Radiative Corrections to Electroweak Observables

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see also: Peskin and Schroeder, Section 21.3
M. Peskin, arXiv:2003.05433, 2003.05435

In the history of QED, it was crucial to test predictions of the theory against high-precision experiments. To do this for the SU(2)xU(1) electroweak theory, we need to know how to derive finite predictions for observables.

basic notation for the electroweak theory:

$$Q = I^3 + Y \qquad Q_Z = I^3 - s_w^2 Q = c_w^2 I^3 - s_w^2 Y$$

$$c_w^2 = \frac{g^2}{g^2 + g'^2} \qquad s_w^2 = \frac{g'^2}{g^2 + g'^2}$$

tree-level values of observables

$$4\pi\alpha = \frac{g^2 {g'}^2}{g^2 + {g'}^2} \qquad \frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} = \frac{1}{2v^2}$$

$$m_W^2 = \frac{g^2 v^2}{4} \qquad m_Z^2 = \frac{(g^2 + g'^2)v^2}{4}$$

actual definition of G_F:

$$\rho = m_e^2 / m_\mu^2$$

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} F(\rho)\left[1 + H_1(\rho)\frac{\alpha}{\pi} + H_2(\rho)\left(\frac{\alpha}{\pi}\right)^2\right]$$

Z boson observables:

$$\Gamma(Z \to f\overline{f}) = \frac{\alpha}{6 s_w^2 c_w^2} m_Z Q_Z^2 \mathcal{N}$$

$$A_f = \frac{Q_{ZL}^2 - Q_{ZR}^2}{Q_{ZL} + Q_{ZR}}$$ polarization asymmetry

for leptons: $$A_\ell = \frac{1/4 - s_w^2}{1/4 - s_w^2 + 2 s_w^4} \approx 8(1/4 - s_w^2)$$

It is useful to define s_*^2 as the value, inserted into this formula, that gives the observed value of A_ℓ .

Some of these quantities are now known very accurately:

$$\begin{aligned} \alpha^{-1} &= 137.035999150\ (33) \\ \alpha^{-1}(m_Z) &= 128.922\ (18) \\ G_F &= 1.1663787\ (6)\ \times 10^{-5}\mathrm{GeV}^{-2} \\ m_Z &= 91.1876\ (21)\ \mathrm{GeV} \\ m_W &= 80.379\ (12)\ \mathrm{GeV} \\ \Gamma_Z &= 2.4955\ (23)\ \mathrm{GeV} \\ s_*^2 &= 0.23153\ (16) \end{aligned}$$

To match this accuracy, we need to generate predictions at the 1- (and, today, 2-) loop level.

To see how this works, compute 1-loop corrections due to an SU(2)xU(1) multiplet of heavy scalars. This is an easy, separately gauge invariant, case.

All of the effects come from vacuum polarization diagrams.

correction to α

$$i\Pi^{\mu\nu}(q) = i(q^2 g^{\mu\nu} - q^\mu q^\nu)\left(-\frac{e^2 \mathrm{tr}[Q^2]}{(4\pi)^2}\right)\int_0^1 dx(1-2x)^2 \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

$$\Delta = M^2 - x(1-x)q^2 \qquad d = 4 - 2\epsilon$$

$q^\mu q^\nu$ give zero when contracted with massless fermions; from here on I will write

$$i\Pi^{\mu\nu} = i g^{\mu\nu}\Pi(q^2) + \cdots$$

$$\Pi_{AA} = q^2 \left(-\frac{e^2 \mathrm{tr}[Q^2]}{(4\pi)^2}\right) \int_0^1 dx (1-2x)^2 \left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log(M^2 - x(1-x)q^2\right]$$

$$= q^2 \left(-\frac{e^2 \mathrm{tr}[Q^2]}{(4\pi)^2}\right) \left[\frac{1}{3} L + \frac{1}{30} \frac{q^2}{M^2} + \cdots\right]$$

where $$L = \frac{1}{\epsilon} - \gamma + \log 4\pi - \log M^2$$

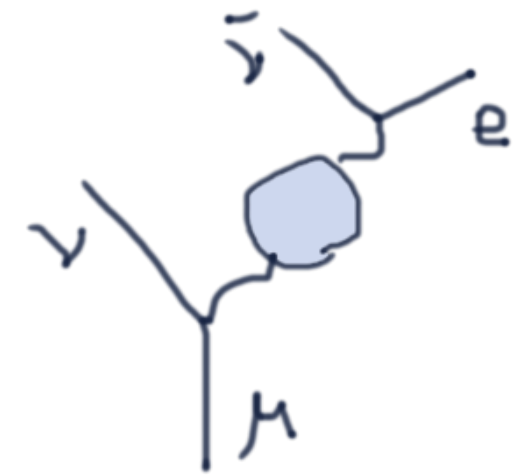
then the measurable value of α is shifted by

$$\alpha \to \alpha + \delta\alpha$$

where

$$\frac{\Delta\alpha}{\alpha} = -\frac{e^2 \mathrm{tr}[Q^2]}{(4\pi)^2} \left[\frac{1}{3} L\right]$$

Let's go through the effects on other processes. The structure of the vacuum polarizations is exactly the same, except for the overall coefficients.



