal{Q} T^c - T^c \mathcal{Q}) \Big\} v \\ &= g_a g_c v^{\mathsf{T}} \big(T^a T^c \mathcal{Q} - \mathcal{Q} T^a T^c \big) v \,. \end{aligned} \tag{27}$$

Using the fact that $\mathcal{Q}$ is an unbroken generator corresponding to the unbroken U(1) subgroup of G, it follows that $v^{\mathsf{T}} \mathcal{Q} = \mathcal{Q} v = 0$. Employing this result in eq. (27) yields

$$[Q \,,\, M^2] = 0 \,.$$

Thus, one can simultaneously diagonalize M^2 and Q. The corresponding simultaneous eigenstates are gauge boson states of definite mass and unbroken U(1)-charge.

2. In class, we examined in detail the structure of a spontaneously broken SU(2)×U(1)$_Y$ gauge theory, in which the symmetry breaking was due to the vacuum expectation value of a complex $Y = 1$, SU(2) doublet of scalar fields. In this problem, a different representation of scalar fields will be employed.

(a) Consider an SU(2)×U(1)$_Y$ gauge theory with a $Y = 0$, SU(2) triplet of *real* scalar fields, Φ. The scalar potential is given by

$$V(\Phi) = -\tfrac{1}{2} m^2 \Phi^{\mathsf{T}} \Phi + \lambda (\Phi^{\mathsf{T}} \Phi)^2 \,,$$

where m^2 and λ are real parameters. After spontaneous symmetry breaking, the electrically neutral ($Q = 0$) member of the scalar triplet acquires a vacuum expectation value (where $Q = T_3 + Y/2$). Identify the subgroup that remains unbroken. Compute the vector boson masses and the physical Higgs scalar masses in this model. Deduce the Feynman rules for the three-point interactions among the Higgs and vector bosons.

Consider a model where the SU(2)×U(1) gauge symmetry is broken by a $Y = 0$ triplet of real scalar fields, whose neutral member acquires a vacuum expectation value. The generators of SU(2) in the triplet (adjoint) representation are $(T^a)_{bc} = -i\epsilon_{abc}$. Explicitly,

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad T^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad T^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

Using eq. (13),

$$L_1 = g\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad L_2 = g\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad L_3 = g\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which act on the scalar field multiplet,

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{pmatrix},$$

where the $\phi_i(x)$ are real scalar fields. In addition, the generator corresponding to U(1) is

$$L_4 = i\tfrac{1}{2}g'Y\,,$$

where Y is the hypercharge operator. Since $\Phi(x)$ is a triplet of scalar fields with zero hypercharge, it follows that $L_4\Phi = 0$.

We now compute the squared-mass matrix of the gauge bosons using eq. (14), where v is the vacuum expectation value of the electrically neutral member of the scalar triplet. The electric charge operator is given by

$$\mathcal{Q} = T^3 + \tfrac{1}{2}Y\,.$$

In particular, when acting on the scalar triplet (which has hypercharge zero),

$$\mathcal{Q}\Phi = (T^3 + \tfrac{1}{2}Y)\Phi = T^3\Phi\,. \tag{29}$$

This implies that the electrically neutral member of the scalar triplet must be an eigenstate of T^3 with zero eigenvalue. Thus, we choose the vacuum expectation value $\boldsymbol{\vec{v}} \equiv \langle\Phi\rangle$ to have the form

$$\boldsymbol{\vec{v}} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \tag{30}$$

in order to ensure that after spontaneous symmetry breaking, the unbroken gauge group preserves the U(1) of electromagnetism. We now can compute $L_a\vec{\boldsymbol{v}}$ for $a = 1, 2, 3, 4$. We already know that $L_4\Phi = 0$, so we need only consider $a = 1, 2, 3$.

$$L_1\vec{\boldsymbol{v}} = g\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ gv \\ 0 \end{pmatrix},$$

$$L_2\vec{\boldsymbol{v}} = g\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = \begin{pmatrix} -gv \\ 0 \\ 0 \end{pmatrix},$$

$$L_3\vec{\boldsymbol{v}} = g\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, eq. (14) yields the squared-mass matrix of the gauge bosons,

$$M^2 = \begin{pmatrix} g^2v^2 & 0 & 0 & 0 \\ 0 & g^2v^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{31}$$

We conclude that the $W^\pm$ has gained a mass $m_W = gv$, whereas W^3 and B remain massless. This means that the SU(2)×U(1) gauge symmetry has broken down to U(1)×U(1). One of the U(1)'s can be identified with the gauge group of electromagnetism. In light of eq. (29), we can choose the electromagnetic charge operator to be $\mathcal{Q} = T^3$. Using eq. (23), we can therefore identify the photon field as $A_\mu = W^3_\mu$ and $e = g$.

The physical Higgs bosons of the model are obtained from the Higgs scalar potential. The most general quartic gauge invariant scalar potential is

$$V(\Phi) = -\tfrac{1}{2}m^2\Phi^{\mathsf{T}}\Phi + \lambda(\Phi^{\mathsf{T}}\Phi)^2 = -\tfrac{1}{2}m^2(\phi_1^2 + \phi_2^2 + \phi_3^2)^2 + \lambda(\phi_1^2 + \phi_2^2 + \phi_3^2)^4. \tag{32}$$

The minimum of the scalar potential corresponds to $\Phi = v$ given by eq. (30). Imposing the minimum condition,

$$\left(\frac{\partial V}{\partial \phi_i}\right)_{\Phi=v} = 0,$$

yields

$$-m^2v + 4\lambda v^3 = 0. \tag{33}$$

Assuming that the symmetry is broken, $v \neq 0$, and we obtain

$$v = \frac{m}{2\sqrt{\lambda}}.$$

Note that this implies that $V(v) = -m^2/(16\lambda)$. A second extremum corresponding to $\Phi = 0$ (the symmetry conserving minimum) yields $V(0) = 0$. Hence, if $m^2 > 0$, it follows that the symmetry-breaking minimum is the global minimum of the scalar potential.

Next, we identify the Goldstone bosons, which were given in class by

$$G_a = \frac{1}{m_a} \sum_j (\widetilde{L}_a \vec{\boldsymbol{v}})_j \eta_j \,, \tag{34}$$

where the m_a are the (non-zero) masses of the gauge bosons and the η_j are the shifted scalar fields defined by

$$\Phi = \begin{pmatrix} \eta_1 \\ \eta_2 \\ v + \eta_3 \end{pmatrix} ,$$

and the $\widetilde{L}_a$ are defined in eq. (18). Since M^2 given in eq. (31) is already diagonal, the diagonalization matrix $\mathcal{O} = \mathbb{1}$ and $\widetilde{L}_a = L_a$. Since $L_a \vec{\boldsymbol{v}} = 0$ for $a = 3$ and 4, it follows that there are precisely two Goldstone bosons, η_1 and η_2. Thus, $H = \eta_3$ is the physical Higgs boson.

In the unitary gauge, we set the Goldstone fields to zero. Then,

$$V(H) = -\tfrac{1}{2} m^2 (H + v)^2 + \lambda (H + v)^4 \,.$$

Using eq. (33), the term linear in H vanishes. The constant term can be removed by redefining the energy of the vacuum to be zero. Thus,

$$V(H) = H^2(-\tfrac{1}{2} m^2 + 6\lambda v^2) + \mathcal{O}(H^3) + \mathcal{O}(H^4) \,.$$

The coefficient of H^2 is identified as $\frac{1}{2} m_H^2$, where m_H is the Higgs mass. Using eq. (33) to eliminate m^2, we obtain

$$m_H^2 = -m^2 + 12\lambda v^2 = 8\lambda v^2 \,.$$

Hence,

$$m_H = 2\sqrt{2\lambda}\, v \,.$$

The three-point interactions among the Higgs and Gauge bosons arise from the kinetic energy term,

$$\mathscr{L}_{\rm KE} = \tfrac{1}{2} (D_\mu \Phi)^{\mathsf{T}} D^\mu \Phi \,,$$

where the covariant derivative acting on the scalar field is given by

$$(D_\mu \Phi)_i = \partial_\mu \phi_i + i g T^a_{ij} W^a_\mu \phi_j + i \tfrac{1}{2} g' B Y \phi_i \,.$$

Since $Y\Phi = 0$, $A_\mu = W^3_\mu$ and $g = e$, we have

$$D_\mu = \partial_\mu + \frac{ie}{\sqrt{2}} \big(T^+ W^+_\mu + T^- W^-_\mu \big) + ie \mathcal{Q} A_\mu \,,$$

where $T^\pm \equiv T_1 \pm i T^2$ and

$$W^\pm_\mu = \frac{1}{\sqrt{2}} \big(W^1_\mu \mp i W^2_\mu \big) \,. \tag{35}$$

In the unitary gauge,

$$\Phi = \begin{pmatrix} 0 \\ 0 \\ v + H \end{pmatrix} .$$

Using

$$ieT^+ = e\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 1 \\ i & -1 & 0 \end{pmatrix}, \qquad ieT^- = e\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & -1 & 0 \end{pmatrix}, \qquad ie\mathcal{Q} = e\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

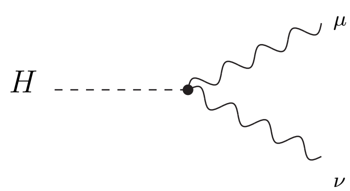
it follows that

$$\left\{\partial_\mu + \frac{ie}{\sqrt{2}}\big(T^+W_\mu^+ + T^-W_\mu^-\big) + ie\mathcal{Q}A_\mu\right\}\begin{pmatrix} 0 \\ 0 \\ v+H \end{pmatrix} = \begin{pmatrix} -\frac{ie}{\sqrt{2}}(W_\mu^+ - W_\mu^-))(v+H) \\ \frac{e}{\sqrt{2}}(W_\mu^+ + W_\mu^-))(v+H) \\ \partial_\mu H \end{pmatrix}.$$

Therefore, we end up with

$$\begin{aligned}\tfrac{1}{2}(D_\mu\Phi)^{\mathsf{T}}D^\mu\Phi &= \tfrac{1}{2}(\partial_\mu H)^2 + \tfrac{1}{4}(v+H)^2\big[(W_\mu^+ + W_\mu^-)^2 - (W_\mu^+ - W_\mu^-)^2\big] \\ &= \tfrac{1}{2}(\partial_\mu H)^2 + e^2\big[v^2 + 2vH + H^2\big]W_\mu^+W^{\mu\,-} \\ &= \tfrac{1}{2}(\partial_\mu H)^2 + (m_W^2 + 2em_W H + e^2H^2)W_\mu^+W^{\mu\,-},\end{aligned}$$

after using $m_W = gv = ev$. We can therefore identify the Feynman rule for the trilinear HW^+W^- interaction,



$$2iem_W g^{\mu\nu}$$

(b) Consider an SU(2)×U(1)$_Y$ gauge theory with a $Y=2$, SU(2) triplet of *complex* scalar fields (again denoted by Φ). The scalar potential is given by

$$V(\Phi) = -m^2\Phi^\dagger\Phi + \lambda_1(\Phi^\dagger\Phi)^2 - \lambda_2\sum_a(\Phi^\dagger\mathcal{T}^a\Phi)(\Phi^\dagger\mathcal{T}^a\Phi),$$

where $m^2 > 0$ and $\lambda_1 > \lambda_2 > 0$. The $\mathcal{T}^a$ are hermitian generators in the 3-dimensional representation of SU(2) in a basis where $\mathcal{T}^3$ is diagonal.

