Classical lumps and their quantum descendants:

A presentation based on the "Erice lecture" given by Sydney Coleman in 1975 [1].

Sydney Coleman with Abdus Salam in 1991 [2].

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Let $\Theta_{00}(\mathbf{x}, t)$ be the energy density of a classical field theory. We call a solution ϕ of the equation of motion (EOM) dissipative if

$$\lim_{t \to \infty} \max_{x} \Theta_{00}(\mathbf{x}, t) = 0.$$

Not all theories need be like this! There exist theories with non-dissipative behavior and these include some spontaneously broken gauge theories.

A simple example

Consider one spatial dimension and ignore derivative interactions

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi).$$

The energy of any field configuration is given by

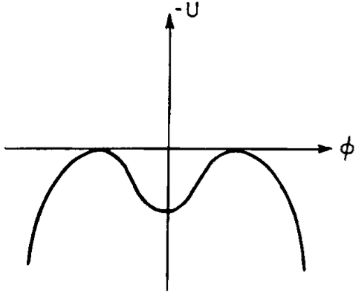
$$E = \int dx \left(\frac{1}{2}\left(\partial_0\phi\right)^2 + \frac{1}{2}\left(\partial_1\phi\right)^2 + U(\phi)\right).$$

A simple example

For a time independent solution, the EOM reduces to

$$\partial_1 \partial_1 \phi = U'(\phi).$$

For $x \leftrightarrow t$, $\phi \leftrightarrow x$ this is $F = ma$ for a particle in potential $-U$.



The equivalent particle problem for the case of ϕ^4 theory [1].

A simple example

1. If U has only one zero, then the only time independent solution is where ϕ is in the ground state forever.

2. If U has more than one zero, we can find a time independent solution of finite energy such that ϕ monotonically increases from one zero of U at $x = -\infty$ to another zero at $x = +\infty$.

To solve the EOM in the particle metaphor, we just set $E = 0$,

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - U(x) = 0$$

which translates back to field theory like,

$$\frac{1}{2}\left(\partial_1 \phi\right)^2 = U(\phi).$$

This has a solution

$$x = \pm \int_{\phi_0}^{\phi} d\phi' \left(2U(\phi')\right)^{-\frac{1}{2}}.$$

Are the solutions stable?

Yes, the EOM for our field is

$$\Box\phi + U'(\phi) = 0$$

Consider a perturbation $\delta(x, t)$ to our time independent solution $f(x)$, $\phi(x, t) = f(x) + \delta(x, t)$ and keep only $O(\delta)$ terms. Our perturbation must satisfy

$$\Box\delta + U''(f)\delta = 0.$$

This is invariant under time translations!

Are the solutions stable?

We can write our solution as a superposition of normal modes[1],

$$\delta(x,t) = \mathrm{Re} \sum_n a_n e^{i\omega_n t} \psi_n(x)$$

Where the ψ_n satisfy the one dimensional Schrödinger equation,

$$-\frac{d^2\psi_n}{dx^2} + U''(f)\psi_n = \omega_n^2 \psi_n.$$

If we can show that this Schrödinger equation has no negative eigenvalues, then we have proved stability.

[1] WLOG we work with discrete modes.

Are the solutions stable?

If $f(x)$ is a solution, then so is $f(x+a)$ (the center of the lump can be anywhere). For infinitesimal a,

$$\begin{aligned} f(x+a) &= f(x) + \frac{df}{dx}a \qquad \forall x \\ &= f(x) + \delta(x,t) \\ &= f(x) + \psi_0(x). \end{aligned}$$

So we found an eigenfunction $\psi_0(x) = \frac{df}{dx}$ with $\omega = 0$. I claim it is the ground state.

Are the solutions stable?

We showed that $f(x)$ is a monotonic function of x. This means $\psi_0 = \frac{df}{dx}$ has no nodes. Recall from QM that, for one-dimensional Schrödinger problems with arbitrary potential, the eigenfunction with no nodes is the eigenfunction of lowest energy [5]. We have bounded the energies to be greater than or equal to zero.