G_F $\quad \Pi_{WW}(0) = 0$ so there is no shift

m_W $\quad m_W^2 \to m_W^2 + \Pi(m_W^2)$

so $$\frac{\Delta m_W^2}{m_W^2} = -\frac{g^2 \mathrm{tr}[I^+ I^-]}{2(4\pi)^2}\left[\frac{1}{3}L + \frac{1}{30}\frac{m_W^2}{M^2}\right]$$

m_Z

similarly $$\frac{\Delta m_Z^2}{m_Z^2} = -\frac{g^2 \mathrm{tr}[Q_Z^2]}{c_w^2(4\pi)^2}\left[\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}\right]$$

s_*^2 the Z coupling is modified to

$$i\frac{gQ_{Zf}}{c_w} + \left(-\frac{eg\,\mathrm{tr}[Q_Z Q]}{c_w(4\pi)^2}\left[\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}\right]\right)\cdot ieQ_f$$

$$= i\frac{g}{c_w}(I_f^3 - s_w^2 Q_f - (\frac{e^2\mathrm{tr}[Q_Z Q]}{(4\pi)^2}\left[\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}\right])\,Q_f\ )$$

so $$\frac{\Delta s_*^2}{s^2} = +\frac{e^2\,\mathrm{tr}[Q_Z Q]}{s^2(4\pi)^2}\left[\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}\right]$$

Now we have many examples of EW radiative corrections. However, all of these expressions are UV divergent.

In QED, we renormalize to fix the measurable constants m_e and α ; then the predictions of the theory are made finite.

In the electroweak theory, it is not so clear what to do.

However, we have a lesson from the SO(N) linear σ model:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi^i)^2 + \frac{1}{2}\mu^2(\Phi^i)^2 - \frac{1}{4}\lambda((\Phi^i)^2)^2$$

spontaneous breaking of SO(N) leads to a large number of possible vertices, but these are made finite by counterterms for the field strength Z , the mass parameter μ^2, and the coupling λ .

Ben Lee's Theorem:

In a renormalizable QFT with a symmetry that is spontaneously broken, the infinities are removed by the counterterms of the symmetric theory.

corollary:

In a renormalizable QFT with a symmetry that is spontaneously broken, any relation of observables that is zero at the tree level will receive only finite radiative corrections.

In the SU(2)xU(1) theory, all of the observables I have discussed depend only on 3 parameters

$$g, g', v$$

at the tree level. Then, if we renormalize to fix the values of 3 observables, any further prediction will be finite. This is called "on-shell renormalization".

An alternative is to perform $\overline{MS}$ subtraction. Then the predictions for any observable in terms of the $\overline{MS}$ couplings is scheme-dependent, but any relation of 4 observables is scheme-independent. This makes calculation easier but is less "physical".

The PDG defines the parameters of the SU(2)xU(1) model as the set of $\overline{MS}$ couplings giving the best fit to the full corpus of electroweak data.

The simplest on-shell scheme is to define a reference value s_0^2 using the three best-measured EW observables.

$$\sin^2 2\theta_0 = 4 s_0^2 c_0^2 = \frac{\alpha(m_Z)}{\sqrt{2} G_F m_Z^2}$$

Any other combination of observables equal to s_w^2 at tree level will differ from s_0^2 at 1-loop by a finite correction.

The measured value of s_0^2 is $s_0^2 = 0.231079(36)$.

$$\frac{\Delta \sin^2 2\theta_0}{\sin^2 2\theta_0} = \frac{4(c_0^2 - s_0^2)\Delta s_0^2}{4 s_0^2 c_0^2} = \frac{\Delta\alpha}{\alpha} - \frac{\Delta G_F}{G_F} - \frac{\Delta m_Z^2}{m_Z^2}$$

then $$\frac{\Delta s_0^2}{s_0^2} = \frac{c_0^2}{c_0^2 - s_0^2}\left[\frac{\Delta\alpha}{\alpha} - \frac{\Delta G_F}{G_F} - \frac{\Delta m_Z^2}{m_Z^2}\right]$$

In the following, I use s^2, c^2 for tree-level quantities.

