2. GENERATING FUNCTIONALS

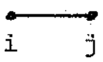
A. Propagators and vertices

A particle (an elementary excitation of a theory) is specified by a list of attributes; its name, its state (spin up, ingoing, ...), its spacetime location, etc. To develop the formalism of field theory, one does not need any specific part of this information, so we hide it in a single collective index:

$$i = \{q, a, \alpha, \mu, x_\mu, \; .... \}$$

q : particle type
a : colour
α : spin
μ : Minkowski indices
x_μ : spacetime coordinates (2.1)

A particle is an interesting particle only if it can do something. The simplest thing it can do is to change its position, its spin or some other attribute. The probability (amplitude) that this happens is described by the (bare) <u>propagators</u>:

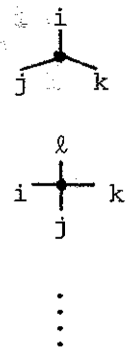
Δ_{ij} =  . (2.2)

Beyond this, many things can happen; a particle can split into two, or three, or many other particles. The probability (amplitude) that this happens is described by (bare) <u>vertices</u>:

γ_{ijk} =

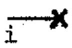
$\gamma_{ijk\ell}$ =

$\gamma_{ijk\ell m}$ =



(2.3)

A particle can also be created (or removed from the system). This is described by a <u>source</u> (or a <u>sink</u>):

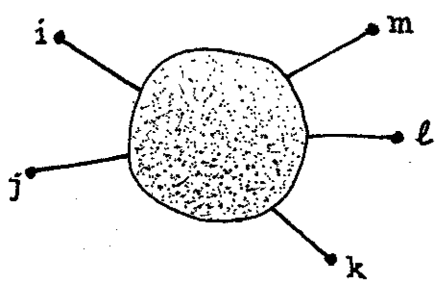
J_i =  . (2.4)

The concept of a particle makes sense only if its persistence probability (2.2) is appreciable, i.e. if (2.3), the probability of its disintegration, is relatively small. In that case the interactions (2.3) may be treated as small corrections, and the perturbation theory applies. If the "particle" described by attributes (2.1) has a negligible persistence probability, the theory should be reformulated in terms of another set of "elementary excitations" which are a better approximation to the physical spectrum of the theory (an easy thing to say).

How many identical particles (particles with all the same labels) can coexist? We shall consider two extremes: infinity (bosons) or at most one (fermions). Other more perverse possibilities cannot be excluded. Assumption of additivity of probabilities/amplitudes then implies that the bosonic propagators and vertices must be symmetric under interchange of indices $\Delta_{ij} = \Delta_{ji}$, $\gamma_{ijk} = \gamma_{jik} = \gamma_{ikj} = \dots$ (The argument is similar to the one we shall use for fermions in chapter 4). For the time being, we assume that the vertices (2.3) are symmetric.

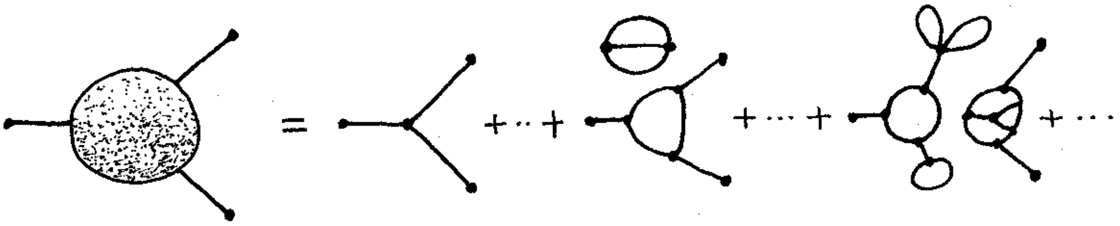
B. Green functions

A typical experiment consists of a setup of the initial particle configuration, followed by a measurement of the final configuration. The theoretical prediction is expressed in terms of the Green functions. For example, if we are considering an experiment in which particles i and j interact, and the outcome is particles k, ℓ, and m, we draw the corresponding Green functions

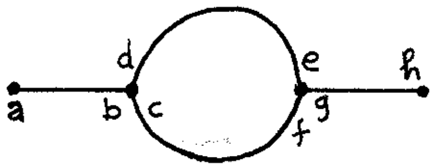
$G_{ijk\ell m} =$  (2.5)

(remember that labels i, j, ... stand for all variables and indices which specify a particle.)

A Green function is a sum of the probabilities (amplitudes) associated with all possible ways in which the final state can be reached. This is represented by an infinite sum of Feynman diagrams:



Each Feynman diagram corresponds to a sum (or an integral). For example, diagram



represents the probability that 1) a particle whose type, location, etc. is described by the collective index a reached any state labeled b; 2) that b splits into any two particles labeled c and d, and so forth. The intermediate states are summed over the entire range of possible index values

$$= \sum_{b,c,d,e,f,g} \Delta_{ab}\gamma_{bcd}\Delta_{cf}\Delta_{de}\gamma_{efg}\Delta_{gh} .$$

Here the summation signs imply sums over discrete indices (such as spin) and integrals over continuous indices (such as position). In the future we shall drop the explicit summation signs, and use instead Einstein's repeated index convention; if an index appears twice in a term, it is summed (integrated) over.

Exercise 2.B.1 Continuous indices. For QCD the collective index i stands for:

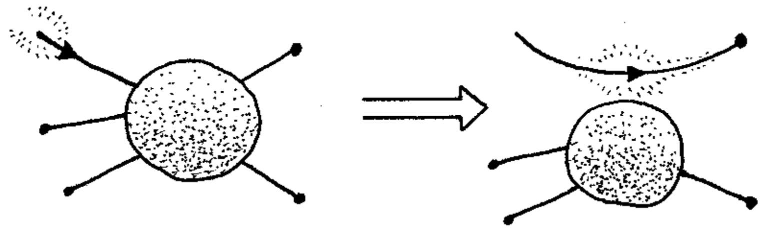
x^μ	spacetime coordinates,
$\mu = 1,2,\ldots,d$	Minkowski indices,
$j = 1,2,\ldots,N$	gluon colours.

If the propagator is denoted by $D^{ij}_{\mu\nu}(x,y)$ and the three-gluon vertex by $\gamma^{ijk}_{\mu\nu\sigma}(x,y,z)$, write down the complete expression for the above self-energy diagram.

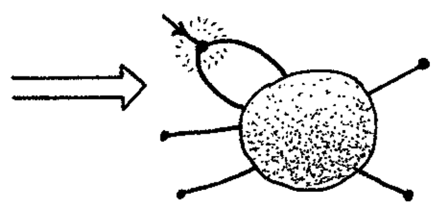
C. Dyson-Schwinger equations

A Green function consists of an infinity of Feynman diagrams. For a theory to be manageable, it is essential that these diagrams can be generated systematically, in order of their relative importance.

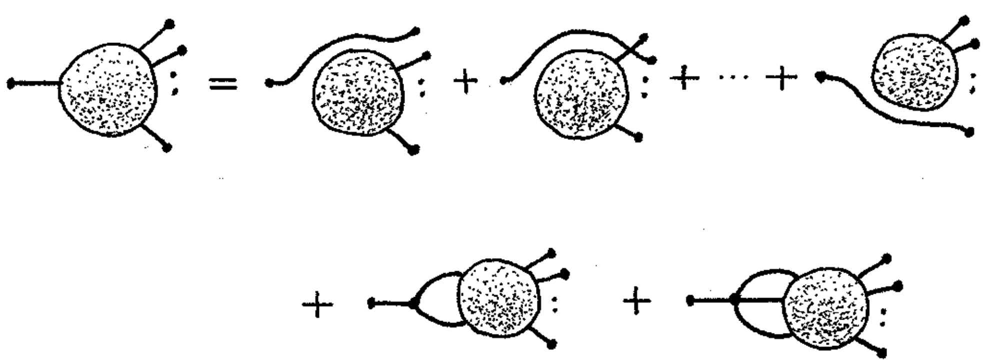
Consider (for simplicity) a theory with only cubic and quartic vertices†. Take a Green function and follow a particle into the blob. Two things can happen; either the particle survives



or it interacts at least once:



More precisely, entering the diagram via leg 1, we either reach leg 2, or leg 3, ... , or hit a three-vertex, or a four-vertex, etc. Adding up all the possibilities, we end up with the Dyson-Schwinger equations:



(2.6)

†Remember that the different particle types are covered by a single collective index, so QCD is also this type.