More dimensions and a discouraging theorem

G.H. Derrick proves that we cannot continue this method for higher dimensions.

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Comments on Nonlinear Wave Equations as Models for Elementary Particles

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It is shown that for a wide class of nonlinear wave equations there exist no stable time-independent solutions of finite energy. The possibility is considered whether elementary particles might be oscillating solutions of some nonlinear wave equation, in which the wavefunction is periodic in the time though the energy remains localized.

1. INTRODUCTION

IN an attempt to find a model for *extended* elementary particles, as opposed to singular *point* particles, Enz[1] has recently considered the nonlinear equation

$$\nabla^2\theta - (1/c^2)(\partial^2\theta/\partial t^2) = \tfrac{1}{2}\sin 2\theta, \qquad (1)$$

which is derived from the variation principle

where the energy is localized about a point on the x axis; if we further require that the solution be stable with respect to small deformations then only certain discrete energy values are permitted. In addition these one-dimensional solutions possess certain symmetry and topological properties which Enz suggests might correspond in the three-dimensional case to such discrete quantum numbers as charge or parity.

The original paper from Derrick [3].

More dimensions and a discouraging theorem

Theorem (*Derrick's Theorem*)

Let ϕ be a set of scalar fields (assembled into a big vector) in one time dimension and D space dimensions. Let the dynamics of these fields be defined by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - U(\phi) \tag{1}$$

and let U be non-negative and equal to zero for the ground state(s) of the theory. Then for $D > 2$ the only non-singular time-independent solutions of finite energy are the ground states.

A Discouraging Theorem

Proof.

Define $V_1 \equiv \frac{1}{2}\int d^D\mathbf{x}(\nabla\phi)^2$ and $V_2 \equiv \int d^D\mathbf{x}U(\phi)$. V_1 and V_2 are both non-negative and are simultaneously equal to zero only for the ground states. Define

$$\phi(\mathbf{x},\lambda) \equiv \phi(\lambda\mathbf{x})$$

where $\lambda \in \mathbb{R}^+$. For these functions the energy is given by

$$V[\phi(x,\lambda)] = \lambda^{2-D}V_1 + \lambda^{-D}V_2.$$

A Discouraging Theorem

Proof (cont.)

This must be stationary at $\lambda = 1$ wrt to all field configurations. So it is necessary (but not sufficient) that $V[\phi(\lambda, x)]$ is stationary wrt to the variation produced by λ.

$$\left.\frac{dV[\phi]}{d\lambda}\right|_{\lambda=1} = 0.$$

such that we find

$$(D-2)V_1 + DV_2 = 0$$

For $D > 2$ we force $V_1 = V_2 = 0$. Coleman suggests Derrick's theorem even holds for D=2, but this is not correct. Please see Appendix 1.

□

What can we do?

- Introduce gauge fields
- Introduce Fermion fields
- Find time-*dependant* non-dissipative behavior

Why do gauge theories get around Derrick's theorem

Alone, they do not.

Theorem (*Gauge Theory Derrick's Theorem*)

For the standard non-Abelian gauge theory with Lagrangian

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

all finite energy time independent solutions are gauge transforms of $A^a_\mu = 0$, except for $D = 4$.

Proof.

For a time-independent solution, L simplifies to

$$L = L_1 - L_2$$

where we have defined

$$L_1 \equiv \frac{1}{2}\int d^D (F_{0i}^a)^2 = \frac{1}{2}\int d^D x \left(\partial_i A_0^a + ef^{abc} A_0^b A_i^c\right)^2$$
$$L_2 \equiv \frac{1}{4}\int d^D x (F_{ij}^a)^2.$$

Proof (cont.)