In the model discussed above:

$$\frac{\Delta s_0^2}{s_0^2} = \frac{c^2}{c^2-s^2}\left[-\frac{e^2\mathrm{tr}[Q^2]}{(4\pi)^2}[\frac{1}{3}L] - 0 + \frac{g^2\mathrm{tr}[Q_Z^2]}{c^2(4\pi)^2}[\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}]\right]$$

$$= \frac{c^2}{c^2-s^2}\frac{g^2}{(4\pi)^2}\left[-s^2\mathrm{tr}[Q^2][\frac{1}{3}L] + \frac{\mathrm{tr}[Q_Z^2]}{c^2}[\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}]\right]$$

$$= \frac{c^2}{c^2-s^2}\frac{g^2}{(4\pi)^2}\left[-s^2(\mathbf{I}+\mathbf{Y})[\frac{1}{3}L] + \frac{(c^4\mathbf{I}+s^4\mathbf{Y})}{c^2}[\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}]\right]$$

using $\mathrm{tr}[Q^2] = \mathrm{tr}[(I^3+Y)^2] = \mathrm{tr}(I^3)^2 + \mathrm{tr}Y^2 = \mathbf{I}+\mathbf{Y}$

$$\mathrm{tr}[Q_Z^2] = \mathrm{tr}[(c^2I^3 - s^2Y)^2] = c^4\mathbf{I} + s^4\mathbf{Y}$$

$$= \frac{g^2}{(4\pi)^2}\left[(c^2\mathbf{I} - s^2\mathbf{Y})[\frac{1}{3}L] + \frac{c^4\mathbf{I}+s^4\mathbf{Y}}{c^2-s^2}[\frac{1}{30}\frac{m_Z^2}{M^2}]\right]$$

compare to $s_W^2 = 1 - \frac{m_W^2}{m_Z^2}$

(the Marciano-Sirlin definition of s_w^2)

$$\frac{\Delta s_W^2}{s_W^2} = -\frac{c^2}{s^2}\left[\frac{\Delta m_W^2}{m_W^2} - \frac{\Delta m_Z^2}{m_Z^2}\right]$$

$$= \frac{c^2}{s^2}\frac{g^2}{(4\pi)^2}\left[\frac{\mathrm{tr}[I^+I^-]}{2}[\frac{1}{3}L + \frac{1}{30}\frac{m_W^2}{M^2}] - \frac{\mathrm{tr}[Q_Z^2]}{c^2}[\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}\right]$$

$$= \frac{g^2}{(4\pi)^2}\left[\frac{c^2}{s^2}\mathbf{I}\,[\frac{1}{3}L + \frac{1}{30}\frac{m_W^2}{M^2}] - (\frac{c^4}{s^2}\mathbf{I} + s^2\mathbf{Y})[\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}\right]$$

using $\mathrm{tr}[I^+I^-] = \mathrm{tr}[(I^1)^2 + (I^2)^2] = 2\mathrm{tr}[(I^3)^2]$

$$= \frac{g^2}{(4\pi)^2}\left[(c^2\mathbf{I} - s^2\mathbf{Y})[\frac{1}{3}L] - s^2\mathbf{Y}\frac{1}{30}\frac{m_Z^2}{M^2}\right]$$

compare to the expression for s_*^2 :

$$\frac{\Delta s_*^2}{s_*^2} = \frac{e^2 \mathrm{tr}[QQ_Z]}{s^2(4\pi)^2}[\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}]$$

$$= \frac{g^2}{(4\pi)^2}\mathrm{tr}[(I^3 + Y)(c^2 I^3 - s^2 Y)][\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}]$$

$$= \frac{g^2}{(4\pi)^2}(c^2\mathbf{I} - s^2\mathbf{Y})[\frac{1}{3}L + \frac{1}{30}\frac{m_Z^2}{M^2}]$$

In both cases, the difference of values of s_w^2 is free of the infinite term L . The finite residue is of order m_Z^2/M^2 ("decoupling").

Let's next discuss a physically more relevant set of radiative corrections, those from a 4th-generation quark or lepton doublet. In this case also, only vacuum polarization corrections are important.

The value of the QED vacuum polarization amplitude is

$$i(q^2 g^{\mu\nu} - q^\mu q^\nu)\left(-\frac{e^2}{(4\pi)^2}\right)\int_0^1 dx\ 8x(1-x)\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

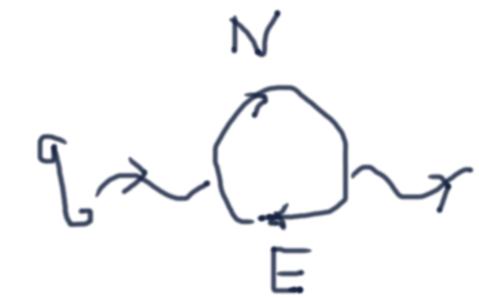
As in the previous example, for heavy fermions, this is composed of an infinite term and a term that is order q^2/M^2, that is, decoupling.