Iteration of the Dyson-Schwinger (DS) equations yields all Feynman diagrams contributing to a given process, ordered by the number of vertices (the order in perturbation theory).

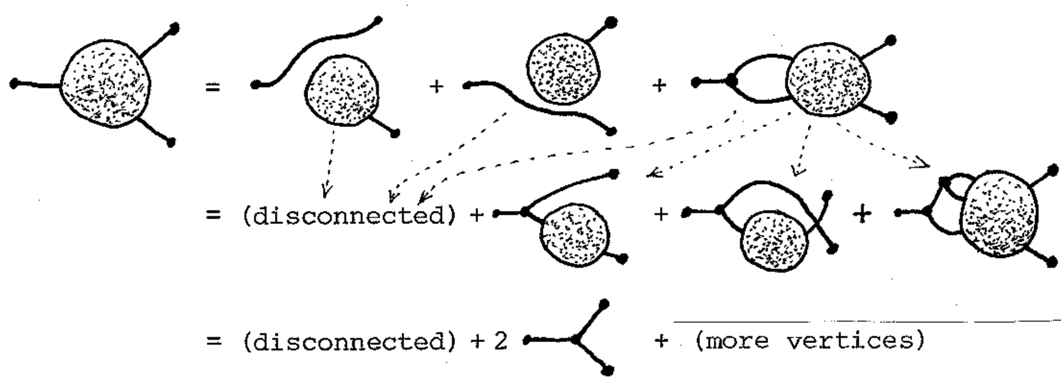
A few words about the diagrammatic notation; a diagrammatic equation like (2.6) contains precisely the same information as its algebraic transcription

$$G_{ij..k\ell} = \Delta_{i\ell}G_{j..k} + \Delta_{ik}G_{j..\ell} + \ldots + \Delta_{ij}G_{..k\ell}$$
$$+ \Delta_{ir}\gamma_{rst}G_{tsj..k\ell} + \Delta_{ir}\gamma_{rstu}G_{utsj..k\ell} \,.$$

Indices can always be omitted. An internal line implies a summation/integration over the corresponding indices, and for external lines the equivalent points on each diagram represent the same index in all terms of a diagrammatic equation. The advantages of the diagrammatic notation are obvious to all those who prefer the comic strip editions of "The greatest story ever told" to the unwieldy, fully indexed version†. Two of the principal benefits are that it eliminates "dummy indices" and that it does not force Feynman integrals into one-dimensional format (both being means whereby identical integrals can be made to look totally different).

D. Combinatoric factors

For a three-leg Green function the DS equations yield



It is rather unnatural that an expansion of a three-leg Green function does not start with the bare three-vertex, but twice

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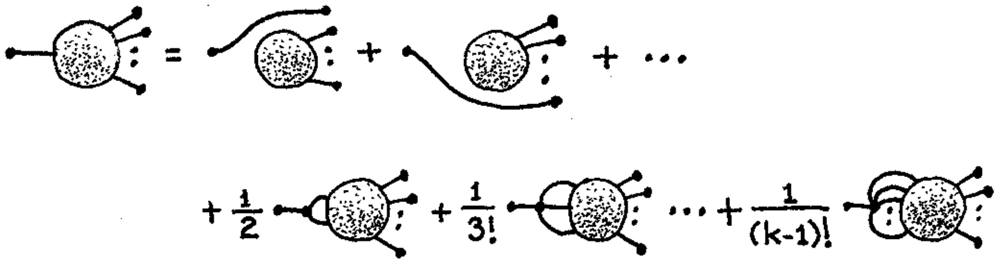
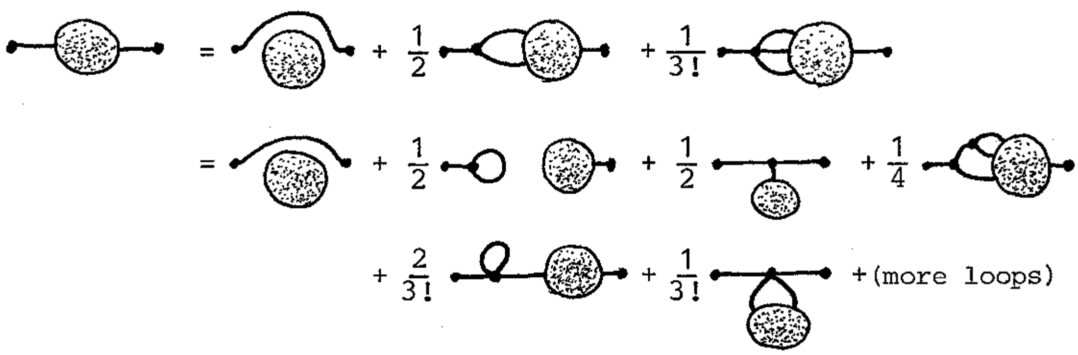
†C. Itzykson and J.-B. Zuber, Quantum Field Theory (McGraw-Hill, N.Y., 1980).

the bare three-vertex. This is easily fixed-up by including compensating combinatorial factors into DS equations:

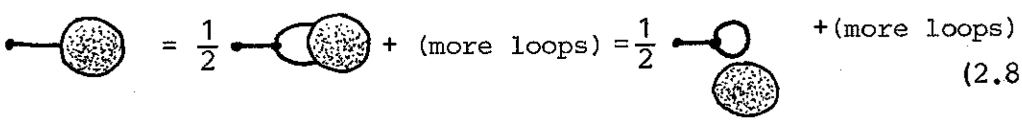


(2.7)

To illustrate how the DS equations generate the perturbation expansion, we expand a two-leg Green function up to one loop:

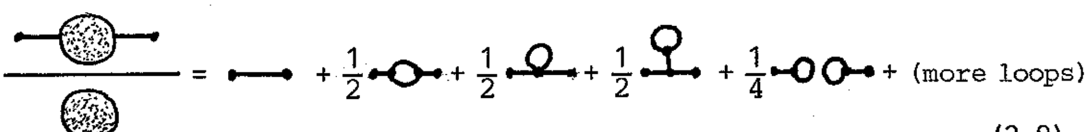


The one-loop tadpole is given by



(2.8)

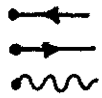
Substituting the tadpole into the above, we finally obtain the self-energy expansion up to two vertices with all the correct combinatoric factors:



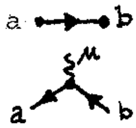
(2.9)

This expansion looks like the usual $\phi^3 + \phi^4$ theory, but it is not only that: the combinatoric factors are correct for any theory with cubic and quartic vertices, such as QCD with its full particle content.