Again, assume that the electrically neutral ($Q=0$) member of the scalar triplet acquires a vacuum expectation value (where $Q = T_3 + Y/2$). After symmetry breaking, identify the subgroup that remains unbroken. Compute the vector boson masses and the physical Higgs scalar masses in this model.

Consider a model where the SU(2)×U(1) gauge symmetry is broken by a $Y=2$ triplet of complex scalar fields $\Phi(x)$, whose neutral member acquires a vacuum expectation value. It is convenient to employ a basis of hermitian generators $\{\mathcal{T}^a\,,\,\frac{1}{2}\mathcal{Y}\}$ where $\mathcal{T}^3$ and $\mathcal{Y}$ are diagonal.

That is, we shall identify the $\mathcal{T}^a$ with the standard spin-1 matrices defined in the $|j\ m\rangle$ basis in quantum mechanics.[3] Explicitly,

$$\mathcal{T}^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathcal{T}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \mathcal{T}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{36}$$

In analogy with eq. (13), we define $\mathcal{L}_a = ig\mathcal{T}^a$. In particular,

$$\mathcal{L}_1 = \frac{g}{\sqrt{2}}\begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix}, \qquad \mathcal{L}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \mathcal{L}_3 = g\begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}, \tag{37}$$

which act on the scalar field multiplet,

$$\Phi(x) = \begin{pmatrix} \Phi^{++}(x) \\ \Phi^{+}(x) \\ \Phi^{0}(x) \end{pmatrix},$$

where $\Phi^{++}(x)$, $\Phi^{+}(x)$ and $\Phi^{0}(x)$ are complex scalar fields. In addition, the generator corresponding to the hypercharge U(1) is

$$\mathcal{L}_4 = \tfrac{1}{2}ig'\mathcal{Y} = ig'\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{38}$$

where the hypercharge operator is normalized such that $\mathcal{L}_4\Phi = ig'\Phi$.

The electric charge operator is given by

$$\mathcal{Q} = \mathcal{T}^3 + \tfrac{1}{2}\mathcal{Y} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This implies that the electrically neutral member of the scalar triplet can be identified with Φ^0. Thus, we choose the vacuum expectation value to have the form

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \tag{39}$$

in order to ensure that after spontaneous symmetry breaking, the unbroken gauge group preserves $\mathrm{U}(1)_{\mathrm{EM}}$.[4]

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[3] For example, see R. Shankar, *Principles of Quantum Mechanics*, 2nd edition (Springer Science, New York, 1994) p. 328.

[4] The factor of $1/\sqrt{2}$ in eq. (39) is conventional. If we write $\phi^0 = (\phi^0_R + i\phi^0_I)/\sqrt{2}$, then the kinetic energy term for ϕ^0_R will be canonically normalized. We can choose the vacuum expectation value to be real without loss of generality, in which case $\langle\phi^0_R\rangle = v$.

Following the class handout on gauge theories, one can employ the complex representation of scalar fields to evaluate the squared-masses of the gauge bosons and Higgs bosons. (Details can be found at the end of the solution to part (b) of this problem.) However, here we will perform the computations by employing a real representation of the scalar fields. In the real representation, the generators are 6×6 matrices. We determine these matrices as follows. First it is convenient to write[5]

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_4 \\ \phi_2 + i\phi_5 \\ \phi_3 + i\phi_6 \end{pmatrix} , \qquad (40)$$

where the ϕ_i are real fields. In light of

$$\mathcal{L}_1 \Phi = \frac{g}{2} \begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \phi_1 + i\phi_4 \\ \phi_2 + i\phi_5 \\ \phi_3 + i\phi_6 \end{pmatrix} = \frac{g}{2} \begin{pmatrix} -\phi_5 + i\phi_2 \\ -\phi_4 - \phi_6 + i(\phi_1 + \phi_3) \\ -\phi_5 + i\phi_2 \end{pmatrix} ,$$

$$\mathcal{L}_2 \Phi = \frac{g}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 + i\phi_4 \\ \phi_2 + i\phi_5 \\ \phi_3 + i\phi_6 \end{pmatrix} = \frac{g}{2} \begin{pmatrix} \phi_2 + i\phi_5 \\ \phi_3 - \phi_1 + i(\phi_6 - \phi_4) \\ -\phi_2 - i\phi_5 \end{pmatrix} ,$$

$$\mathcal{L}_3 \Phi = \frac{g}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} \begin{pmatrix} \phi_1 + i\phi_4 \\ \phi_2 + i\phi_5 \\ \phi_3 + i\phi_6 \end{pmatrix} = \frac{g}{\sqrt{2}} \begin{pmatrix} -\phi_4 + i\phi_1 \\ 0 \\ \phi_6 - i\phi_3 \end{pmatrix} ,$$

$$\mathcal{L}_4 \Phi = ig' \frac{\mathcal{Y}}{2} \Phi = ig' \Phi = \frac{g'}{\sqrt{2}} \begin{pmatrix} -\phi_4 + i\phi_1 \\ -\phi_5 + i\phi_2 \\ -\phi_6 + i\phi_3 \end{pmatrix} ,$$

it follows that the corresponding generators in the real representation (denoted by L_a below) must satisfy,

$$L_1 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{pmatrix} = \frac{g}{\sqrt{2}} \begin{pmatrix} -\phi_5 \\ -\phi_4 - \phi_6 \\ -\phi_5 \\ \phi_2 \\ \phi_1 + \phi_3 \\ \phi_2 \end{pmatrix} , \qquad L_2 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{pmatrix} = \frac{g}{\sqrt{2}} \begin{pmatrix} \phi_2 \\ \phi_3 - \phi_1 \\ -\phi_2 \\ \phi_5 \\ \phi_6 - \phi_4 \\ -\phi_5 \end{pmatrix} , \qquad (41)$$

[5] Note in particular the choice of subscripts in eq. (40). This choice is motivated by the observation that the corresponding 6×6 real antisymmetric matrix generators $L_a = igT^a$ given in eq. (43) can be expressed in block diagonal form in terms of the hermitian generators $\mathcal{T}^a$ via eq. (44). If one instead employs

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \\ \phi_5 + i\phi_6 \end{pmatrix} ,$$

then the resulting expressions for the L^a are related to the generators given in eq. (43) by an appropriate permutation of rows and columns, thereby losing the nice block form for the matrices given in eq. (44).

$$
L_3 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{pmatrix} = g \begin{pmatrix} -\phi_4 \\ 0 \\ \phi_6 \\ \phi_1 \\ 0 \\ -\phi_3 \end{pmatrix} , \qquad L_4 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{pmatrix} = g' \begin{pmatrix} -\phi_4 \\ -\phi_5 \\ -\phi_6 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} , \tag{42}
$$

It immediately follows that

$$
L_1 = \frac{g}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} , \qquad L_2 = \frac{g}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} ,
$$

$$
L_3 = g \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} . \qquad L_4 = g' \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} . \tag{43}
$$

Note that the real antisymmetric generators $L_a = ig_a T^a$ are 6×6 real antisymmetric matrices that can be written in block form in terms of the real and imaginary parts of the 3×3 hermitian generators $\mathcal{T}^a$,

$$
L_a = ig_a T^a = g_a \begin{pmatrix} -\mathrm{Im}\ \mathcal{T}^a & -\mathrm{Re}\ \mathcal{T}^a \\ \mathrm{Re}\ \mathcal{T}^a & -\mathrm{Im}\ \mathcal{T}^a \end{pmatrix} . \tag{44}
$$

This convenient form for the L_a provides the motivation for the choice of subscripts in eq. (40) [cf. footnote 5].

Likewise, the vacuum expectation value [cf. eq. (39)] in the real representation is given by

$$
\vec{\boldsymbol{v}} = \begin{pmatrix} 0 \\ 0 \\ v \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{45}
$$

To determine the vector boson squared-mass matrix, we first compute

$$
L_1 \vec{\boldsymbol{v}} = \frac{g}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v \\ 0 \end{pmatrix} , \qquad L_2 \vec{\boldsymbol{v}} = \frac{g}{\sqrt{2}} \begin{pmatrix} 0 \\ v \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \qquad L_3 \vec{\boldsymbol{v}} = g \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -v \end{pmatrix} , \qquad L_4 \vec{\boldsymbol{v}} = g' \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ v \end{pmatrix} . \tag{46}
$$

Using eq. (14), it then follows that

$$M_{ab}^2 = (L_a \vec{\boldsymbol{v}}\,,\, L_b \vec{\boldsymbol{v}}) = \begin{pmatrix} \frac{1}{2}g^2v^2 & 0 & 0 & 0 \\ 0 & \frac{1}{2}g^2v^2 & 0 & 0 \\ 0 & 0 & g^2v^2 & -gg'v^2 \\ 0 & 0 & -gg'v^2 & g'^2v^2 \end{pmatrix} . \tag{47}$$

We see that there are two degenerate gauge bosons, which we identify as $W^\pm$ [defined in eq. (35)] with

$$m_W^2 = \tfrac{1}{2}g^2v^2 \,. \tag{48}$$

The diagonalization of the lower 2×2 block of eq. (47) is nearly identical to the computation of the Standard Model (with a complex, hypercharge-one Higgs doublet). Indeed, the only difference is the minus sign that appears in the off-diagonal term. The Z corresponds to the eigenvector,

$$\frac{1}{\sqrt{g^2+g'^2}}\begin{pmatrix} 0 \\ 0 \\ g \\ -g' \end{pmatrix} , \qquad\qquad m_Z^2 = (g^2 + g'^2)v^2 \,, \tag{49}$$

and the massless photon corresponds to

$$\frac{1}{\sqrt{g^2+g'^2}}\begin{pmatrix} 0 \\ 0 \\ g' \\ g \end{pmatrix} , \qquad\qquad m_\gamma = 0 \,. \tag{50}$$

If we define $\sin\theta_W \equiv g'/\sqrt{g^2+g'^2}$, then we can identify,

$$\begin{aligned} Z_\mu &= W_\mu^3 \cos\theta_W - B_\mu \sin\theta_W \,, \\ A_\mu &= W_\mu^3 \sin\theta_W + B_\mu \cos\theta_W \,. \end{aligned}$$

Note that that SU(2)×U(1) has spontaneously broken down to U(1), which we identify as the gauge group of electromagnetism.