However, when we compute the vacuum polarization amplitudes of chiral currents, the result is not so simple.

Consider a heavy lepton doublet with masses M_N and M_E ; write

$$M_N^2 = M^2 + \frac{1}{2}\Delta M^2 \qquad\qquad M_E^2 = M^2 - \frac{1}{2}\Delta M^2$$

Compute the W vacuum polarization, keeping only the $g^{\mu\nu}$ terms:



$$= (i\frac{g}{\sqrt{2}}) \int \frac{d^dk}{(2\pi)^d}(-1)\mathrm{tr}\left[\gamma^\mu P_L \frac{i(\not{k}+\not{q}+M_N)}{(k+q)^2-M_N^2}\gamma^\nu P_L \frac{i(\not{k}+M_E)}{k^2-M_E^2}\right]$$

$$= -\frac{g^2}{2}\int_0^1 dx \int \frac{d^d\mathbf{k}^2}{(2\pi)^d}\frac{1}{(\mathbf{k}-\Delta)^2}$$

$$\cdot 2\big\{((\mathbf{k}+(1-x)q)^\mu(\mathbf{k}-xq)^\nu+(\mathbf{k}-xq)^\mu(\mathbf{k}+(1-x)q)^\nu-g^{\mu\nu}(\mathbf{k}+(1-x)q)(\mathbf{k}-xq)\big\}$$

$$= -\frac{g^2}{2}g^{\mu\nu}\int_0^1 dx \frac{2i}{(4\pi)^{d/2}}\{\frac{-(2/2-d/2)\Gamma(1-d/2)}{\Delta^{1-d/2}}+\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}x(1-x)q^2\}$$

$$= -\frac{g^2}{2}g^{\mu\nu}\int_0^1 dx \frac{2i}{(4\pi)^{d/2}}\{\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}(-\Delta+x(1-x)q^2)\}$$

Now $\Delta = xM_N^2 + (1-x)M_E^2 - x(1-x)q^2$

$$= M^2 + (x - \frac{1}{2})\Delta M^2 - x(1-x)q^2$$

so the W vacuum polarization becomes

$$ig^{\mu\nu}(-\frac{g^2}{(4\pi)^{d/2}}) \int_0^1 dx \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}(2x(1-x)q^2 - M^2 - (x-1/2)\Delta M^2)$$

$$= ig^{\mu\nu}(-\frac{g^2}{(4\pi)^2}) \int_0^1 dx\ (2x(1-x)q^2 - M^2 - (x-1/2)\Delta M^2)$$

$$[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log M^2 - (x - \frac{1}{2})\frac{\Delta M^2}{M^2} + \frac{1}{2}(x-\frac{1}{2})^2(\frac{\Delta M^2}{M^2})^2 + x(1-x)\frac{q^2}{M^2} + \cdots]$$

$$= ig^{\mu\nu}(-\frac{g^2}{(4\pi)^2})\left[(\frac{1}{3}q^2 - M^2)L - \frac{1}{6}q^2 + \frac{1}{24}\frac{(\Delta M^2)^2}{M^2} + \cdots\right]$$

We know that the infinite term will cancel, so look at the finite terms. Unlike the previous case, these persist in the limit $M^2 \to \infty$. There are two separate "nondecoupling" effects, one that vanishes at $q^2 = 0$ and one that does not vanish at $q^2 = 0$ but depends on the mass difference within an electroweak doublet.

The presence of these terms should not be a surprise. The mass of a heavy fermion is given by

$$M = \lambda v/\sqrt{2}$$

It is perfectly possible to have terms proportional to

$$\lambda^2/M^2$$

These are finite as $M^2 \to \infty$ because the Yukawa coupling becomes extremely large in that limit.