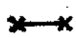
Exercise 2.D.1 Feynman diagrams in the collective index notation look like diagrams for scalar field theories. Nevertheless, they do contain the perturbative expansion for theories with arbitrary particle content. As an example, consider a QED-type theory with an "in" particle (electron), and "out" particle (positron) and a scalar particle (photon). The collective index (2.1) now ranges over an array of three sub-collective indices

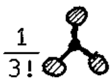
$$i = \begin{bmatrix} a, \text{ in} \\ a, \text{ out} \\ \mu \end{bmatrix} =$$  electron, positron, photon

Index a stands for the charged particle's position and spin, and index μ stands for all labels characterizing the neutral particle. The "in" - "out" labels can be eliminated by taking a to be an upper index for "in" particles, and a lower index for "out" particles. Diagrammatically they are distinguished by drawing arrows pointing away from upper indices and down into lower indices:

$$\Delta^a{}_b = $$ $$\gamma_{\mu a}{}^b =$$ 

Show that if the sources and fields are replaced by $J = (\eta^a, \eta_b, J_\mu)$, $\phi = (\psi_a, \psi^b, A^\mu)$, the combinatoric factors in (2.9) cancel, and the vertices such as the electron-positron-photon vertex have no combinatoric weight:

$\frac{1}{2}$  $= \frac{1}{2} J_i \Delta_{ij} J_j =$  $+ \frac{1}{2}$ 

$\frac{1}{3!}$  $= \frac{1}{3!} \gamma_{ijk} \phi_i \phi_j \phi_k = \gamma_{\mu b}{}^a \psi^b A_\mu \psi_a$.

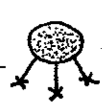
Exercise 2.D.2 Write the Dyson-Schwinger equations for QED-like theories. (We say "QED-like" because electrons are fermions. We shall return to the fermion DS equations later.)

Exercise 2.D.3 Determine the one-loop self-energy diagrams (2.9) for QED-like theories.

E. Generating functionals

The structure of the DS equations is very general; still, at present we have to write them separately for two-leg Green function, three-leg Green function, and so on. To state relations between Green functions in a more compact way we introduce generating functionals. A generating functional is the vacuum (legless) Green function for a theory with sources (2.4):

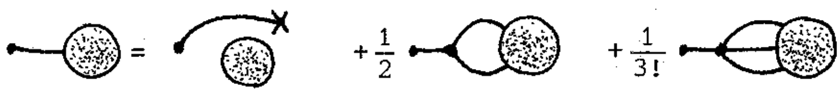
$$Z[J] = \sum_{m=0}^{\infty} \frac{1}{m!} G_{i_1 i_2 \ldots i_m} J_{i_1} J_{i_2} \ldots J_{i_m}$$

 $= 1 +$  $+ \frac{1}{2}$  $+ \frac{1}{3!}$  $+ \ldots$, (2.10)

(as J_i is a function which depends on both discrete and continuous indices, $Z[J]$ is a functional). The coefficients in this expansion are the usual Green functions. They can be retrieved from the generating functional by differentiation:

$$G_{ijk} = \frac{d}{dJ_i}\frac{d}{dJ_j}\frac{d}{dJ_k} Z[J]\bigg|_{J=0} \text{, etc.} \tag{2.11}$$

The DS equations (2.7) can be written as



$$\frac{d}{dJ_i} Z[J] = \Delta_{ij}\left\{J_j + \frac{1}{2}\gamma_{jk\ell}\frac{d}{dJ_\ell}\frac{d}{dJ_k} + \frac{1}{3!}\gamma_{jk\ell m}\frac{d}{dJ_m}\frac{d}{dJ_\ell}\frac{d}{dJ_k}\right\}Z[J] \ . \tag{2.12}$$

The bare propagators and vertices can themselves be collected in a functional called the <u>action</u>:

$$S[\phi] = -\frac{1}{2}\phi_i\Delta_{ij}^{-1}\phi_j + S_I[\phi] \ , \tag{2.13}$$

$$S_I[\phi] = \sum_m \underbrace{\gamma_{ijk..\ell}}_{m \text{ legs}} \frac{\phi_i\phi_j\cdots\phi_\ell}{m!} \ . \tag{2.14}$$

Now the Dyson-Schwinger equations can be stated in an even more elegant way:

$$0 = \left(\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right] + J_i\right)Z[J] \ , \tag{2.15}$$

where

$$\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right] \equiv \frac{dS[\phi]}{d\phi_i}\bigg|_{\phi=\frac{d}{dJ}} \ .$$

The action (or the Lagrangian) is just another way of defining the propagators and vertices for a given theory. Giving the Lagrangian or listing the Feynman rules is one and the same thing.

Exercise 2.E.1 Functional derivatives. For continuous indices the Kronecker deltas are replaced by Dirac deltas. For example, check that in d-dimensions

$$\frac{dJ(x)}{dJ(y)} = \delta^d(x-y) \ ,$$

is the correct definition of the derivative in (2.11).

Exercise 2.E.2 Feynman rules. Consider ϕ^3 theory given by the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}\,\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}\,m^2\phi(x)^2 - \frac{\mu}{3!}\,\phi(x)^3$$

$$S = \int d^dx\mathcal{L}(x) \ .$$

Read off the bare propagators and vertices (the Feynman rules) from the Lagrangian.

Hint: $\gamma_{ij..k} = \frac{d}{d\phi_i}\frac{d}{d\phi_j}\cdot\cdot\frac{d}{d\phi_k}\,S[\phi]\Big|_{\phi=0}$,

and the derivatives are in this case functional derivatives.

Exercise 2.E.3. Zero-dimensional field theory. Consider a ϕ^3 theory defined by trivial Feynman rules

The value of a graph with k vertices is g^k, and k-th order contribution to Green function is basically the number of contributing diagrams. More precisely, if

$$Z[J] = \sum_{k,m} G_k^{(m)} g^k \frac{J^m}{m!}$$

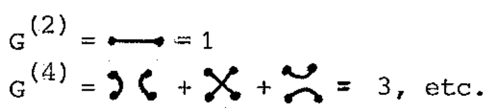
the Green function

$$G_k^{(m)} = \sum_G C_G$$

is the sum of combinatoric factors of all diagrams with m legs and k vertices. Use the Dyson-Schwinger equation (2.7) to show that for a free field theory

$$G_0^{(m)} = (m-1)!! \qquad \text{m even}$$
$$= 0 \qquad \text{m odd} \ .$$

Diagrammatically

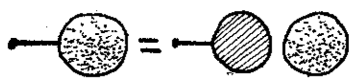


The zero-dimensional field theory is about the only field theory which is easily computable to all orders. We shall use it often to illustrate in a concrete way various field-theoretic relations.

F. Connected Green functions

Generating functionals are a powerful tool for stating relations between Green functions. For example, we can use them to derive relations between the full and the connected Green functions:

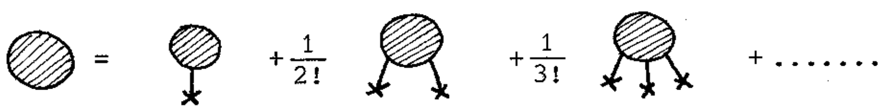
Pick out a leg and follow it into a full Green function. This separates all associated Feynman diagrams into two parts - the part that is connected to the initial leg, and the remainder:



$$\frac{d}{dJ_i} Z[J] = \frac{dW[J]}{dJ_i} Z[J] \tag{2.16}$$

The generating functional for the connected Green functions is defined in the same way as (2.10), the generating functional for the full Green functions:

$$W[J] = \sum_{m=1}^{\infty} \frac{1}{m!} G^{(c)}_{i_1 i_2 \ldots i_m} J_{i_1} J_{i_2} \ldots J_{i_m} ,$$

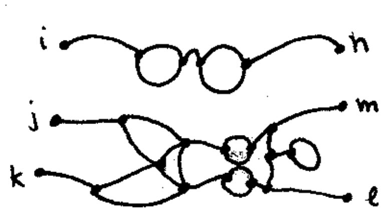


(2.17)