Following eqs. (16)–(18), we define a new basis for the Lie algebra, $\widetilde{L}_a$. The computation is the same as the one performed in class in the case of the electroweak Standard Model. Thus, we simply employ the results obtained in class,

$$\begin{aligned} \widetilde{L}_1 &= L_1 \,, \\ \widetilde{L}_2 &= L_2 \,, \\ \widetilde{L}_3 &= L_3 \cos\theta_W - L_4 \sin\theta_W = \frac{ig}{\cos\theta_W}\left(T^3 - Q\sin^2\theta_W\right) \,, \\ \widetilde{L}_4 &= L_3 \sin\theta_W + L_4 \cos\theta_W = ieQ \,, \end{aligned}$$

where $e = g\sin\theta_W = g'\cos\theta_W$.

The explicit forms for $\widetilde{L}_3$ and $\widetilde{L}_4$ are as follows,

$$\widetilde{L}_3 = \frac{g}{\cos\theta_W}\begin{pmatrix} 0 & 0 & 0 & -\cos 2\theta_W & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin^2\theta_W & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \cos 2\theta_W & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sin^2\theta_W & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \tag{51}$$

$$\widetilde{L}_4 = e\begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{52}$$

To analyze the scalar sector of this model, we must specify the Higgs scalar potential. In this case, the most general quartic gauge invariant scalar potential is

$$V(\Phi) = -m^2\Phi^\dagger\Phi + \lambda_1(\Phi^\dagger\Phi)^2 - \lambda_2\sum_a(\Phi^\dagger\mathcal{T}^a\Phi)(\Phi^\dagger\mathcal{T}^a\Phi)\,. \tag{53}$$

Note that this is somewhat more complicated than eq. (32), since there are two independent gauge invariant quartic interactions for the case of a complex hypercharge-two scalar field.[6]

One can minimize eq. (53) and demonstrate that for $m^2 > 0$, there exists a global minimum corresponding to eq. (39). Here, let us assume that such a global minimum exists. We can then identify the Goldstone bosons, which are given by eq. (34). That is,

$$G_a = \frac{1}{m_a}\sum_j(\widetilde{L}_a\boldsymbol{\vec{v}})_j\eta_j\,, \tag{54}$$

where the m_a are the (non-zero) masses of the gauge bosons and the η_j are the shifted scalar fields defined by

$$\Phi = \begin{pmatrix} \eta_1 \\ \eta_2 \\ v+\eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{pmatrix}.$$

Then, eq. (54) yields

$$G_1 = \eta_5 = \sqrt{2}\,\text{Im}\ \Phi^+\,, \qquad G_2 = \eta_2 = \sqrt{2}\,\text{Re}\ \Phi^+\,, \qquad G_3 = -\eta_6 = -\sqrt{2}\,\text{Im}\ \Phi^0\,, \tag{55}$$

where we have used eqs. (48) and (49) to simplify our results.

[6] In the case of a real triplet of scalar fields, $(\Phi^{\mathsf{T}}\Phi)^2$ is the only quartic invariant. Indeed, for a real multiplet of scalar fields, the generators T^a are antisymmetric matrices, and it follows that $\Phi^{\mathsf{T}}T^a\Phi = \Phi_i T^a_{ij}\Phi_j = 0$.

The physical Higgs states are orthonormal to the G_a and can be determined by inspection,

$$H^{++} = \frac{1}{\sqrt{2}}(\eta_1 + i\eta_4) = \Phi^{++}\,, \qquad H^{--} = [\Phi^{++}]^\dagger\,, \qquad H = \eta_3 = \sqrt{2}\,\mathrm{Re}\,\Phi^0 - v\,.$$

Thus, the complex scalar triplet takes the form

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\,H^{++} \\ G_2 + iG_1 \\ v + H - iG_3 \end{pmatrix} \equiv \begin{pmatrix} H^{++} \\ G^+ \\ \frac{1}{\sqrt{2}}\,[v + H + iG^0] \end{pmatrix}\,,$$

which defines the Goldstone states of definite charge, $G^\pm \equiv (G_2 \pm iG_1)/\sqrt{2}$ and G^0, where $G^- \equiv (G^+)^\dagger$ and $G^0 = -G_3$.

In the unitary gauge, we can set $G^\pm = G^0 = 0$. Then,

$$\begin{aligned}
\Phi^\dagger\Phi &= |H^{++}|^2 + \tfrac{1}{2}|H|^2 + vH + \tfrac{1}{2}v^2\,, \\
\Phi^\dagger\mathcal{T}^1\Phi &= 0\,, \\
\Phi^\dagger\mathcal{T}^2\Phi &= 0\,, \\
\Phi^\dagger\mathcal{T}^3\Phi &= |H^{++}|^2 - \tfrac{1}{2}|H|^2 - vH - \tfrac{1}{2}v^2\,.
\end{aligned}$$

Hence, eq. (53) yields

$$\begin{aligned}
V(H^{++}, H) &= -m^2\big(|H^{++}|^2 + \tfrac{1}{2}|H|^2 + vH + \tfrac{1}{2}v^2\big) + \lambda_1\big(|H^{++}|^2 + \tfrac{1}{2}|H|^2 + vH + \tfrac{1}{2}v^2\big)^2 \\
&\qquad -\lambda_2\big(|H^{++}|^2 - \tfrac{1}{2}H^2 - vH - \tfrac{1}{2}v^2\big)^2 \\
&= \text{constant} + v\big[(\lambda_1 - \lambda_2)v^2 - m^2\big] + \big[\tfrac{3}{2}(\lambda_1 - \lambda_2)v^2 - \tfrac{1}{2}m^2\big]H^2 \\
&\qquad + \big[(\lambda_1 + \lambda_2)v^2 - m^2\big]|H^{++}|^2 + \text{cubic terms} + \text{quartic terms}\,. \qquad (56)
\end{aligned}$$

The terms linear in H must vanish at the minimum of the scalar potential, which implies that

$$m^2 = (\lambda_1 - \lambda_2)v^2\,. \qquad (57)$$

I leave it as an exercise for the reader to check that an extremum of the scalar potential given by eq. (53) exists with the vacuum expectation value of Φ given by eq. (57). Inserting the result of eq. (57) into eq. (56), we obtain

$$V(H^{++}, H) = \text{constant} + (\lambda_1 - \lambda_2)v^2H^2 + 2\lambda_2 v^2|H^{++}|^2 + \text{cubic terms} + \text{quartic terms}\,.$$

Comparing the terms quadratic in the Higgs field with $\frac{1}{2}m_H^2 H^2 + m_{H^{++}}^2|H^{++}|^2$, we conclude that

$$m_H^2 = 2(\lambda_1 - \lambda_2)v^2\,, \qquad m_{H^{++}}^2 = m_{H^{--}}^2 = 2\lambda_2 v^2\,.$$

Since the squared-masses of the physical Higgs bosons must be positive, we must demand that $\lambda_1 > \lambda_2 > 0$ in order to guarantee that the extremum of the scalar potential corresponding to eq. (39) is a local minimum. These conditions also require that $m^2 > 0$, in light of eq. (57).

REMARKS: Using the complex representation of the scalar fields to obtain the gauge boson and scalar mass eigenstates:

Following Section 4.6 of the class handout entitled *Gauge Theories and the Standard Model*, one can make use of the following formulae for the gauge boson squared-mass matrix,

$$M^2_{ab} = g_a g_b \nu^\dagger(\mathcal{T}^a\mathcal{T}^b + \mathcal{T}^b\mathcal{T}^a)\nu = -\nu^\dagger(\mathcal{L}^a\mathcal{L}^b + \mathcal{L}^b\mathcal{L}^a)\nu\,, \tag{58}$$

where the $\mathcal{T}^a$ are hermitian generators and $\mathcal{L}_a \equiv ig_a\mathcal{T}^a$. Using eq. (58), the gauge boson squared-mass matrix is easily computed. In the complex representation of the scalar fields, the vacuum expectation value $\nu \equiv \langle\Phi\rangle$ is given by eq. (39). Employing the explicit matrices of the generators $\mathcal{L}^a$ given in eqs. (37) and (38), the gauge boson squared-mass matrix obtained from eq. (58) reproduces eq. (47).[7] We can diagonalize M^2 using

$$(\mathcal{O}M^2\mathcal{O}^T)_{ab} = (\widetilde{\mathcal{L}}_a\nu)^\dagger(\widetilde{\mathcal{L}}_b\nu) + (\widetilde{\mathcal{L}}_b\nu)^\dagger(\widetilde{\mathcal{L}}_a\nu) = m_a^2\delta_{ab}\,.$$

In light of eqs. (49) and (50), we identify $\widetilde{\mathcal{L}}_1 = \mathcal{L}_1$, $\widetilde{\mathcal{L}}_2 = \mathcal{L}_2$, and

$$\widetilde{\mathcal{L}}_3 = \frac{1}{\sqrt{g^2+g'^2}}\bigl(g\mathcal{L}_3 - g'\mathcal{L}_4\bigr) = \frac{ig}{\cos\theta_W}\bigl(\mathcal{T}^3 - \mathcal{Q}\sin^2\theta_W\bigr) = \frac{ig}{\cos\theta_W}\begin{pmatrix}\cos 2\theta_W & 0 & 0\\ 0 & -\sin^2\theta_W & 0\\ 0 & 0 & -1\end{pmatrix}\,,$$

$$\widetilde{\mathcal{L}}_4 = \frac{1}{\sqrt{g^2+g'^2}}\bigl(g'\mathcal{L}_3 + g\mathcal{L}_4\bigr) = ie\mathcal{Q} = ie\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\,.$$

Note that $\widetilde{\mathcal{L}}_a\nu \neq 0$ for $a = 1, 2, 3$ and $\widetilde{\mathcal{L}}_4\nu = ie\mathcal{Q}\nu = 0$, which implies that three Goldstone bosons are present and provide masses for the $W^\pm$ and Z. It is convenient to use eqs. (48) and (49) to write

$$m_W = gv/\sqrt{2}\,, \qquad m_Z = \frac{gv}{\cos\theta_W}\,. \tag{59}$$

It is then straightforward to evaluate,

$$\widetilde{\mathcal{L}}_1\nu = \tfrac{1}{2}igv\begin{pmatrix}0\\1\\0\end{pmatrix}\,, \qquad \widetilde{\mathcal{L}}_1\nu = \tfrac{1}{2}gv\begin{pmatrix}0\\1\\0\end{pmatrix}\,, \qquad \widetilde{\mathcal{L}}_3\nu = -\frac{igv}{\sqrt{2}\cos\theta_W}\begin{pmatrix}0\\0\\1\end{pmatrix}\,.$$

Using the complex representation of scalar fields, the Goldstone boson fields are given by

$$G_a = \frac{1}{m_a}\bigl[\overline{\phi}^\dagger\widetilde{\mathcal{L}}_a\nu + (\widetilde{\mathcal{L}}_a\nu)^\dagger\overline{\Phi}\bigr]\,, \tag{60}$$

where $\Phi_i \equiv \nu_i + \overline{\Phi}_i$ and $\nu_i \equiv \langle\Phi_i\rangle$ is the (complex) scalar field vacuum expectation value. Then, eq. (60) yields

$$G_1 = \sqrt{2}\,\mathrm{Im}\;\Phi^+\,, \qquad G_2 = \sqrt{2}\,\mathrm{Re}\;\Phi^+\,, \qquad G_3 = -\sqrt{2}\,\mathrm{Im}\;\Phi^0\,,$$

where we have used eq. (59) to simplify our results. Thus, we have reproduced eq. (55).