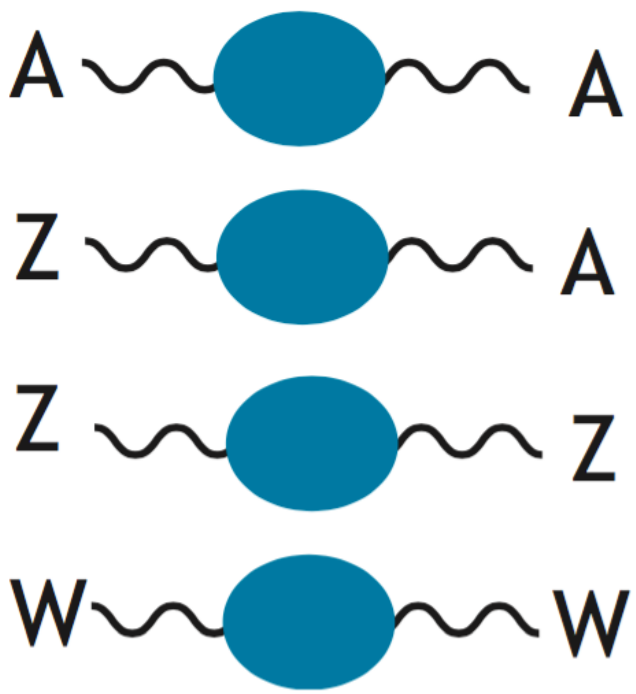
How do these corrections affect EW observables?

It is possible to do a quite general analysis for the class of models in which 1-loop corrections come only from vacuum polarization diagrams.   Particles that modify the SM Higgs sector without (large) couplings to the light fermions satisfy this description. Supersymmetry is not in this class.

In a paper with the late lamented Bryan Lynn, we called these “oblique corrections”.

Theories with non-oblique corrections require a framework with more parameters, the “Standard Model Effective Field Theory” (SMEFT).   See   arXiv:2003.05435.

There are 4 relevant vacuum polarization amplitudes



$$
\begin{aligned}
A\text{–}A &= ie^2\Pi_{QQ}g^{\mu\nu} \\
Z\text{–}A &= i\frac{e^2}{s_w c_w}(\Pi_{3Q} - s_w^2\Pi_{QQ})g^{\mu\nu} \\
Z\text{–}Z &= i\frac{e^2}{s_w^2 c_w^2}(\Pi_{33} - 2s_w^2\Pi_{3Q} + s_w^2\Pi_{QQ})g^{\mu\nu} \\
W\text{–}W &= i\frac{e^2}{s_w^2}\Pi_{11}g^{\mu\nu}
\end{aligned}
$$

The possibly divergent terms are the leading terms in a Taylor series in q^2:

$$\Pi_{QQ}(q^2) = Aq^2 + \cdots$$

$$\Pi_{3Q}(q^2) = Bq^2 + \cdots$$

$$\Pi_{33}(q^2) = C + Dq^2 + \cdots$$

$$\Pi_{11}(q^2) = E + Fq^2 + \cdots$$

Note that Π_{QQ} and Π_{3Q} must vanish at $q^2 = 0$ due to the QED Ward identity.

Of these 6 coefficients, only 3 linear combinations can be divergent. Finite predictions of the EW theory will depend only on the orthogonal 3 combinations,

$$S = \frac{16\pi}{m_Z^2}[\Pi_{33}(m_Z^2) - \Pi_{33}(0) - \Pi_{3Q}(m_Z^2)]$$

$$T = \frac{4\pi}{s_w^2 m_W^2}[\Pi_{11}(0) - \Pi_{33}(0)]$$

$$U = \frac{16\pi}{m_Z^2}[\Pi_{11}(m_Z^2) - \Pi_{11}(0) - \Pi_{33}(m_Z^2) + \Pi_{33}(0)]$$

The W and Z masses are renormalized by the same counterterm, which preserves $m_W^2 = m_Z^2 c_w^2$. T is the correction to this relation. The q^2 terms in the 33 and 3Q vacuum polarizations are corrected by the same counterterm, which is the SU(2) field strength renormalization. S is the residual finite effect. U has a double suppression and is very small in most models.

It is a nice exercise to plug the general vacuum polarization amplitudes into the expressions I derived for the tree-level zero relations

$$s_W^2 - s_0^2 \qquad\qquad s_*^2 - s_0^2$$

and watch these expressions reduce to a linear combination of S, T, and U. The results are

$$s_W^2 - s_0^2 = \frac{\alpha c^2}{c^2 - s^2}\left[\frac{1}{2}S - c^2 T - \frac{c^2 - s^2}{4s^2}U\right]$$

$$s_*^2 - s_0^2 = \frac{\alpha}{c^2 - s^2}\left[\frac{1}{4}S - s^2 c^2 T\right]$$

There are formulae like this for all finite electroweak observables.

At the same time, each model yields a relatively simple formula for the (non-decoupling) terms in S, T, and U.