The differential equation (2.16) is easily solved

$$Z[J] = e^{W[J]} \tag{2.18}$$

A disconnected Feynman diagram such as



represents a product of two independent processes; one could take place on the moon, and the other in Aarhus. The physically interesting processes are described by the connected Green functions. To obtain a systematic perturbation series which includes only the connected Feynman diagrams, we use the identity[†]

$$\frac{1}{Z[J]}\frac{d}{dJ_i}Z[J] = \frac{dW[J]}{dJ_i} + \frac{d}{dJ_i} \tag{2.19}$$

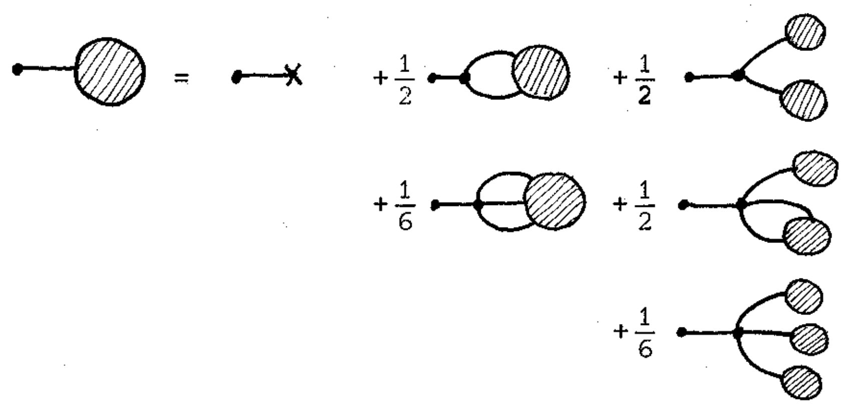
to rewrite the DS equations (2.15) in terms of the connected Green functions:

$$0 = \frac{dS}{d\phi_i}\left[\frac{dW[J]}{dJ} + \frac{d}{dJ}\right] + J_i \ . \tag{2.20}$$

This is very elegant, but possibly not too transparent. To get a feeling for these equations, take the $\phi^3 + \phi^4$ DS equations (2.12) and substitute $Z[J] = \exp(W[J])$. The result is, in the functional notation

$$\frac{dW[J]}{dJ_i} = \Delta_{ij}\left\{ J_j + \frac{1}{2}\gamma_{jk\ell}\left(\frac{d^2W[J]}{dJ_\ell dJ_k} + \frac{dW[J]}{dJ_k}\frac{dW[J]}{dJ_\ell}\right) + \frac{1}{6}\gamma_{jk\ell m}\left(\frac{d^3W[J]}{dJ_m dJ_\ell dJ_k} + 3\frac{dW[J]}{dJ_k}\frac{d^2W[J]}{dJ_m dJ_\ell} + \frac{dW[J]}{dJ_k}\frac{dW[J]}{dJ_\ell}\frac{dW[J]}{dJ_m}\right)\right\}, \tag{2.21}$$

and in the longleggedy notation



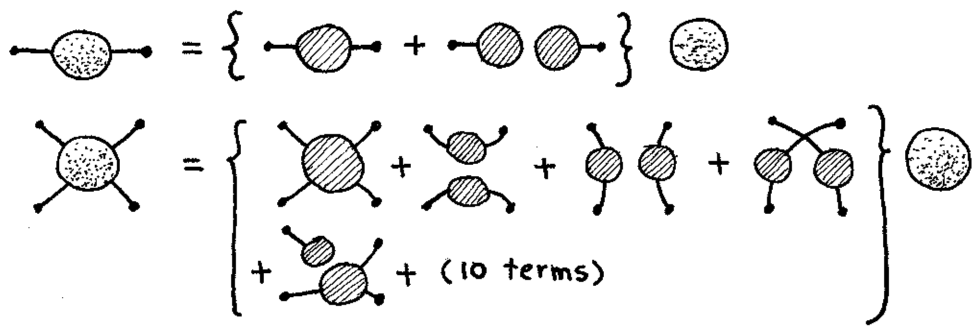
After reaching a vertex, one continues into diagrams that are either mutually disconnected, or connected - that is the reason that there are extra terms in the connected DS equations, compared with the full Green functions equations (2.12).

[†] more explicitly

$$\frac{1}{Z[J]}\frac{d}{dJ}(Z[J]f[J]) = \left(\frac{dW[J]}{dJ} + \frac{d}{dJ}\right)f[J].$$

Exercise 2.F.1 Use DS equations (2.21) to compute self-energy to one loop. How does the result differ from (2.9)?

Exercise 2.F.2 Expand some full Green functions in terms of the connected ones:

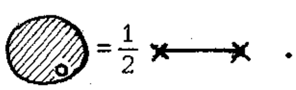


Hint: iterating (2.19) is probably the fastest way.

G. Free field theory

The connected generating functional for a free field theory is trivial: there are no interactions, so the only connected Feynman diagram is the propagator:

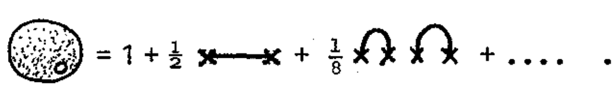
$$W_0[J] = \frac{1}{2} J_i \Delta_{ij} J_j$$



(2.22)

For the free field theory (2.18) gives an explicit expression for the generating functional:

$$Z_0[J] = e^{\frac{1}{2} J_i \Delta_{ij} J_j}$$

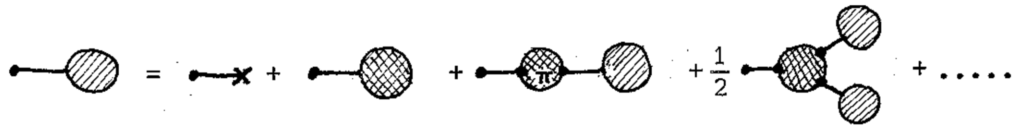


(2.23)

H. One-particle irreducible Green functions

A one-particle irreducible (1PI) diagram cannot be cut into two disconnected parts by cutting a single internal line. An arbitrary connected diagram has in general a number of such lines. The connected and the 1PI Green functions can be related by our usual diagrammatic trick:

Pick out a leg of a connected diagram. This pulls out a 1PI piece, which ends in 0, 1, 2, ... lines whose cutting would disconnect the diagram. Those lines continue into further connected pieces:



$$\phi_i = \Delta_{ij}(J_j + \Gamma_j + \pi_{jk}\phi_k + \tfrac{1}{2}\,\Gamma_{jk\ell}\phi_k\phi_\ell + \ldots) \, . \qquad (2.24)$$

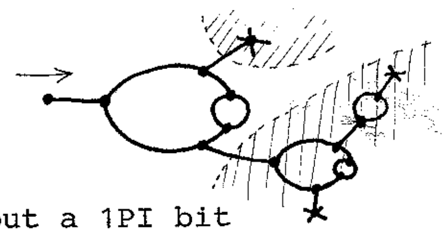
Here the "field" ϕ is defined by

$$= \phi_i = \frac{dW[J]}{dJ_i} \, . \qquad (2.25)$$

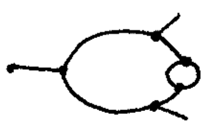
We draw the 1PI Green functions as cross-hatched blobs

$\Gamma_{ij..k} =$  .