Define

$$A_0^a(x;\sigma,\lambda) = \sigma\lambda A_0^a(\lambda x)$$
$$A_i^a(x;\sigma,\lambda) = \lambda A_i^a(\lambda x)$$

and after substitution we find

$$L(\sigma,\lambda) = \sigma^2\lambda^{4-D}L_1 - \lambda^{4-D}L_2.$$

This must be stationary wrt to the variations produced by λ *and* σ at $\lambda = \sigma = 1$. This gives

$$(4-D)L_1 - (4-D)L_2 = 0$$
$$2L_1 = 0$$

and so for $D \neq 4$ we force $L_1 = L_2 = 0$. This implies $F_{\mu\nu}^a = 0$, so that there exists a gauge where $A_\mu^a = 0$ [1]. □

Why gauge theories get around Derrick's theorem

In a gauge theory with a scalar field, the general form of the total energy is

$$\begin{aligned} E &= \int dx \left(|F|^2 + |D\phi|^2 + U(\phi) \right) \\ &\equiv V_4 + V_2 + V_0. \end{aligned}$$

We scale the independent scalar and gauge fields as we did in the previous proofs:

$$V(\lambda) = \lambda^{4-D} V_4 + \lambda^{2-D} V_2 + \lambda^{-D} V_0.$$

We find that for $D = 2, 3$ being on shell at $\lambda = 1$ leaves

$$\begin{aligned} V_4 - V_1 = 0 \qquad & D = 2 \\ V_4 - V_2 + 3V_1 = 0 \qquad & D = 3. \end{aligned}$$

Another path

Another way to find non-dissipative behavior in field theories is to discover & use topological conservation laws.

Another path: Topological Conservation laws

Again, finite energy solutions push the fields to a zero of U at infinity. That is,

$$U\left(\phi(\pm\infty, t)\right) = 0,$$

and since the zeroes of U form a finite set,

$$\partial_0 \phi(\pm\infty, t) = 0.$$

If U has multiple zeros, this equation can be used to prove the existence of non-dissipative solutions!

Another path: Topological Conservation laws

Consider the 1D ϕ^4 theory again, where we shift the minima to occur at $U = 0$:

$$U = \frac{\lambda}{2}\phi^4 - \mu^2\phi^2 + \frac{\mu^4}{2\lambda}.$$

If we demand the initial condition at one time,

$$\phi(\infty, t) = -\phi(-\infty, t)$$

we have it for all time. By continuity in x, for all t there exists some x s.t. $\phi = 0$. At this point, $\Theta_{00}(x, t) \geq U(0)$. But $U(0) = \frac{\mu^2}{2\lambda}$ such that for all t

$$\max_x \Theta_{00}(x, t) \geq \frac{\mu^2}{2\lambda}.$$

And so we have found non-dissipative behavior.

Another path: Topological Conservation laws

What just happened? Recall the conservation law:

$$\partial_0 \phi(\pm\infty, t) = 0.$$

It did not come from a symmetry in the Lagrangian. Instead, it arose because we had a discrete set of zeros of the potential. In another sense, we have divided the space of non-singular finite-energy solutions at a fixed time into subspaces (labeled by $\phi(\pm\infty, t)$) which are disconnected “in the normal topological sense”[1].

“In the normal topological sense”

Definition (Topological Space).

A topological space is an ordered pair (X, τ) where X is a set and τ is a collection of subsets of X that satisfy

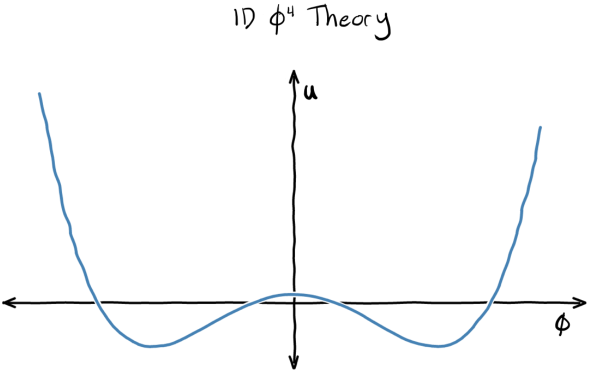
1. The empty set $\emptyset$ and X belong to τ.
2. Any (finite or infinite) union of members of τ belong to τ.
3. Any finite intersection of members of τ belong to τ.

The members of τ are called open.

Definition (Disconnected).