For example, the heavy lepton doublet gives

$$S = \frac{1}{6\pi} \qquad T = \frac{1}{48\pi s^2 c^2}\frac{(\Delta M^2)^2}{m_Z^2 M^2} \qquad U = 0$$

S and T turn out also to be a simple way to describe the relatively large EW corrections due to the top quark and the Higgs boson. If we define the SM with specific reference values, the effect of changing those values is summarized as

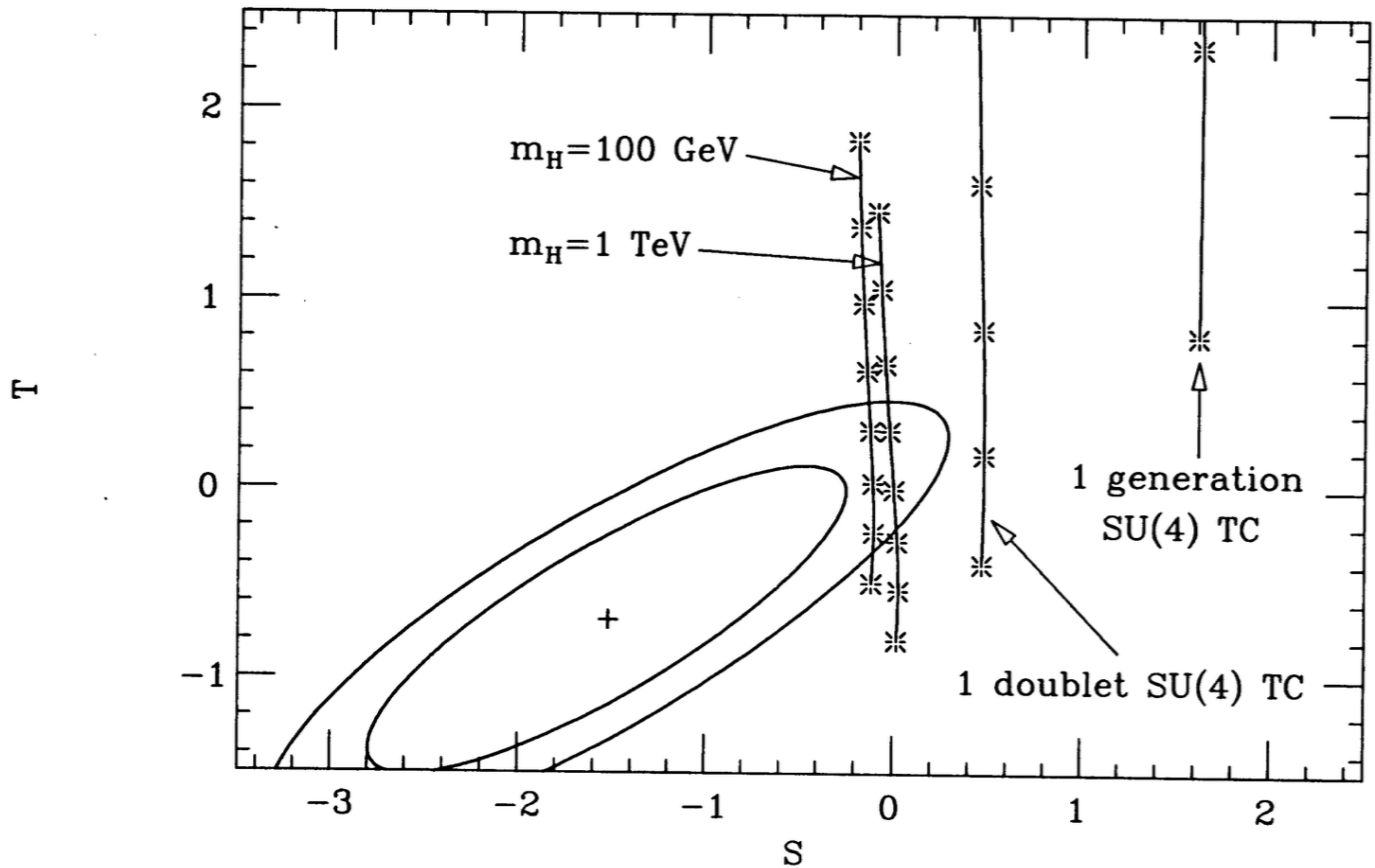
$$S = \frac{1}{6\pi}\log\frac{m_t^2}{m_{t\,ref}^2} \qquad T = \frac{3}{16\pi s^2 c^2}\frac{m_t^2 - m_{t\,ref}^2}{m_Z^2} \qquad U = \frac{1}{2\pi}\log\frac{m_t^2}{m_{t\,ref}^2}$$

$$S = \frac{1}{12\pi}\log\frac{m_h^2}{m_{h\,ref}^2} \qquad T = -\frac{3}{16\pi c^2}\log\frac{m_h^2}{m_{h\,ref}^2} \qquad U = 0$$

A way to search for new physics beyond the SM is to fit the corpus of EW data to the SM (with fixed m_t , m_h), plus arbitrary values of S and T. Note that different observables have linear combinations of S and T with different slopes, so, in principle, the data can determine nonzero values of these two parameters independently.