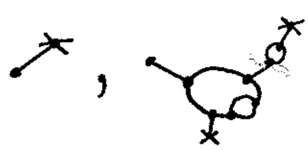
Unlike the full and the connected Green functions, the 1PI ones do not have propagators on external legs - the external indices always belong to a vertex of an 1PI diagram. This is indicated by drawing dots on the edges of 1PI Green functions. Any connected diagram belongs to one and only one term in the expansion (2.24). For example, going into connected diagram



we pull out a 1PI bit



followed by connected bits



Multiplying both sides of (2.24) by the inverse of the bare propagator we obtain

$$0 = J_i + \Gamma_i + (-\Delta^{-1} + \pi)_{ij}\phi_j + \tfrac{1}{2}\,\Gamma_{ijk}\phi_k\phi_j + \ldots \, .$$

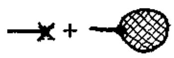
(For reasons which should soon be clear, it is convenient to define the two-leg Γ as $\Gamma_{ij} = -\Delta^{-1}_{ij} + \pi_{ij}$, where π_{ij} is the 1PI two-leg Green function, or the proper self-energy.)

Collecting all 1PI Green functions into the <u>effective action</u> functional

$$\Gamma[\phi] = \sum_{m=1} \Gamma_{ij..k} \frac{\phi_k \cdot\cdot \phi_j \phi_i}{m!} \quad , \tag{2.26}$$

we can write (2.24), the relation between the connected and the 1PI Green functions, as:

$$0 = J_i + \frac{d\Gamma[\phi]}{d\phi_i} \; ,$$

0 =  . (2.27)

This, together with (2.25), can be summarized by a <u>Legendre transformation</u>

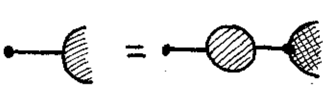
$$W[J] = \Gamma[\phi] + \phi_i J_i \; . \tag{2.28}$$

(2.27) guarantees that W is independent of ϕ, and (2.25) guarantees that Γ is independent of J:

$$\frac{dW[J]}{d\phi} = 0, \qquad \frac{d\Gamma[\phi]}{dJ} = 0 \; .$$

This is elegant, but how does it help us to get 1PI Green functions? The point is that we are not interested in extracting 1PI Green functions from the connected ones; what we need are the 1PI Dyson-Schwinger equations, i.e. the systematics of generating 1PI diagrams (and only 1PI diagrams). To achieve this, we must first eliminate J-derivatives in favour of ϕ-derivatives (cf. (2.25)):

$$\frac{d}{dJ_i} = \frac{d\phi_j}{dJ_i}\frac{d}{d\phi_j} = \frac{d^2W[J]}{dJ_i dJ_j}\frac{d}{d\phi_j}$$

 (2.29)

This accounts for all self-energy insertions. The right-hand side can be expressed in terms of 1PI Green functions by taking a derivative of (2.27):

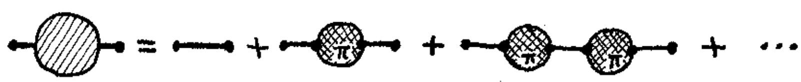
$$0 = \delta_{ij} + \frac{d}{dJ_j}\frac{d\Gamma[\phi]}{d\phi_i} = \delta_{ij} + \frac{d^2W[J]}{dJ_j dJ_k}\frac{d^2\Gamma[\phi]}{d\phi_k d\phi_i} \ . \qquad (2.30)$$

In order to understand this relation diagrammatically, we separate out the bare propagator in (2.26) by defining the "interaction" part of Γ:

$$\Gamma[\phi] = -\tfrac{1}{2}\phi_i \Delta^{-1}_{ij}\phi_j + \Gamma_I[\phi] \ . \qquad (2.31)$$

Now (2.30) can be written as

$$\frac{d^2W[J]}{dJ_i dJ_j} = \Delta_{ij} + \Delta_{ik}\frac{d^2\Gamma_I[\phi]}{d\phi_k d\phi_\ell}\Delta_{\ell j} + \ldots .$$



$$W[J]'' = \frac{1}{\Delta^{-1} - \Gamma_I[\phi]''} \ . \qquad (2.32)$$

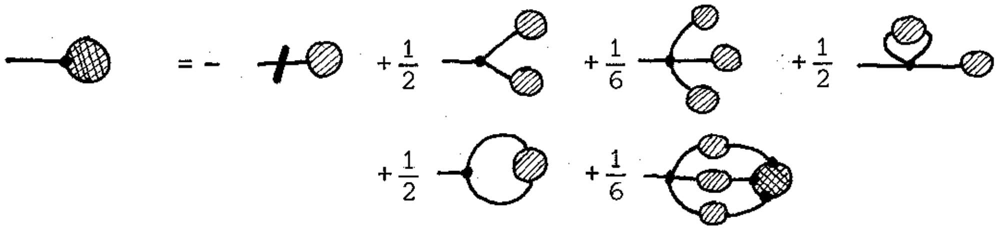
Diagrammatically W'' is a complete propagator which sums up all proper self-energies.

We can use (2.25) and (2.27) to eliminate source-dependent functionals in favour of field-dependent functionals, and (2.29) to replace J-derivatives by ϕ-derivatives, in order to rewrite (2.20) as the 1PI Dyson-Schwinger equation:

$$\frac{d\Gamma[\phi]}{d\phi_i} = \frac{dS}{d\phi_i}\left[\phi + W''[J]\frac{d}{d\phi}\right] . \qquad (2.33)$$

The form of this equation is one of the reasons why the generating functional for 1PI Green functions is called the effective action. If the derivatives are dropped, the effective action reduces to the classical action. The role of the derivatives is to generate loops, i.e. quantum corrections (or statistical fluctuations). We shall return to this in our discussion of path integrals.

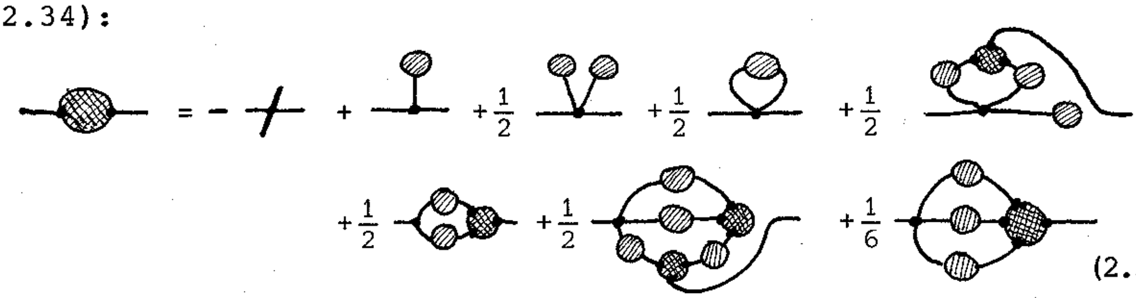
DS equations (2.33) are again so elegant that one is probably at a loss as to what to do with them. To get a feeling for their utility, we write them out for the $\phi^3 + \phi^4$ example (2.21):



$$\frac{d\Gamma[\phi]}{d\phi_i} = -\Delta^{-1}_{ij}\phi_j + \tfrac{1}{2}\gamma_{ijk}\phi_k\phi_j + \tfrac{1}{6}\gamma_{ijk\ell}\phi_\ell\phi_k\phi_j + \tfrac{1}{2}\gamma_{ijk\ell}\phi_\ell \frac{d^2W[J]}{dJ_k dJ_j}$$

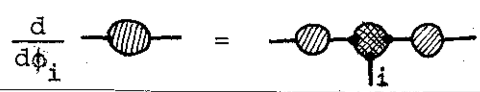
$$+ \tfrac{1}{2}\gamma_{ijk}\frac{d^2W[J]}{dJ_k dJ_j} + \tfrac{1}{6}\gamma_{ijk\ell}\frac{d^2W[J]}{dJ_j dJ_m}\frac{d^2W[J]}{dJ_k dJ_n}\frac{d^2W[J]}{dJ_\ell dJ_\sigma}\frac{d^3\Gamma[\phi]}{d\phi_m d\phi_n d\phi_\sigma} \quad . \tag{2.34}$$

Such equations are used iteratively. For example, to obtain the DS equation for the proper self-energy†, take a derivative of (2.34):



(2.35)

Exercise 2.H.1 Use (2.32) to show that



(2.36)

This is a useful identity for deriving relations such as (2.34) and (2.35).