[7] One can repeat this calculation using the generators given in eq. (28). In this case, in order to preserve $\mathrm{U}(1)_{\mathrm{EM}}$, one must choose the vacuum expectation value of the form $\nu = \frac{1}{2}v(1\,,\,-i\,,\;0)$, so that $\mathcal{Q}\nu = 0$. One can check that with this choice, eq. (58) yields the gauge boson squared-mass matrix obtained in eq. (47).

(c) If both doublet and triplet Higgs fields exist in nature, what does this exercise imply about the parameters of the Higgs Lagrangian?

In the Standard Model with a complex hypercharge-one Higgs doublet with $\langle\Phi^0\rangle = v/\sqrt{2}$, one finds

$$m_W^2 = \tfrac{1}{4}g^2v^2\,, \qquad m_Z^2 = \tfrac{1}{4}(g^2+g'^2)v^2 \quad\Longrightarrow\quad \rho \equiv \frac{m_W^2}{m_Z^2\cos^2\theta_W} = 1\,.$$

This can be compared with the results of parts (a) and (b). In a model with a real hypercharge-zero Higgs triplet with $\langle\Phi^0\rangle = v$,

$$m_W^2 = g^2v^2\,, \qquad m_Z^2 = 0 \quad\Longrightarrow\quad \rho \equiv \frac{m_W^2}{m_Z^2\cos^2\theta_W} = \infty\,.$$

In a model with a complex hypercharge-two Higgs triplet with $\langle\Phi^0\rangle = v/\sqrt{2}$,

$$m_W^2 = \tfrac{1}{2}g^2v^2\,, \qquad m_Z^2 = (g^2+g'^2)v^2 \quad\Longrightarrow\quad \rho \equiv \frac{m_W^2}{m_Z^2\cos^2\theta_W} = \frac{1}{2}\,.$$

In a model with multiple Higgs bosons, each vacuum expectation value contributes to the W and the Z mass. Since experimental observation confirms that $\rho \simeq 1$, the conclusion of this analysis is that if Higgs triplet fields also exist, then there are two possibilities. Either, the vacuum expectation values of the triplet fields are much smaller than that of the doublet field, in which case we would expect that the relation $\rho = 1$ would be minimally disturbed. A second possibility is that the vacuum expectation values are arranged such that the contribution of the triplet fields to the W and Z masses cancels almost exactly. An example of such a model was proposed by H. Georgi and M. Machacek in 1985.[8]

For your amusement, I provide a general formula for ρ in a model with an arbitrary number of Higgs multiplets of isospin T and hypercharge Y (note that a scalar field with weak isospin T has $2T+1$ components),[9]

$$\rho \equiv \frac{m_W^2}{m_Z^2\cos^2\theta_W} = \frac{\sum_{T,Y}\left[4T(T+1)-Y^2\right]|v_{T,Y}|^2 c_{T,Y}}{\sum_{T,Y} 2Y^2|v_{T,Y}|^2}\,, \tag{61}$$

where $\langle\Phi^0(T,Y)\rangle \equiv v_{T,Y}$ defines the vacuum expectation value of each neutral Higgs field of weak isospin T and hypercharge Y. In addition, we have introduced the notation,

$$c_{T,Y} = \begin{cases} 1\,, & (T,Y) \in \text{complex representation}, \\ \tfrac{1}{2}\,, & (T,Y) \in \text{real representation}. \end{cases}$$

Here, we employ a rather narrow definition of a real representation, which consists of a real multiplet of scalar fields with integer weak isospin and $Y=0$.

It is a simple matter to check that eq. (61) reproduces the cases considered above. Note that Higgs doublet and triplet fields have weak isospins $T=\frac{1}{2}$ and $T=1$, respectively.

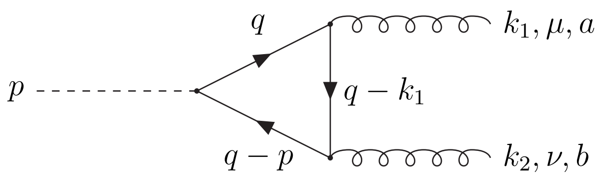
[8]H. Georgi and M. Machacek, *Nucl. Phys.* **B262**, 463 (1985).

[9]See eq. (4.1) of J.F. Gunion, H.E. Haber, G.L. Kane and S. Dawson, *The Higgs Hunter's Guide* (Westview Press, Boulder, CO, 2000).

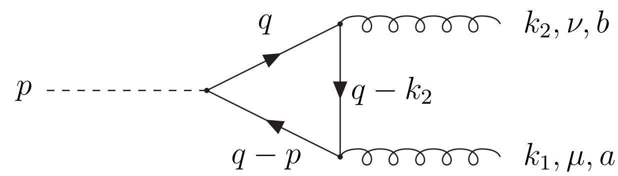
3. In the Standard Model, the Higgs boson H couples to two gluons via a one-loop triangle diagram containing top quarks in the loop.[10]

(a) Compute the amplitude for the decay of the Higgs boson to two gluons ($H \to gg$), as a function of m_t, m_H, G_F (the Fermi constant) and $\alpha_s \equiv g_s^2/(4\pi)$, using perturbation theory in the one loop approximation. Simplify your answer by invoking the kinematics of the problem, *i.e.* the conservation of four-momentum and the on-shell conditions for the external particles.

There are two Feynman diagrams contributing to $H \to gg$ at one loop:

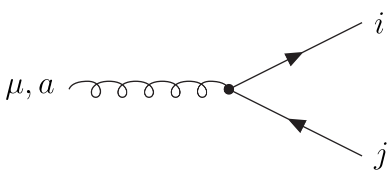


(A)

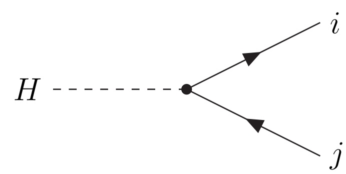


(B)

Diagrams (A) and (B) differ in that the outgoing gluons are interchanged. The relevant Feynman rules for the vertices are:



$-ig_s\gamma^\mu T^a_{ij}$



$-\dfrac{igm_t}{2m_W}\,\delta_{ij}$

where i and j are the color indices of the top quark and a is the (adjoint) color index of the gluon. It is convenient to rewrite g in terms of the Fermi constant G_F [cf. eq. (29.74) of Schwartz],

$$\sqrt{2}\,G_F \equiv \frac{g^2}{4m_W^2}\,.$$

Applying the Feynman rules, and recalling the minus sign for the closed fermion loop,

$$i\mathcal{M}_A = -(\sqrt{2}\,G_F)^{1/2}\int\frac{d^4q}{(2\pi)^4}\,\frac{i^3\,\mathrm{Tr}\big[(-im_t)(\not{q}-\not{p}+m_t)(-ig_s\gamma^\nu)(\not{q}-\not{k}_1+m_t)(-ig_s\gamma^\mu)(\not{q}+m_t)\big]}{(q^2-m_t^2+i\varepsilon)\big[(q-p)^2-m_t^2+i\varepsilon\big]\big[(q-k_1)^2-m_t^2+i\varepsilon\big]}$$
$$\times\,\mathrm{Tr}(T^bT^a)\,\epsilon^*_{\mu\,a}(k_1,\lambda_1)\epsilon^*_{\nu\,b}(k_2,\lambda_2)\,, \tag{62}$$

where the factor of i^3 arises from the three fermion propagators. Next, $\mathcal{M}_b$ is obtained from $\mathcal{M}_a$ by interchanging $k_1 \leftrightarrow k_2$, $\mu \leftrightarrow \nu$ and $a \leftrightarrow b$,

$$i\mathcal{M}_B = -(\sqrt{2}\,G_F)^{1/2}\int\frac{d^nq}{(2\pi)^n}\,\frac{i^3\,\mathrm{Tr}\big[(-im_t)(\not{q}-\not{p}+m_t)(-ig_s\gamma^\mu)(\not{q}-\not{k}_2+m_t)(-ig_s\gamma^\nu)(\not{q}+m_t)\big]}{(q^2-m_t^2+i\varepsilon)\big[(q-p)^2-m_t^2+i\varepsilon\big]\big[(q-k_2)^2-m_t^2+i\varepsilon\big]}$$
$$\times\,\mathrm{Tr}(T^aT^b)\,\epsilon^*_{\mu\,a}(k_1,\lambda_1)\epsilon^*_{\nu\,b}(k_2,\lambda_2)\,. \tag{63}$$

[10] In this problem, we shall work in the approximation where all quarks are massless, with the exception of the top quark, in which case only triangle diagrams with top quarks in the loop contribute.

We now evaluate the trace that appears in the numerator in eq. (62). First, the trace over color yields $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. Next,

$$
\begin{aligned}
&\mathrm{Tr}\big[(\not{q}-\not{p}+m_t)\gamma^\nu(\not{q}-\not{k}_1+m_t)\gamma^\mu(\not{q}+m_t)\big] \\
&\quad = m_t^3\,\mathrm{Tr}(\gamma^\mu\gamma^\nu) + m_t\Big\{\mathrm{Tr}\big[(\not{q}-\not{p})\gamma^\nu(\not{q}-\not{k}_1)\gamma^\mu\big] + \mathrm{Tr}\big[(\not{q}-\not{p})\gamma^\nu\gamma^\mu\not{q}\big] + \mathrm{Tr}\big[\gamma^\nu(\not{q}-\not{k}_1)\gamma^\mu\not{q}\big]\Big\} \\
&\quad = 4m_t^3 g^{\mu\nu} + 4m_t\Big\{(q-p)^\mu(q-k_1)^\nu + (q-p)^\nu(q-k_1)^\mu - g^{\mu\nu}(q-p)\cdot(q-k_1) \\
&\qquad\quad +(q-p)^\nu q^\mu + g^{\mu\nu}q\cdot(q-p) - (q-p)^\mu q^\nu + (q-k_1)^\nu q^\mu + (q-k_1)^\mu q^\nu - g^{\mu\nu}q\cdot(q-k_1)\Big\} \\
&\quad = 4m_t\Big\{g^{\mu\nu}\big[m_t^2 - q^2 + 2q\cdot k_1 - p\cdot k_1\big] + 4q^\mu q^\nu - 2q^\mu(k_1+p)^\nu - 2k_1^\mu q^\nu + p^\mu k_1^\nu + k_1^\mu p^\nu\Big\}\,. \qquad (64)
\end{aligned}
$$