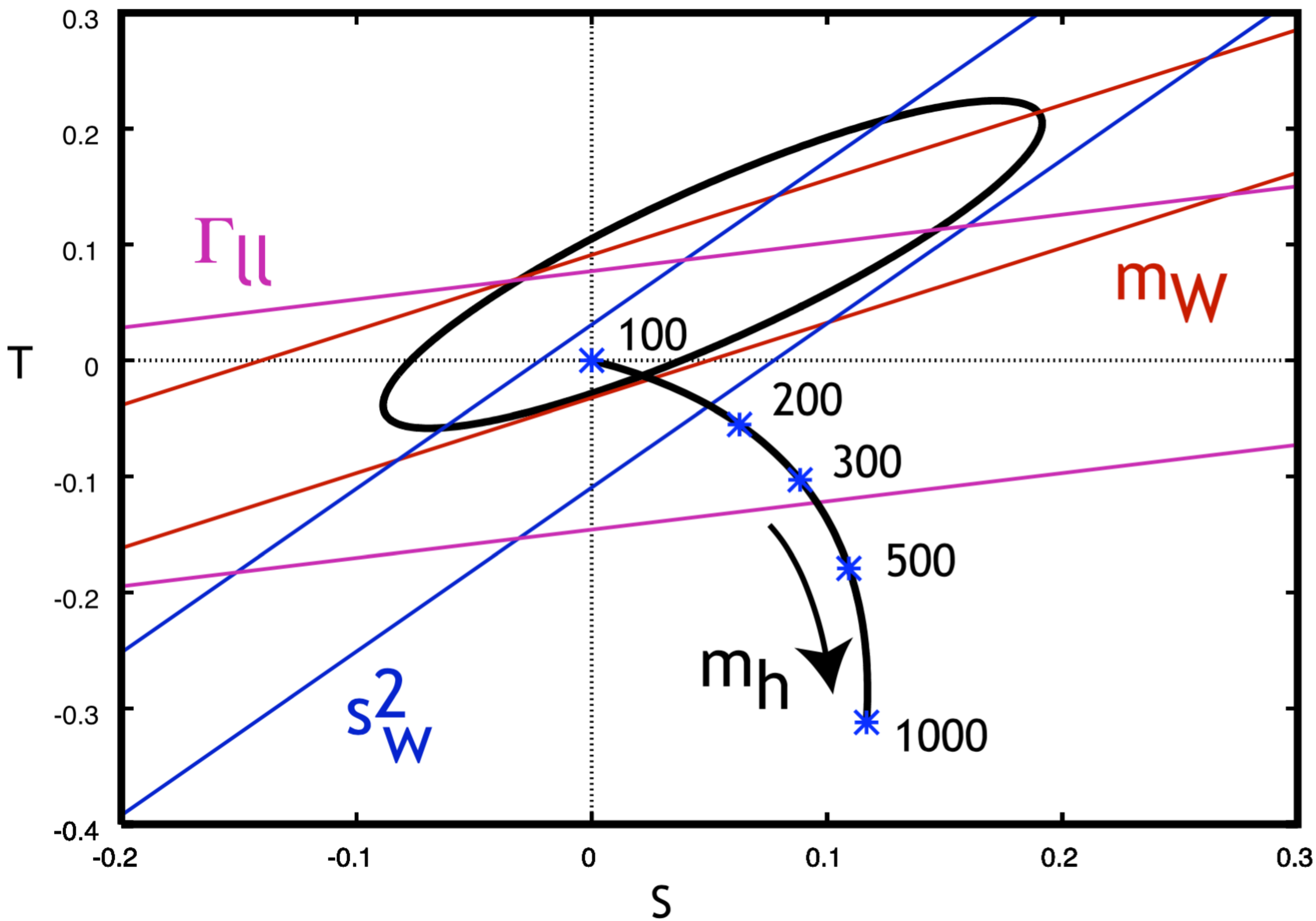
Here are results from three different eras.