Exercise 2.H.2 Take successive derivatives of (2.30) to show that the connected Green functions can be expanded in terms of 1PI Green functions as

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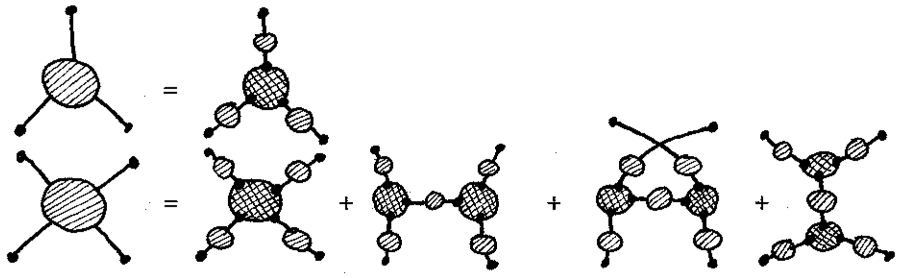
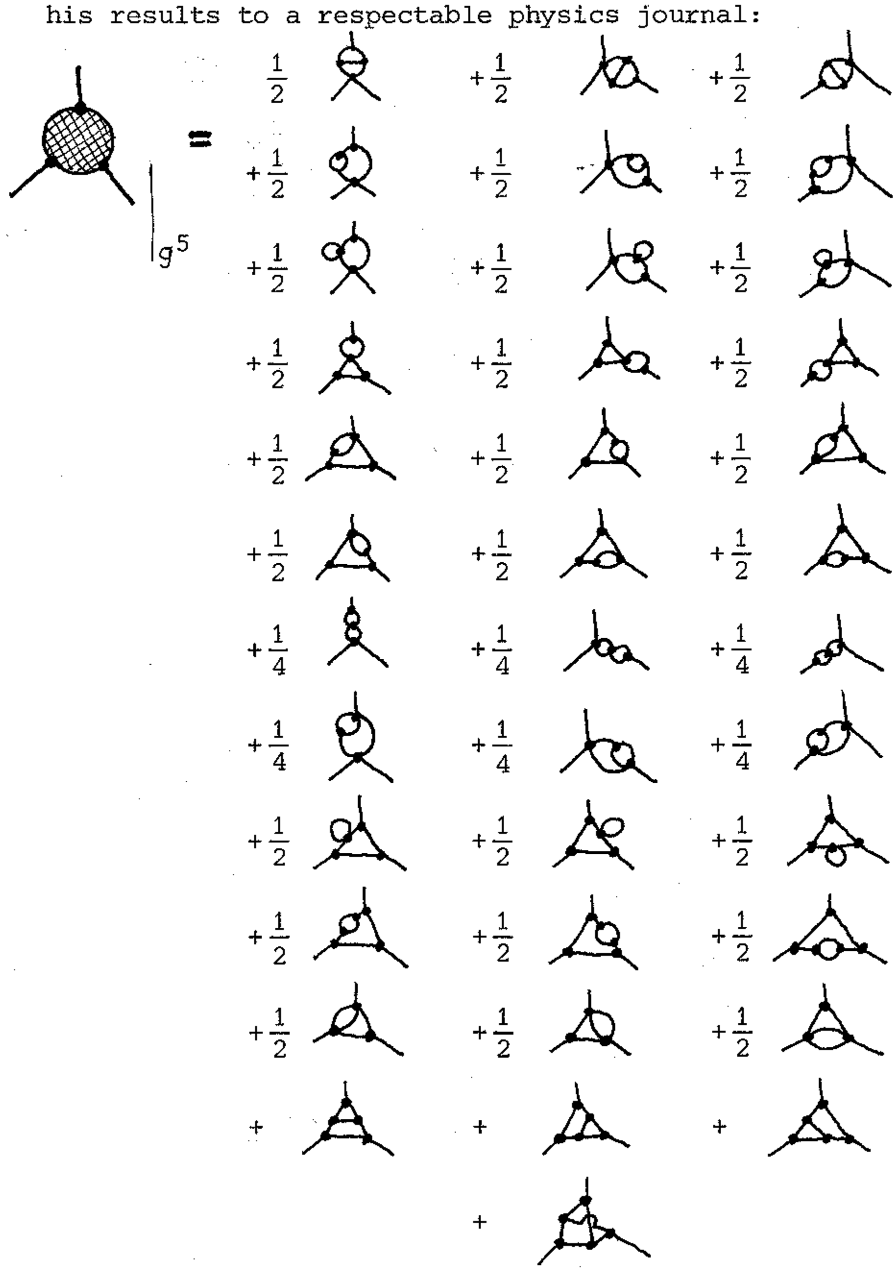
†Here the slash stands for inverse propagator; diagrammatically it is a two-leg vertex. Other vertices are denoted by dots, and a line connecting two vertices is always a propagator, so that $\Delta^{-1}_{ij}\Delta_{jk}$ = i—✚—k = i——k = δ_{ik}.



(2.37)

<u>Exercise 2.H.3</u> Jens J. Jensen, a serious young student of field theory, is getting set to compute the two-loop QCD beta-function. He has drawn up a list of gluon corrections to the three-gluon vertex. Use the 1PI Dyson-Schwinger equations to check this list and make Jens aware of 7 (seven) errors before he rushes his results to a respectable physics journal:



I. Vacuum bubbles

The Green function formalism we have developed so far is tailored to scattering problems; all the Green functions we have considered had external legs. Processes without external particles (the corresponding legless diagrams are called vacuum bubbles) are also physically interesting. For example, if a particle is propagating through a hot, dense soup†, a particle-particle scattering experiment would be a hopeless and messy undertaking. Such systems are probed by varying bulk parameters, such as temperature. Indeed, the generating functionals do not depend only on the single-particle sources J_i, but on all interaction parameters

$$Z[J] \equiv Z[J,\gamma_{ij},\gamma_{ijk},\gamma_{ijk\ell},\ldots] \quad . \tag{2.38}$$

Any of these, or any combination of these, can be varied. Diagrammatically we view an n-vertex as an n-particle source. For example, if we rescale $\gamma_{ij..k} \to g\gamma_{ij..k}$ and vary infinitesimally the coupling constant g, we "touch" each $\gamma_{ij..k}$ vertex in a Green function:

$g \frac{d}{dg} Z[J] = \frac{1}{k!}$  $= \frac{g}{k!}\gamma_{ij..k} \frac{d}{dJ_k} \cdots \frac{d}{dJ_j}\frac{d}{dJ_i} Z[J] \quad .$ (2.39)

We can use such generalizations of the Dyson-Schwinger equations (from varying single-particle sources J_i to varying many-particle sources $\gamma_{ijk..\ell}$) to compute hosts of physically significant quantities. One such quantity is the expectation value of the action. We rescale the entire action (2.13)

$$\frac{1}{\hbar} S[\phi] = -\frac{1}{2\hbar}\phi_i \Delta^{-1}_{ij}\phi_j + \frac{1}{3!\hbar}\gamma_{ijk}\phi_k\phi_j\phi_k + \ldots , \tag{2.40}$$

and vary $\hbar$ (depending on the context, $\hbar$ could be the Planck constant, coupling constant, inverse temperature or something else):

$\hbar \frac{d}{d\hbar} Z[J] = -\frac{1}{\hbar}\Big(-\frac{1}{2}$  $+\frac{1}{3!}$  $+\frac{1}{4!}$  $+\ldots\Big)$

$$= -\frac{1}{\hbar} S\left[\frac{d}{dJ}\right] Z[J] \quad . \tag{2.41}$$

---

† minestrone, to be specific.