To perform the integral over q, we introduce Feynman parameters. Following Appendix B.1.1 of Schwartz,

$$
\frac{1}{ABC} = 2\int_0^1 dx\int_0^1 dy\int_0^1 dz\;\delta(x+y+z-1)\,\frac{1}{[xA+yB+zC]^3}\,.
$$

Integrating over z yields,

$$
\frac{1}{ABC} = 2\int_0^1 dx\int_0^{1-x} dy\,\frac{1}{[xA+yB+(1-x-y)C]^3}\,. \qquad (65)
$$

Identifying A$\equiv (q-p)^2-m_t^2$, $B\equiv (q-k_1)^2-m_t^2$ and $C\equiv q^2-m_t^2$, the resulting denominator factor in eq. (65), denoted by D below, is given by

$$
D = (1-x-y)(q^2-m_t^2) + \big[(q-p)^2-m_t^2\big]x + \big[(q-k_1)^2-m_t^2\big]y = q^2-2q\cdot(px+k_1y)-m_t^2+p^2x+k_1^2y+i\varepsilon\,.
$$

For the physical $H\to gg$ decay, we have $p^2=m_H^2$ and $k_1^2=0$, where m_H is the mass of the Higgs boson. Then,

$$
D = q^2 - 2q\cdot(px+k_1y) + m_H^2x - m_t^2 + i\varepsilon\,.
$$

Hence, the factor of 2 from the Feynman parameter integration cancels the factor of $\frac{1}{2}$ from the color trace, and we obtain

$$
\begin{aligned}
\mathcal{M}_A = {}& 4ig_s^2m_t^2(G_F\sqrt{2})^{1/2}\,\epsilon^*_{\mu a}(k_1,\lambda_1)\epsilon^*_{\nu a}(k_2,\lambda_2)\int_0^1 dx\int_0^{1-x} dy \\
&\times\int\frac{d^nq}{(2\pi)^n}\,\frac{g^{\mu\nu}\big[m_t^2-q^2+2q\cdot k_1-p\cdot k_1\big]+4q^\mu q^\nu-2q^\mu(k_1+p)^\nu-2k_1^\mu q^\nu+p^\mu k_1^\nu+k_1^\mu p^\nu}{\big[q^2-2q\cdot(px+k_1y)+m_H^2x-m_t^2+i\varepsilon\big]^3}\,.
\end{aligned}
$$

It is convenient to isolate the numerator term that is quadratic in q, since this term yields a potential divergence. Let us write

$$
\mathcal{M}_A = \big[\mathcal{M}_A^{(1)\,\mu\nu} + \mathcal{M}_A^{(2)\,\mu\nu}\big]\,\epsilon^*_{\mu a}(k_1,\lambda_1)\epsilon^*_{\nu a}(k_2,\lambda_2)\,, \qquad (66)
$$

where

$$\mathcal{M}_A^{(1)\,\mu\nu} = 4ig_s^2 m_t^2 (G_F\sqrt{2})^{1/2}\int_0^1 dx\int_0^{1-x} dy\int\frac{d^4q}{(2\pi)^4}\,\frac{4q^\mu q^\nu - g^{\mu\nu}q^2}{\bigl[q^2-2q\cdot(px+k_1y)+m_H^2x-m_t^2+i\varepsilon\bigr]^3}$$

$$\mathcal{M}_A^{(2)\,\mu\nu} = 4ig_s^2 m_t^2 (G_F\sqrt{2})^{1/2}\int_0^1 dx\int_0^{1-x} dy\int\frac{d^4q}{(2\pi)^4}$$

$$\times\;\frac{g^{\mu\nu}(m_t^2+2q\cdot k_1-p\cdot k_1)-2q^\mu(k_1+p)^\nu-2k_1^\mu q^\nu+p^\mu k_1^\nu+k_1^\mu p^\nu}{\bigl[q^2-2q\cdot(px+k_1y)+m_H^2x-m_t^2+i\varepsilon\bigr]^3}\,.$$

Using the formulae given in the class handout entitled *Useful formulae for computing one-loop integrals*,[11]

$$\int\frac{d^nq}{(2\pi)^n}\,\frac{4q^\mu q^\nu-g^{\mu\nu}q^2}{\bigl[q^2-2q\cdot P-M^2+i\varepsilon\bigr]^3}=\frac{-i\Gamma(\epsilon)(4\pi)^\epsilon}{32\pi^2}(P^2+M^2)^{-1-\epsilon}\bigl[4\epsilon P^\mu P^\nu-\epsilon(2P^2+M^2)g^{\mu\nu}\bigr]\,,$$

where $\epsilon\equiv 2-\tfrac{1}{2}n$. Using $\epsilon\,\Gamma(\epsilon)=\Gamma(1+\epsilon)$, we see that the above integral is finite as $\epsilon\to 0$. Hence taking the $n\to 4$ limit,

$$\lim_{n\to 4}\int\frac{d^nq}{(2\pi)^n}\,\frac{4q^\mu q^\nu-g^{\mu\nu}q^2}{\bigl[q^2-2q\cdot P-M^2+i\varepsilon\bigr]^3}=\frac{-i\bigl[4P^\mu P^\nu-g^{\mu\nu}(2P^2+M^2)\bigr]}{32\pi^2(P^2+M^2)}\,,$$

where one can now set $\varepsilon=0$ in the denominator.

In computing $\mathcal{M}_a^{(1)\,\mu\nu}$, we identify $P=px+k_1y$ and $M^2=m_H^2x-m_t^2$. Hence,

$$P^2+M^2=m_t^2-m_H^2x(1-x)+2p\cdot k_1xy=m_t^2-m_H^2x(1-x-y)\,.$$

At the final step above, we evaluated $p\cdot k_1=\tfrac{1}{2}m_H^2$ using the kinematic constraints of the $H\to gg$ decay.[12] Hence,

$$\mathcal{M}_A^{(1)\,\mu\nu}=\frac{g_s^2m_t^2(G\sqrt{2})^{1/2}}{8\pi^2}\int_0^1dx\int_0^{1-x}dy\left\{-2g^{\mu\nu}+\frac{4(px+k_1y)^\mu(px+k_1y)^\nu+g^{\mu\nu}(m_t^2-m_H^2x)}{m_t^2-m_H^2x(1-x-y)}\right\}.\tag{67}$$

Further simplification can be achieved by using the properties of the gluon polarization vectors,

$$k_1^\mu\epsilon_\mu(k_1,\lambda_1)=k_2^\nu\epsilon_\nu(k_2,\lambda_2)=0\,.\tag{68}$$

By writing $p=k_1+k_2$ in the numerator of the integrand in eq. (67), we can then omit any terms proportional to k_1^μ and/or k_2^ν. The end result is,

$$\mathcal{M}_A^{(1)\,\mu\nu}=\frac{g_s^2m_t^2(G_F\sqrt{2})^{1/2}}{8\pi^2}\int_0^1dx\int_0^{1-x}dy\,\frac{\bigl[-m_t^2+m_H^2x(1-2x-2y)\bigr]g^{\mu\nu}+4x(x+y)k_2^\mu k_1^\nu}{m_t^2-m_H^2x(1-x-y)}\,.\tag{69}$$

[11] Do not confuse $\epsilon\equiv 2-\tfrac{1}{2}n$ with the infinitesimal number ε that appears in the propagator denominators.

[12] Since four-momentum conservation implies that $k_2=p-k_1$, we have

$$0=k_2^2=(p-k_1)^2=p^2-2p\cdot k_1+k_1^2=m_H^2-2p\cdot k_1\,,$$

after using $k_1^2=k_2^2=0$ and $p^2=m_H^2$. Hence, we conclude that $m_H^2=2p\cdot k_1$.

To evaluate $\mathcal{M}_A^{(2)\,\mu\nu}$ we can set $n = 4$ (or equivalently set $\epsilon = 0$) when evaluating the relevant integral given in the class handout entitled *Useful formulae for computing one-loop integrals*, since this loop integral is manifestly finite.

$$\mathcal{M}_A^{(2)\,\mu\nu} = \frac{g_s^2 m_t^2 (G_F\sqrt{2})^{1/2}}{8\pi^2} \int_0^1 dx \int_0^{1-x} dy$$

$$\times \frac{g^{\mu\nu}\big[m_t^2 - p\cdot k_1 + 2k_1\cdot(px+k_1 y)\big] + p^\mu k_1^\nu + p^\nu k_1^\mu - 2(px+k_1y)^\mu (k_1+p)^\nu - 2(px+k_1y)^\nu (k_1+p)^\mu}{m_t^2 - m_H^2 x(1-x-y)}.$$

We can simplify this result by imposing the kinematical constraints [cf. footnote 12],

$$k_1^2 = k_2^2 = 0\,, \qquad p^2 = m_H^2 = 2p\cdot k_1\,.$$

In addition, we write $p = k_1 + k_2$ and drop terms proportional to k_1^μ and/or k_2^ν, as noted below eq. (68). The end result it,

$$\mathcal{M}_A^{(2)\,\mu\nu} = \frac{g_s^2 m_t^2 (\sqrt{2}G_F)^{1/2}}{8\pi^2} \int_0^1 dx \int_0^{1-x} dy\, \frac{g^{\mu\nu}\big[m_t^2 + m_H^2(x-\tfrac{1}{2})\big] + (1-4x)k_2^\mu k_1^\mu}{m_t^2 - m_H^2 x(1-x-y)}\,. \qquad (70)$$

Adding up eqs. (69) and (70) yields,

$$\mathcal{M}_A^{\mu\nu} = \frac{g_s^2 m_t^2 (\sqrt{2}G_F)^{1/2}}{16\pi^2}\big(m_H^2 g^{\mu\nu} - 2k_2^\mu k_1^\nu\big) \int_0^1 dx \int_0^{1-x} dy\, \frac{4x(1-x-y)-1}{m_t^2 - m_H^2 x(1-x-y)}\,.$$

We can immediately write down the result for $\mathcal{M}_B^{\mu\nu}$ by interchanging $k_1 \leftrightarrow k_2$, $\mu \leftrightarrow \nu$ and $a \leftrightarrow b$. It follows that $\mathcal{M}_A^{\mu\nu} = \mathcal{M}_B^{\mu\nu}$. Hence, the sum of the amplitudes resulting from the two contributing one-loop Feynman diagrams is

$$\mathcal{M} = \frac{\alpha_s m_t^2 (\sqrt{2}G_F)^{1/2}}{2\pi}\big(m_H^2 g^{\mu\nu} - 2k_2^\mu k_1^\nu\big)\, \epsilon^*_{\mu\, a}(k_1,\lambda_1)\epsilon^*_{\nu\, a}(k_2,\lambda_2) \int_0^1 dx \int_0^{1-x} dy\, \frac{4x(1-x-y)-1}{m_t^2 - m_H^2 x(1-x-y)}\,,$$

after writing $\alpha_s \equiv g_s^2/(4\pi)$.