S,T fit c. 1991



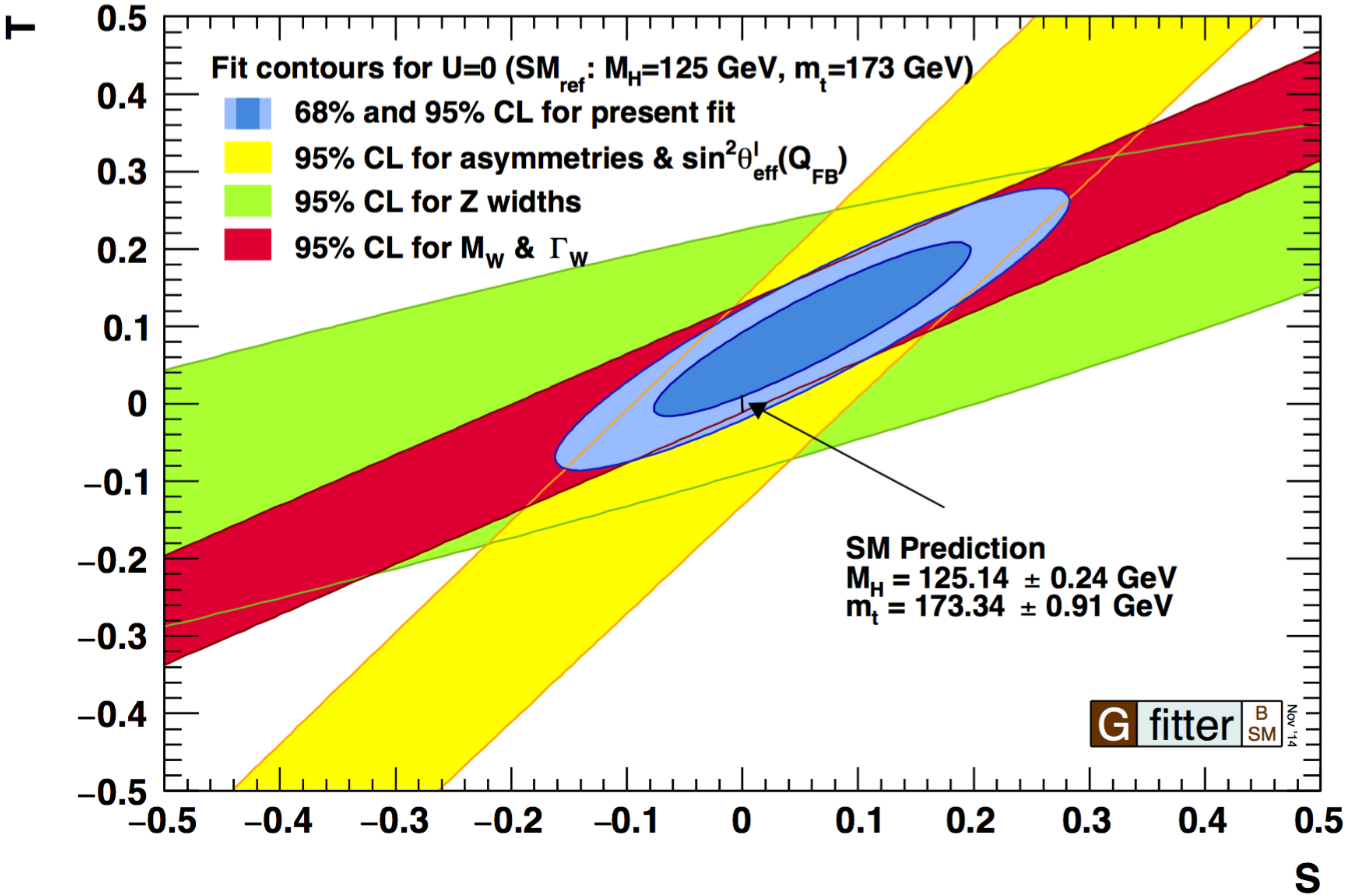
m_H=100 GeV
m_H=1 TeV
1 generation
SU(4) TC
1 doublet SU(4) TC
T
S
2
1
0
−1
−3
−2
−1
0
1
2

S,T fit c. 2008



LEP EWWG: within the MSM $m_h < 144\ (182)\ \text{GeV}\ \ (95\%\ \text{CL})$

S,T fit c. 2014



Fit contours for U=0 (SM$_{ref}$: M$_H$=125 GeV, m$_t$=173 GeV)
68% and 95% CL for present fit
95% CL for asymmetries & $\sin^2\theta^l_{eff}(Q_{FB})$
95% CL for Z widths
95% CL for M_W & Γ_W
SM Prediction
M_H = 125.14 ± 0.24 GeV
m_t = 173.34 ± 0.91 GeV
T
S
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
G fitter B SM
Nov '14

The SM with $m_t = 173\ \text{GeV},\ m_h = 125\ \text{GeV}$ gives an excellent fit to electroweak data today. You can view this either as a success or as a challenge.