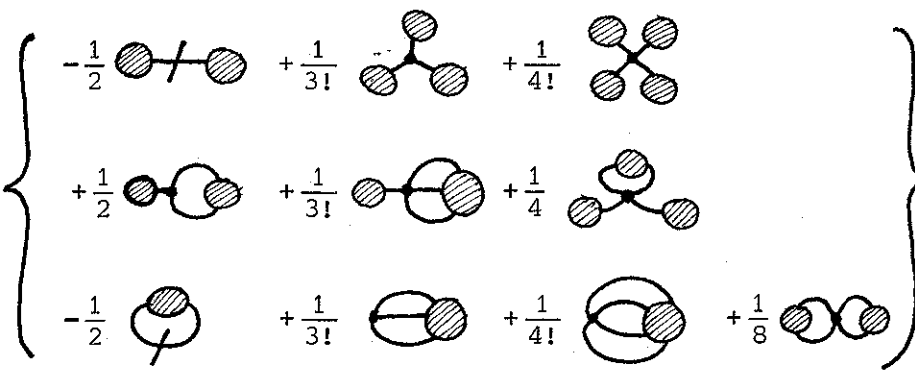
To normalize the expectation value properly, we divide by Z[J]:

$$\langle S[\phi]\rangle = \frac{1}{Z[J]}\, S\left[\frac{d}{dJ}\right] Z[J] \quad . \tag{2.42}$$

That this is really an expectation value will perhaps be easier to grasp in the path-integral formalism, cf. (3.11) in the next chapter. Anyway, we can use (2.19) to rewrite the above in terms of connected Green functions:

$$\left\langle \frac{1}{\hbar}\, S[\phi]\right\rangle = -\hbar \frac{dW[J]}{d\hbar} = \frac{1}{\hbar}\, S\left[\frac{dW[J]}{dJ_i} + \frac{d}{dJ_i}\right]$$

$$= \frac{1}{\hbar}$$



(2.43)

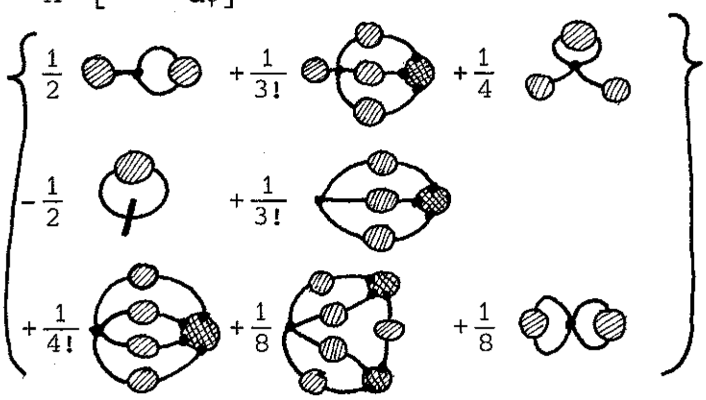
(the diagrammatic expansion is for the $\phi^3 + \phi^4$ theories). Even better, we can use (2.25) and (2.29) together with the identity (follows from (2.28))

$$\frac{dW[J]}{d\hbar} = \frac{d\Gamma[\phi]}{d\hbar} \tag{2.44}$$

to relate the $\langle S[\phi]\rangle$ to the effective action:

$$\frac{1}{\hbar}\langle S[\phi]\rangle = -\hbar\, \frac{d\Gamma[\phi]}{d\hbar} = \frac{1}{\hbar}\, S\left[\phi + W'' \frac{d}{d\phi}\right]$$

$$= \frac{1}{\hbar}\, S[\phi] + \frac{1}{\hbar}$$

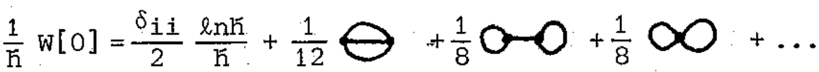


(2.45)

The above expansions can be used to compute the perturbative expansions for the connected and 1PI vacuum bubbles (see exercises). Their physical significance will become clearer in the next chapter.

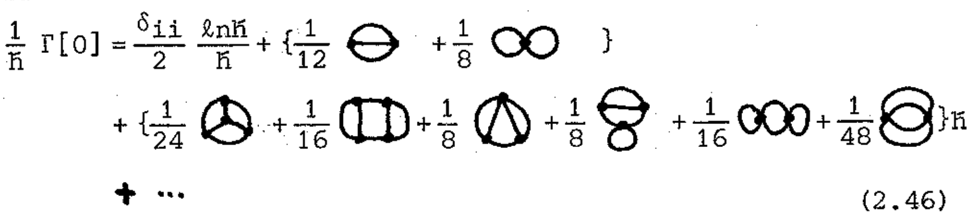
<u>Exercise 2.I.1</u> <u>Loop expansion.</u> Show that with action (2.40) the expansion in powers of $\hbar$ is the loop expansion, i.e. that each loop in a Feynman diagram carries a factor $\hbar$. Hence the loop expansion offers a systematic way of computing quantum corrections (or thermal fluctuations in statistical mechanics). Hint: each propagator carries a factor $\hbar$, while each vertex carries $\hbar^{-1}$.

<u>Exercise 2.I.2</u> <u>Free energy W[0].</u> Compute



for $\phi^3 + \phi^4$ theory. Hint: use (2.43) and the DS equations (2.21).

<u>Exercise 2.I.3</u> <u>Gibbs free energy Γ[0].</u> Compute



(2.46)

for $\phi^3 + \phi^4$ theory. Hint: use (2.45) and the DS equations (2.34). Note that the one-particle reducible diagrams from W[0] are indeed missing. The vacuum-bubble combinatoric weights are not always obvious - equation (2.45) provides the fastest way of computing them, as far as I know.

<u>Exercise 2.I.4</u> Show that for the <u>zero-dimensional ϕ^3 theory</u> (continuation of exercise 2.E.3)

$$G^{(m)} = \left(m - 1 + 3g \frac{d}{dg}\right) G^{(m-2)} \quad .$$

Hint: use (2.39) together with the Dyson-Schwinger equations (2.12).
Show also that

$$G^{(1)} = \frac{g}{2}\, G^{(2)} \quad .$$

Hence all Green functions can be computed from $Z \equiv G^{(0)}$, the vacuum bubbles. Show that these satisfy

$$g\frac{d}{dg}Z = g^2\left(\frac{5}{12} + \frac{9}{4}\, g\, \frac{d}{dg} + \frac{3}{4}\, g^2 \frac{d^2}{dg^2}\right)Z \quad .$$

Compute the first few terms of the expansion in powers of g. The complete solution is given in exercise 3.C.1.

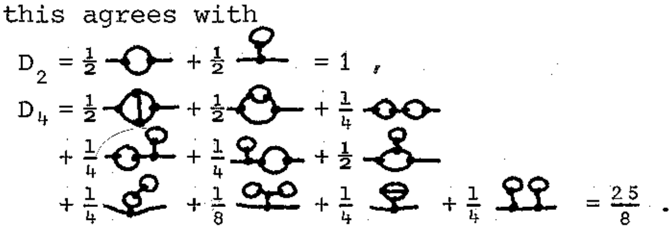
Exercise 2.I.5 Zero-dimensional field theory. Show that the connected vacuum bubbles $W \equiv W[0]$ satisfy

$$g\frac{d}{dg}W = g^2\left[\frac{5}{12} + \frac{9}{4}\,g\frac{dW}{dg} + \frac{3}{4}\,g^2\left(\frac{d^2W}{dg^2} + \left(\frac{dW}{dg}\right)^2\right)\right] .$$

Use this equation to derive recursion relations for connected m-leg Green functions. Compute the exact propagator $D = G^{c\,(2)}$

$$D = 1 + g^2 + \frac{25}{8}\,g^4 + \frac{390}{32}\,g^6 + \ldots..$$

and check that this agrees with



Hint: establish first that

$$\frac{d}{dJ}W[J] = \frac{g}{2} + J + \frac{g}{2}\left(J\frac{d}{dJ} + 3g\frac{d}{dg}\right)W[J] .$$

That relates $G^{c\,(m)}$ to the vacuum bubbles W.