(b) Denote the amplitude for $H \to gg$ by $\mathcal{M}_{\mu\nu}$, where μ and ν are the Lorentz indices of the two gluons. Gauge invariance implies that $k_1^\mu \mathcal{M}_{\mu\nu} = k_2^\nu \mathcal{M}_{\mu\nu} = 0$, where k_1 and k_2 are the respective gluon momenta.[13] Check that your amplitude obtained in part (a) respect this requirement.

The result from part (a) yields

$$\mathcal{M}_{\mu\nu} = \frac{\alpha_s g m_t^2 (\sqrt{2}G_F)^{1/2}}{2\pi}\big(m_H^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\big) \int_0^1 dx \int_0^{1-x} dy\, \frac{4x(1-x-y)-1}{m_t^2 - m_H^2 x(1-x-y)}\,. \qquad (71)$$

[13]In this computation, no three gluon vertex appears since the gluon does not couple directly to the Higgs boson. Consequently, the Ward identities of QED also apply here.

It is straightforward to verify that $k_1^\mu \mathcal{M}_{\mu\nu} = k_2^\nu \mathcal{M}_{\mu\nu} = 0$. For example,

$$k_1^\mu \bigl(m_H^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\bigr) = (m_H^2 - 2k_1\!\cdot\! k_2)k_{1\nu} = 0\,,$$

after noting that

$$2k_1\!\cdot\! k_2 = (k_1+k_2)^2 - k_1^2 - k_2^2 = p^2 = m_H^2\,, \tag{72}$$

where we have used $p = k_1 + k_2$ and $k_1^2 = k_2^2 = 0$. Likewise,

$$k_2^\nu \bigl(m_H^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\bigr) = (m_H^2 - 2k_1\!\cdot\! k_2)k_{2\mu} = 0\,.$$

(c) Work out all integrals explicitly and evaluate the imaginary part of $\mathcal{M}_{\mu\nu}$. For what range of m_t/m_H is the amplitude purely real? Check your result for the imaginary part by using Cutkosky's rules [cf. problem 2 of Problem Set 2].

HINT: You may find the following integral useful:

$$\int_0^1 \frac{dy}{y}\,\log\bigl[1 - 4Ay(1-y)\bigr] = -2(\sin^{-1}\sqrt{A})^2$$

for $0 \leq A \leq 1$. For values of A outside this region, you may analytically continue the above result. The imaginary part of this integral is easily computed once the $i\epsilon$ factor is restored in the argument of the logarithm.

We examine the integral,

$$I = \int_0^1 dx \int_0^{1-x} dy\, \frac{4x(1-x-y)-1}{1-Rx(1-x-y)}\,, \tag{73}$$

where $R \equiv m_H^2/m_t^2$. Rewrite the numerator as

$$4x(1-x-y) - 1 = \frac{4\bigl[Rx(1-x-y)-1\bigr] + 4 - R}{R}\,.$$

Then,

$$\begin{aligned} I &= -\frac{4}{R}\int_0^1 dx \int_0^{1-x} dy + \frac{4-R}{R}\int_0^1 dx \int_0^{1-x} dy\, \frac{1}{1-Rx(1-x-y)} \\ &= -\frac{2}{R} + \frac{R-4}{R^2}\int_0^1 \frac{dx}{x}\,\ln\bigl[1-Rx(1-x)\bigr]\,. \end{aligned} \tag{74}$$

Thus, we must now evaluate

$$J \equiv \int_0^1 \frac{dx}{x}\,\ln\bigl[1-Rx(1-x)\bigr]\,.$$

Using the hint provided,

$$J = -2\bigl[\sin^{-1}\bigl(\tfrac{1}{2}\sqrt{R}\,\bigr)\bigr]^2\,, \qquad \text{for } 0 \leq R \leq 4\,.$$

To analytically continue beyond $R = 4$, we make use of

$$\sin^{-1} z = -i \ln\left[iz + \sqrt{1-z^2}\,\right]$$

For $z > 1$, we have $\sqrt{1-z^2} = \pm i\sqrt{z^2-1}$, where the sign ambiguity will be addressed shortly. Then,

$$\begin{aligned}\sin^{-1} z = &- i \ln\left[i\left(z \pm \sqrt{z^2-1}\,\right)\right] = -i \ln\left[e^{i\pi/2}\left(z \pm \sqrt{z^2-1}\,\right)\right] \\ &= -i\left[\tfrac{1}{2}i\pi + \ln\left(z \pm \sqrt{z^2-1}\,\right)\right] = \tfrac{1}{2}\pi \mp i \ln\left(z \pm \sqrt{z^2-1}\,\right).\end{aligned}$$

To obtain the final result above, we used the fact that

$$z - \sqrt{z^2-1} = \frac{1}{z + \sqrt{z^2-1}}\,,$$

which implies that

$$\ln\left(z - \sqrt{z^2-1}\right) = -\ln\left(z + \sqrt{z^2-1}\right).$$

To resolve the sign ambiguity, we shall compute $\operatorname{Im} J$ directly following the procedure of Problem 2 of Problem Set 2. Here, we will need to put back the factor of $i\varepsilon$ by replacing $m_t^2 \to m_t^2 - i\varepsilon$. Since $R \equiv m_H^2/m_t^2$, this means that we should replace $R \to R + i\varepsilon$. Noting that $x(1-x) > 0$ for $0 < x < 1$, we examine,

$$J \equiv \int_0^1 \frac{dx}{x} \ln\left[1 - Rx(1-x) - i\varepsilon\right].$$

Following eqs. (25)–(27) of Solution Set 2, the roots of the argument of the logarithm are given by

$$x_\pm = \tfrac{1}{2}\left[1 \pm \sqrt{1 - \frac{4}{R}}\,\right]. \tag{75}$$

Thus,

$$\begin{aligned}\operatorname{Im} J = \Theta(R-4) \int_{x_-}^{x_+} \frac{dx}{x} \operatorname{Im}\ln\left[1 - Rx(1-x) - i\varepsilon\right] = -\Theta(R-4)\pi \int_{x_-}^{x_+} \frac{dx}{x} \\ = -\Theta(R-4)\pi \ln\left(\frac{x_+}{x_-}\right) = -\Theta(R-4)\pi \ln\left(\frac{1 + \sqrt{1 - \frac{4}{R}}}{1 - \sqrt{1 - \frac{4}{R}}}\right).\end{aligned} \tag{76}$$

It follows that for $R > 4$, the correct analytic continuation is

$$J = -2\left[\tfrac{1}{2}\pi + i \ln\left(\frac{\sqrt{R}}{2} + \sqrt{\frac{R}{4} - 1}\right)\right]^2.$$

We check this by computing $\operatorname{Im} J$,

$$\operatorname{Im} J = -2\pi \ln\left(\frac{\sqrt{R}}{2} + \sqrt{\frac{R}{4} - 1}\right) = -\pi \ln\left(\frac{\sqrt{R}}{2} + \sqrt{\frac{R}{4} - 1}\right)^2 = -\pi \ln\left(\frac{1 + \sqrt{1 - \frac{4}{R}}}{1 - \sqrt{1 - \frac{4}{R}}}\right),$$

in agreement with eq. (76). We can thus rewrite J in the following form,

$$J = \begin{cases} -2\left[\sin^{-1}\left(\tfrac{1}{2}\sqrt{R}\right)\right]^2, & \text{for } 0 \le R \le 4 \\ -\tfrac{1}{2}\left[\pi + i\ln\left(\frac{1+\sqrt{1-\frac{4}{R}}}{1-\sqrt{1-\frac{4}{R}}}\right)\right]^2 . & \text{for } R > 4 . \end{cases}$$

It is convenient to introduce a function $f(R)$ defined by

$$f(R) = \begin{cases} \sin^{-1}\left(\tfrac{1}{2}\sqrt{R}\right), & \text{for } 0 \le R \le 4 \\ \tfrac{1}{2}\left[\pi + i\ln\left(\frac{1+\sqrt{1-\frac{4}{R}}}{1-\sqrt{1-\frac{4}{R}}}\right)\right] . & \text{for } R > 4 . \end{cases} \tag{77}$$

Then $J = -2[F(R)]^2$, and eq. (74) yields,

$$I = -\frac{2}{R}\left\{1 + \left(1 - \frac{4}{R}\right)\left[f(R)\right]^2\right\} .$$

In light of eq. (73), we see that eq. (71) yields

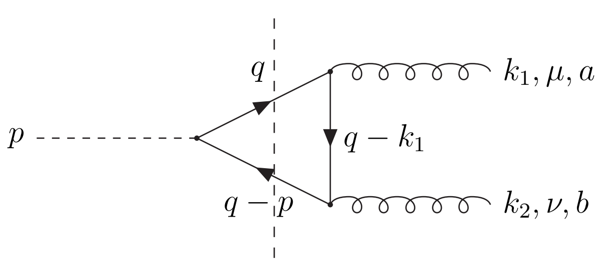
$$\mathcal{M}_{\mu\nu} = -\frac{\alpha_s m_t^2(\sqrt{2}G_F)^{1/2}}{\pi m_H^2}\left(m_H^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\right)\left\{1 + \left(1 - \frac{4}{R}\right)\left[f(R)\right]^2\right\} . \tag{78}$$

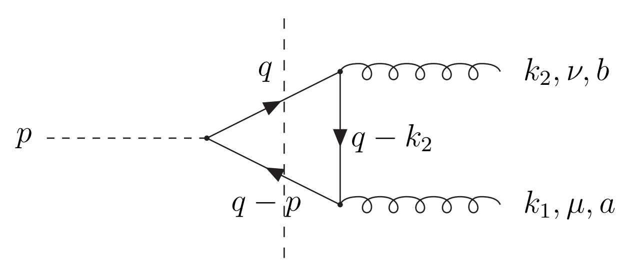
In particular,

$$\operatorname{Im}\mathcal{M}_{\mu\nu} = \frac{\alpha_s m_t^2(\sqrt{2}G_F)^{1/2}}{2m_H^2}\left(m_H^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\right)\left(\frac{4}{R} - 1\right)\ln\left(\frac{1+\sqrt{1-\frac{4}{R}}}{1-\sqrt{1-\frac{4}{R}}}\right)\Theta(R-4) . \tag{79}$$

Thus, $\operatorname{Im}\mathcal{M}_{\mu\nu} \neq 0$ when $R = m_H^2/m_t^2 > 4$, which corresponds to $m_H > 2m_t$. In this case, the kinematics allows the Higgs boson to decay into a $t\bar{t}$ pair. Thus, we can cut the triangle diagrams to reveal the on-shell top quarks.