A topological space (X, τ) is said to be disconnected iff a pair of disjoint, non-empty open subsets X_1, X_2 exists, such that $X = X_1 \cup X_2$.

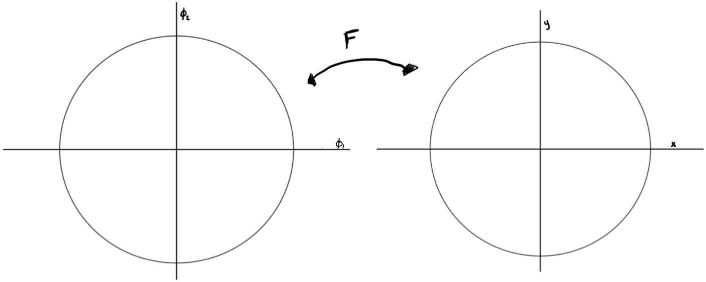
How many subspaces are there for our simple example? 4.



Recall the 1D ϕ^4 symmetry breaking theory.

Some vague but beautiful final points

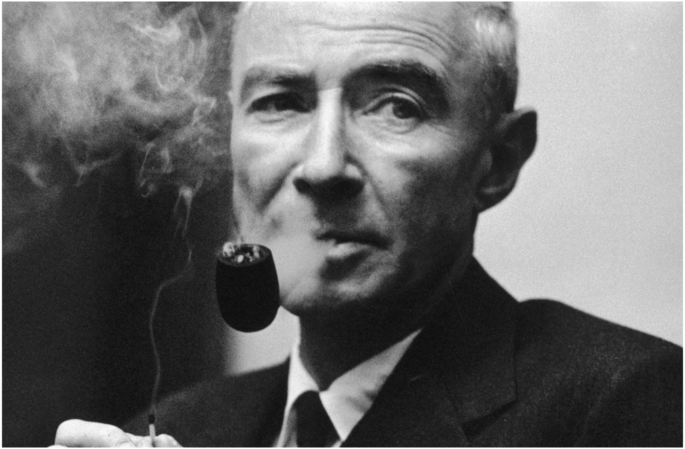
It generalizes, but it's not obvious how. Consider, for instance, that in two ϕ dimensions and two space dimensions there is no discrete set of zeroes to work with. However, the question of whether two solutions are connected is intimately related to the question of whether the corresponding functions from space at infinity (an $r = \infty$ circle in two dimensions, $r = \infty$ sphere in 3 dimensions, etc.) to the set of zeroes of U are homotopic.



Some vague but beautiful final points

1. If the theory has no spontaneous symmetry breakdown, the space of non-singular finite energy solutions has only one component, and there are no non-trivial topological conservation laws.
2. If the symmetry breakdown is total there is also only one component of finite energy solutions.
3. If one Goldstone survives, there are two cases:
   3.1 The gauge group when written as a product of simple Lie groups contains a U(1) factor, and the generator of this U(1) factor enters the expression for the electric charge (this occurs in Weinberg-Salam). In this case there are no non-trivial topological conservation laws.
   3.2 In other cases, there are interesting topological conservation laws (e.g. model of Glashow and Georgi), and it can be shown that some solutions emanate magnetic flux (but these are not Dirac monopoles!).

"I find the work hard, thank God, & almost pleasant." [7]
– Oppenheimer, on physics.



J Robert Oppenheimer [6].

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Appendix 1: the problem with D=2

The argument that Coleman makes for the case of $D = 2$ is as follows. For $D = 2$, our previous result

$$(D-2)V_1 + DV_2 = 0$$

only determines that $V_2 = 0$. Then, "if V_2 vanishes it is stationary, since zero is its minimum value. Thus we may apply Hamilton's principle to V_1 alone, from which it trivially follows that V_1 also vanishes."[1] This is true in the case when $U(\phi)$ has a set of *discrete* minima: in the discrete case, if $\phi(x)$ has to be at a minimum of $U(\phi)$ for all x, it has to be at the same minimum for all x. But there are cases of interest when the zeros of U are not discrete (e.g. the baby-skyrmion in a two dimensional O(3) model).