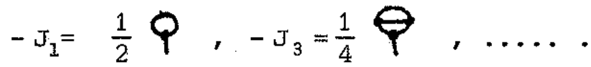
Exercise 2.I.6 Zero-dimensional ϕ^3 theory. Combine the DS equation (2.34) and the previous results to relate 1PI Green functions with different numbers of legs:

$$\frac{d}{d\phi}\Gamma[\phi] = \frac{g}{2} - \phi + \frac{g}{2}\left(-\phi\frac{d}{d\phi} + 3g\frac{d}{dg}\right)\Gamma[\phi] ,$$

and show that the proper tadpoles $J = -\Gamma^{(1)}$ satisfy

$$J = -\frac{g}{2} + \frac{g}{2}\left(1 - \frac{3}{2}g\frac{d}{dg}\right)J^2$$

$$= -\frac{1}{2}\,g - \frac{1}{4}\,g^3 - \frac{5}{8}\,g^5 - \ldots.$$

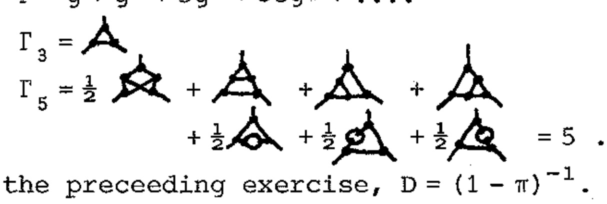


Compute the proper self-energy

$$\pi = \frac{1}{2}\,g^2 + g^4 + \frac{35}{8}\,g^6 + \ldots..$$

and the proper three-vertex $\Gamma \equiv \Gamma^{(3)}$

$$\Gamma = g + g^3 + 5g^5 + 35g^7 + \ldots.$$

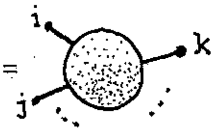


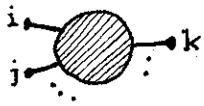
Compare π with the preceeding exercise, $D = (1 - \pi)^{-1}$.

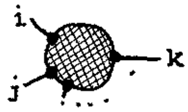
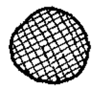
Exercise 2.I.7 Check (2.44).

J. Summary of the generating functional formalism

generating functionals

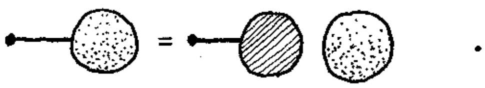
Full Green functions: $G_{ij..k} =$  $Z[J] =$ 

Connected Green functions: $G^{(c)}_{ij..k} =$  $W[J] =$ 

1PI Green functions: $\Gamma_{ij..k} =$  $\Gamma[\phi] =$ 

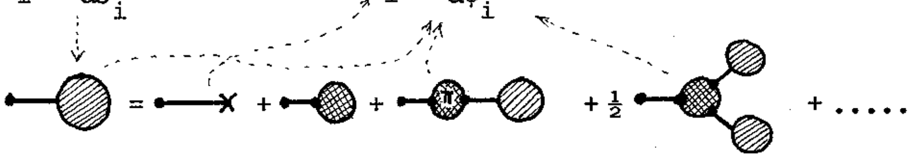
Full ↔ connected relation:

$$\frac{1}{Z[J]} \frac{d}{dJ_i} Z[J] = \frac{dW[J]}{dJ_i} + \frac{d}{dJ_i}$$



Connected ↔ 1PI relations:

$$\phi_i = \frac{dW[J]}{dJ_i} \qquad\qquad 0 = J_i + \frac{d\Gamma[\phi]}{d\phi_i}$$



$$\frac{d}{dJ_i} = \frac{d^2W[J]}{dJ_i dJ_k} \frac{d}{d\phi_k} \quad \leftrightarrow$$ 

Dyson-Schwinger equations:

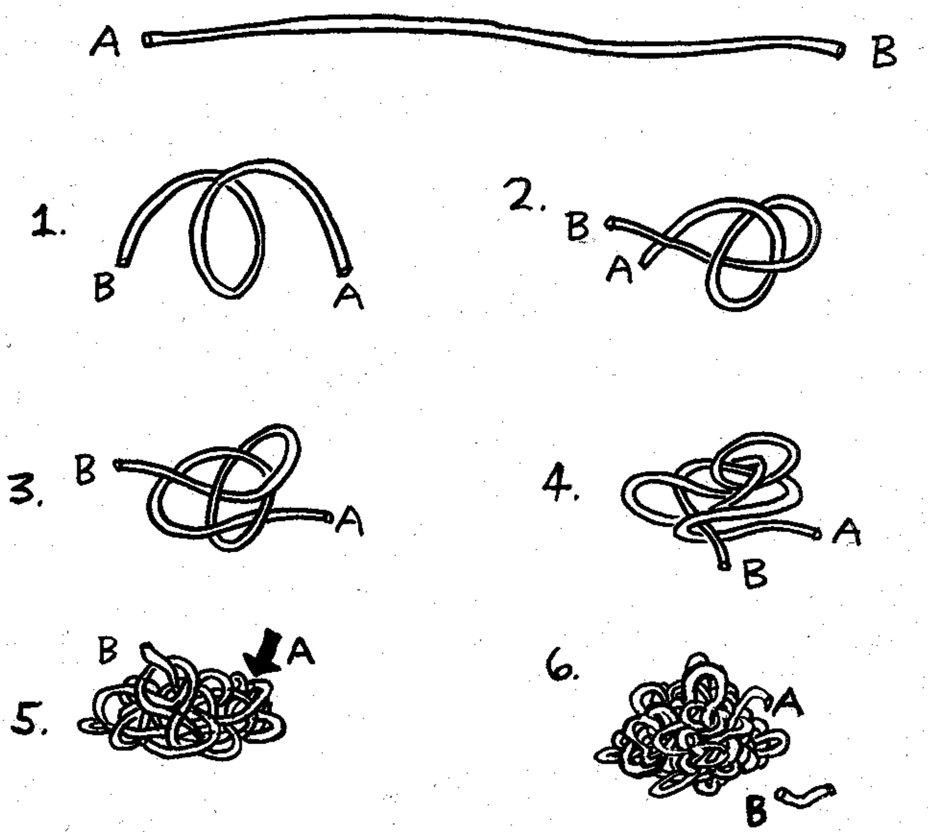
full $$\left(\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right] + J_i\right) Z[J] = 0 \quad ,$$

connected $$\frac{dS}{d\phi_i}\left[\frac{dW[J]}{dJ} + \frac{d}{dJ}\right] + J_i = 0 \quad ,$$

1PI $$\frac{d\Gamma[\phi]}{d\phi_i} = \frac{dS}{d\phi_i}\left[\phi + W''[J]\frac{d}{d\phi}\right] \; .$$

By now we are thoroughly fed up with longleggedy beasties, and the diagrammatic manipulations:

TYING THE NUDO DEL DIABLO OR DEVIL'S KNOT



(CONTINUED NEXT WEEK)

Let us now see whether the crow's vision of Quefithe is any more fun than the mole's version.