As a check of our calculation, we can evaluate $\operatorname{Im}\mathcal{M}_{\mu\nu}$ directly using Cutkosky's rules. There is one way to "cut" each of the two Feynman diagrams contributing to $H \to gg$ at one loop such that the internal cut lines are on-shell:





The cut propagators are evaluated according to Cutkosky's cutting rules,

$$\frac{1}{q^2 - m^2 + i\varepsilon} \quad\longrightarrow\quad -2\pi i\delta(q^2 - m^2)\Theta(q_0) ,$$
$$\frac{1}{(q-p)^2 - m^2 + i\varepsilon} \quad\longrightarrow\quad -2\pi i\delta\big((q-p)^2 - m^2\big)\Theta(p_0 - q_0) .$$

Inserting these replacements in eqs. (62) and (63), we obtain expressions for $2i\,\mathrm{Im}\,\mathcal{M}_A$ and $2i\,\mathrm{Im}\,\mathcal{M}_B$. As above, it is convenient to write $\mathcal{M} = \mathcal{M}^{\mu\nu}\,\epsilon^*_{\mu\,a}(k_1,\lambda_1)\epsilon^*_{\nu\,a}(k_2,\lambda_2)$ after evaluating $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta_{ab}$ and summing over colors. Since the contributions of both diagrams are equal, it is sufficient to compute $2i\,\mathrm{Im}\,\mathcal{M}_A$ and multiply by 2. This factor of 2 is canceled by the $\frac{1}{2}$ from the color trace, and we obtain

$$\mathrm{Im}\,\mathcal{M}^{\mu\nu} = 2(\sqrt{2}\,G_F)^{1/2}g_s^2 m_t^2(-2\pi i)^2\int\frac{d^4q}{(2\pi)^4}\,\delta(q^2-m_t^2)\Theta(q_0)\delta\big((q-p)^2-m_t^2\big)\Theta(p_0-q_0)$$
$$\times\,\frac{g^{\mu\nu}\big[m_t^2-q^2+2q\cdot k_1-p\cdot k_1\big]+4q^\mu q^\nu-2q^\mu(k_1+p)^\nu-2k_1^\mu q^\nu+p^\mu k_1^\nu+k_1^\mu p^\nu}{(q-k_1)^2-m_t^2+i\varepsilon}\,.$$

after using eq. (64). In light of eq. (68) we can omit any terms proportional to k_1^μ and/or k_2^ν (after writing $p = k_1 + k_2$). We may use the delta function to set $q^2 = m_t^2$ in the numerator and denominator above. In addition, $k_1^2 = 0$ for the massless gluon. The end result is

$$\mathrm{Im}\,\mathcal{M}^{\mu\nu} = 32\pi^3(\sqrt{2}\,G_F)^{1/2}\alpha_s m_t^2\int\frac{d^4q}{(2\pi)^4}\,\delta(q^2-m_t^2)\Theta(q_0)\delta\big((q-p)^2-m_t^2\big)\Theta(q_0-p_0)$$
$$\times\,\frac{g^{\mu\nu}(2q-p)\cdot k_1+4q^\mu(q-k_1)^\nu+k_2^\mu k_1^\nu}{2q\cdot k_1}\,, \tag{80}$$

after writing $g_s^2 = 4\pi\alpha_s$. Note that it is now safe to drop the $i\varepsilon$ term. The integral over q in eq. (80) was evaluated in the solution to problem 2(b) of Problem Set 2. Using eq. (33) of Solution Set 2 and adapting this solution to the integral of eq. (80), we obtain

$$\int\frac{d^4q}{(2\pi)^4}\,\delta(q^2-m_t^2)\Theta(q_0)\delta\big((q-p)^2-m_t^2\big)\Theta(p_0-q_0) = \frac{\beta}{128\pi^4}\,\Theta\big(m_H-2m_t\big)\int d\Omega_q\,, \tag{81}$$

where

$$\beta \equiv \left(1-\frac{4m_t^2}{m_H^2}\right)^{1/2}\,. \tag{82}$$

Note that in the solution to problem 2(b) of Problem Set 2, there was no dependence on the angles of the unit three-vector $\hat{\boldsymbol{q}}$, so the integration over $d\Omega_q$ was replaced by 4π. In eq. (80), the integrand does depend on angles, so we have retained the integration over $d\Omega_q$. Hence,

$$\mathrm{Im}\,\mathcal{M}^{\mu\nu} = \frac{(\sqrt{2}\,G_F)^{1/2}\alpha_s m_t^2\beta}{4\pi}\int d\Omega_q\,\frac{g^{\mu\nu}(2q-p)\cdot k_1+4q^\mu(q-k_1)^\nu+k_2^\mu k_1^\nu}{2q\cdot k_1}\,.$$

Thus, we need to evaluate three integrals,

$$I \equiv \int d\Omega_q\,\frac{1}{2q\cdot k_1}\,,\qquad J^\mu \equiv \int d\Omega_q\,\frac{q^\mu}{2q\cdot k_1}\,,\qquad K^{\mu\nu} \equiv \int d\Omega_q\,\frac{q^\mu q^\nu}{2q\cdot k_1}\,. \tag{83}$$

It then follows that

$$\mathrm{Im}\,\mathcal{M}^{\mu\nu} = \frac{(\sqrt{2}\,G_F)^{1/2}\alpha_s m_t^2\beta}{4\pi}\left[4K^{\mu\nu}+2g^{\mu\nu}k_1\cdot J-4J^\mu k_1^\nu+(k_2^\mu k_1^\nu-\tfrac{1}{2}m_H^2 g^{\mu\nu})I\right]\Theta(m_H-2m_t)\,, \tag{84}$$

after using $p\cdot k_1 = \frac{1}{2}m_H^2$ [cf. footnote 12].

To evaluate the integrals listed in eq. (83), it is convenient to work in the rest frame of the Higgs bosons where the gluons are emitted along the z-direction. Since q and $p-q$ represent incoming on-shell top quarks, it follows that

$$\begin{aligned} q &= \tfrac{1}{2}m_H(1\,;\,\beta\sin\theta\cos\phi\,,\,\beta\sin\theta\sin\phi\,,\,\beta\cos\theta)\,,\\ k_1 &= \tfrac{1}{2}m_H(1\,;\,0\,,\,0\,,\,1)\,,\\ k_2 &= \tfrac{1}{2}m_H(1\,;\,0\,,\,0\,,\,-1)\,,\\ p &= m_H(1\,;\,0\,,\,0\,,\,0)\,, \end{aligned} \tag{85}$$

where β is defined in eq. (82). All relevant dot products can now be computed. For example, $2q\cdot k_1 = \frac{1}{2}m_H^2(1-\beta\cos\theta)$. Thus,

$$I = \frac{4\pi}{m_H^2}\int_{-1}^{1}\frac{d\cos\theta}{1-\beta\cos\theta} = \frac{4\pi}{m_H^2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\,. \tag{86}$$

Due to the integration over ϕ, we see that $J^1 = J^2 = 0$, whereas

$$\begin{aligned} J^0 &= \frac{2\pi}{m_H}\int_{-1}^{1}\frac{d\cos\theta}{1-\beta\cos\theta} = \frac{2\pi}{m_H\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\,.\\ J^3 &= \frac{2\pi\beta}{m_H}\int_{-1}^{1}\frac{\cos\theta\, d\cos\theta}{1-\beta\cos\theta} = \frac{2\pi}{m_H\beta}\left[\ln\left(\frac{1+\beta}{1-\beta}\right) - 2\beta\right]\,. \end{aligned}$$

That is,

$$J^\mu = \frac{4\pi}{m_H^2}\left[(k_2-k_1)^\mu + \frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)k_1^\mu\right]\,. \tag{87}$$

There is an alternative method for deriving eq. (87), which is based in the observation that covariance with respect to Lorentz transformations implies that

$$J^\mu = Ak_1^\mu + Bk_2^\mu\,, \tag{88}$$

since k_1 and k_2 are only two independent four-vectors in the problem. Multiplying eq. (88) by $k_{1\mu}$ and $k_{2\mu}$, respectively, yields

$$B = \frac{4\pi}{m_H^2}\,,$$

and

$$A = \frac{2\pi}{m_H^2}\int_{-1}^{1}d\cos\theta\,\frac{q\cdot k_1}{q\cdot k_2} = \frac{2\pi}{m_H^2}\int_{-1}^{1}\frac{1+\beta\cos\theta}{1-\beta\cos\theta}\,d\cos\theta = \frac{4\pi}{m_H^2\beta}\left[\ln\left(\frac{1+\beta}{1-\beta}\right) - \beta\right]\,.$$

Inserting the above expressions for A and B back into eq. (88) confirms the result of eq. (87). Using $k_1\cdot k_2 = \frac{1}{2}m_H^2$, it follows that $k_1\cdot J = 2\pi$.

Likewise, covariance with respect to Lorentz transformations implies that

$$K^{\mu\nu} = Ag^{\mu\nu} + \frac{1}{m_H^2}\left[Bk_1^\mu k_1^\nu + Ck_1^\mu k_2^\nu + Dk_2^\mu k_1^\nu + Ek_2^\mu k_2^\nu\right]\,.$$

Multiplying this equation by $g_{\mu\nu}$, $k_{1\mu}k_{1\nu}$, $k_{1\mu}k_{2\nu}$, $k_{2\mu}k_{1\nu}$, and $k_{2\mu}k_{2\nu}$, respectively, yields five equations and five unknowns. First,

$$4A + \tfrac{1}{2}(C+D) = \int d\Omega_q \, \frac{q^2}{2q\cdot k_1} = m_t^2 I = \frac{4\pi m_t^2}{m_H^2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right), \tag{89}$$

after using $q^2 = m_t^2$ (since the top quark is on-shell) and $k_1^2 = k_2^2 = 0$ (corresponding to the massless gluons). The next three equations are:

$$E = \frac{4\pi}{m_H^2}\int_{-1}^{1} q\cdot k_1 \, d\cos\theta = \pi \int_{-1}^{1} (1-\beta\cos\theta)\, d\cos\theta = 2\pi\,, \tag{90}$$

$$A + \tfrac{1}{2}D = \frac{2\pi}{m_H^2}\int_{-1}^{1} q\cdot k_2 \, d\cos\theta = \tfrac{1}{2}\pi \int_{-1}^{1} (1+\beta\cos\theta)\, d\cos\theta = \pi\,, \tag{91}$$

$$A + \tfrac{1}{2}C = \frac{2\pi}{m_H^2}\int_{-1}^{1} q\cdot k_2 \, d\cos\theta = \tfrac{1}{2}\pi \int_{-1}^{1} (1+\beta\cos\theta)\, d\cos\theta = \pi\,. \tag{92}$$

Eqs. (91) and (92) imply that $C = D = 2(\pi - A)$. Subtracting the sum of eqs. (91) and (92) from eq. (89) yields

$$A = \frac{2\pi m_t^2}{m_H^2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - \pi\,. \tag{93}$$

The fifth equation is

$$B = \frac{4\pi}{m_H^2}\int_{-1}^{1} \frac{(q\cdot k_2)^2}{q\cdot k_1}\, d\cos\theta = 2\pi\int_{-1}^{1}\frac{(1+\beta\cos\theta)^2}{1-\beta\cos\theta}\, d\cos\theta = \frac{8\pi}{\beta}\left[\ln\left(\frac{1+\beta}{1-\beta}\right) - \tfrac{3}{2}\beta\right]. \tag{94}$$

Combining the results obtained above, we conclude that

$$\begin{aligned} K^{\mu\nu} = {} & \frac{2\pi m_t^2}{m_H^2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\left[g^{\mu\nu} - \frac{2}{m_H^2}\left(k_1^\mu k_2^\nu + k_2^\mu k_1^\nu\right)\right] + \frac{8\pi}{m_H^2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) k_1^\mu k_1^\nu \\ & -\pi g^{\mu\nu} + \frac{2\pi}{m_H^2}\left(k_2^\mu k_2^\nu + 2k_1^\mu k_2^\nu + 2k_2^\mu k_1^\nu - 6k_1^\mu k_1^\nu\right). \end{aligned} \tag{95}$$

We may omit all terms proportional to k_1^μ and/or k_2^μ, since these terms will vanish when contracted with the gluon polarization vectors. Hence, it is sufficient to employ the following expressions in eq. (84),

$$J^\mu = \frac{4\pi}{m_H^2}k_2^\mu\,, \tag{96}$$

$$K^{\mu\nu} = \frac{2\pi m_t^2}{m_H^2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\left[g^{\mu\nu} - \frac{2}{m_H^2}k_2^\mu k_1^\nu\right] - \pi\left(g^{\mu\nu} - \frac{4}{m_H^2}k_2^\mu k_1^\nu\right). \tag{97}$$

Noting that

$$2g^{\mu\nu}k_1\cdot J - 4J^\mu k_1^\nu = 4\pi\left(g^{\mu\nu} - \frac{4}{m_H^2}k_2^\mu k_1^\nu\right),$$

it then follows from eq. (84) that

$$\operatorname{Im}\mathcal{M}^{\mu\nu} = \frac{(\sqrt{2}\,G_F)^{1/2}\alpha_s m_t^2}{2m_H^2}\left(\frac{4m_t^2}{m_H^2} - 1\right)\ln\left(\frac{1+\beta}{1-\beta}\right)\left(m_H^2 g^{\mu\nu} - 2k_2^\mu k_1^\nu\right)\Theta(m_H - 2m_t)\,. \tag{98}$$

This result coincides with that of eq. (79), which completes the check of our calculation.

(d) Evaluate $\mathcal{M}_{\mu\nu}$ in the limit of $m_t \to \infty$.

The limit of $m_t \to \infty$ corresponds to $R = m_H^2/m_t^2 \to 0$. Using eq. (77), it follows that in the limit of $R \to 0$,

$$1 + \left(1 - \frac{4}{R}\right)\big[f(R)\big]^2 = 1 + \left(1 - \frac{4}{R}\right)\big[\sin^{-1}(\tfrac{1}{2}\sqrt{R})\big]^2 \simeq 1 + \left(1 - \frac{4}{R}\right)\left[\frac{\sqrt{R}}{2} + \frac{1}{6}\left(\frac{\sqrt{R}}{2}\right)^3\right]^2$$

$$\simeq 1 + \left(1 - \frac{4}{R}\right)\frac{R}{4}\left(1 + \frac{R}{24}\right)^2 \simeq 1 + \left(\frac{R}{4} - 1\right)\left(1 + \frac{R}{12}\right) \simeq \frac{R}{6}\,.$$

Hence, eq. (78) yields

$$\mathcal{M}_{\mu\nu}(H \to gg)\bigg|_{m_t \to \infty} = -\frac{\alpha_s(\sqrt{2}G_F)^{1/2}}{6\pi}\left(m_H^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\right). \tag{99}$$

(e) The dominant decay of the Higgs boson is into a pair of bottom quarks, $H \to b\bar{b}$. Evaluate the ratio of decay rates:

$$\frac{\Gamma(H \to gg)}{\Gamma(H \to b\bar{b})}$$

in the limit where $m_t \gg m_H$. In obtaining the decay rates into $b\bar{b}$ and gg respectively, you should sum the squared-amplitude over the final state spins and colors, and then evaluate the results numerically.

Using the results of Section 5.1 of Schwartz, the decay rate in the rest frame of the Higgs boson is given by

$$\Gamma = \frac{|\vec{\boldsymbol{k}}|}{32\pi^2 m_H^2}\sum_{\substack{\text{colors}\\ \text{spins}}}\int |\mathcal{M}|^2\,d\Omega\,, \tag{100}$$

where $|\vec{\boldsymbol{k}}| = \frac{1}{2}\lambda^{1/2}(m_H^2, m_1^2, m_2^2)/m_H$ is the magnitude of the three-momentum of one of the decaying particles in the rest frame of the Higgs boson. Here, m_1 and m_2 are the final state particle masses and $\lambda(a,b,c) \equiv (a+b-c)^2 - 4ab$ is the triangle function of relativistic kinematics. After the sum over final state spins and colors, the resulting squared-amplitude is independent of angles. Thus, the integration over $d\Omega$ is trivial and yields 4π.

For $H \to gg$, the identical particles in the final state imply that the integration over 4π steradians constitutes double counting, so we must divide by 2. Using $|\vec{\boldsymbol{k}}| = \frac{1}{2}m_H$ (since the gluons are massless), and including the factor of $\frac{1}{2}$ for identical final state particles,

$$\Gamma(H \to gg) = \frac{1}{32\pi m_H}\sum_{\substack{\text{colors}\\ \text{spins}}}|\mathcal{M}|^2\,. \tag{101}$$

Hence, employing eq. (99) in the limit of $m_t \gg m_H$,

$$\mathcal{M} = -\frac{\alpha_s(\sqrt{2}G_F)^{1/2}}{6\pi}\left(m_H^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\right)\epsilon_a^{\mu\,*}(k_1,\lambda_1)\epsilon_a^{\nu\,*}(k_2,\lambda_2)\,.$$

To sum over spins and colors, we make the following replacement[14]

$$\sum_\lambda \epsilon_a^{\nu\,*}(k,\lambda)\,\epsilon_b^{\beta}(k,\lambda) \longrightarrow -\delta_{ab}\,g^{\nu\beta}\,. \tag{102}$$

It then follows that

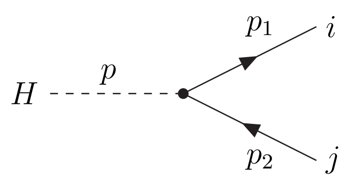
$$\sum_{\substack{\text{colors}\\ \text{spins}}} |\mathcal{M}|^2 = \frac{\sqrt{2}G_F\alpha_s}{36\pi^2}\left(m_H^2 g_{\mu\nu} - 2k_{2\mu}k_{1\nu}\right)\left(m_H^2 g_{\alpha\beta} - 2k_{2\alpha}k_{1\beta}\right)\left[-\delta_{ab}g^{\mu\alpha}\right]\left[-\delta_{ab}g^{\nu\beta}\right], \tag{103}$$

$$= \frac{\sqrt{2}\,G_F\alpha_s^2}{36\pi^2}4(m_H^4 - m_H^2 k_1\!\cdot\!k_2 + k_1^2k_2^2)\delta_{aa} = \frac{4\sqrt{2}\,G_F\alpha_s^2 m_H^4}{9\pi^2}\,, \tag{104}$$

after using $k_1\!\cdot\!k_2 = \frac{1}{2}m_H^2$ [cf. eq. (72)], $k_1^2 = k_2^2 = 0$ and $\sum_a \delta_{aa} = 8$ (where we have summed over the color SU(3) adjoint indices). Inserting the above result into eq. (101), we end up with

$$\Gamma(H \to gg) = \frac{\sqrt{2}\,G_F\alpha_s^2 m_H^3}{72\pi^3}\,. \tag{105}$$

The decay $H \to b\bar{b}$ arises at tree-level. We employ the Feynman rule for the $Hb\bar{b}$ vertex,



$$-\frac{igm_b}{2m_W}\,\delta_{ij}$$

where the momentum of the decaying Higgs boson is p and the two final state momenta p_1 and p_2 shown above should be taken as outgoing. The indices i and j label the color indices of the b quarks. The decay amplitude is then given by

$$i\mathcal{M} = -i(\sqrt{2}G_F)^{1/2}m_b\overline{u}_i(p_1,\lambda_1)v_i(p_2,\lambda_2)\,,$$

Squaring the amplitude and summing over spins and colors yields

$$\sum_{\substack{\text{colors}\\ \text{spins}}} |\mathcal{M}|^2 = \sqrt{2}\,G_F\,m_b^2\,\delta_{ij}\delta_{ij}\,\mathrm{Tr}\left[(\not{p}_2 - m_b)(\not{p}_1 + m_b)\right]$$

$$= 4\sqrt{2}\,G_F\delta_{ii}(p_1\!\cdot\!p_2 - m_b^2) = 6\sqrt{2}\,G_F\,m_b^2(m_H^2 - 4m_b^2)\,, \tag{106}$$

In the final step, we made use of the kinematics of the decay, where

$$2p_1\!\cdot\!p_2 = (p_1+p_2)^2 - p_1^2 - p_2^2 = m_H^2 - 2m_b^2\,,$$

[14]As discussed in the solution to problem 2(b) in Solution Set 3, eq. (102) is a valid replacement given that $k_1^\mu\mathcal{M}_{\mu\nu} = k_2^\nu\mathcal{M}_{\mu\nu} = 0$ has been verified in part (b) of this problem.

and we summed over the b quark colors using $\sum_i \delta_{ii} = 3$. Inserting the result of eq. (106) into eq. (100), and using $|\vec{\boldsymbol{k}}| = \frac{1}{2}\sqrt{m_H^2 - 4m_b^2}$, we end up with

$$\Gamma(H \to b\bar{b}) = \frac{3\sqrt{2}\, G_F m_b^2 (m_H^2 - 4m_b^2)^{3/2}}{8\pi m_H^2}\,. \tag{107}$$

Since $m_H \gg m_b$, we can approximate

$$\Gamma(H \to b\bar{b}) = \frac{3\sqrt{2}\, G_F m_b^2 m_H}{8\pi}\,. \tag{108}$$

Dividing eqs. (105) and (108), and using $\alpha_s \simeq 0.12$, $m_H \simeq 125$ GeV and $m_b \simeq 4.5$ GeV, we obtain

$$\frac{\Gamma(H \to gg)}{\Gamma(H \to b\bar{b})} \simeq \frac{\alpha_s^2 m_H^2}{27\pi^2 m_b^2} \simeq 0.04\,. \tag{109}$$

<u>*REMARK*</u>: It turns out that the tree-level predictions for $\Gamma(H \to gg)$ and $\Gamma(H \to b\bar{b})$ are significantly modified by QCD radiative corrections. The numerator of eq. (109) is underestimated by nearly a factor of 2 and the denominator is overestimated by nearly a factor of 2. Consequently the ratio given in eq. (109) is underestimated by a factor of about 3.5. The results of a more complete computation that takes these radiative corrections into effect are provided at `https://twiki.cern.ch/twiki/bin/view/LHCPhysics/CERNYellowReportPageBR` and yield $\Gamma(H \to gg)/\Gamma(H \to b\bar{b}) \simeq 0.14$ for $m_H = 125$ GeV.