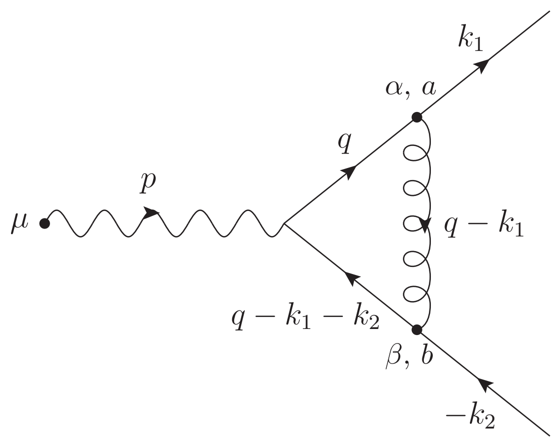
1. Introduction

In class, we worked out the one-loop QCD correction to the process $e^+e^- \to$ hadrons. We showed how to reframe the calculation as the one-loop QCD correction to the process $\gamma^* \to q\bar{q}$, where γ^* is an off-shell photon of timelike four-momentum p. One of the ingredients of this computation was the analysis of the one-loop QCD virtual correction to the $\gamma q\bar{q}$ vertex. In these notes, I will provide a derivation of the result that was quoted in class.

The diagram that is to be evaluated is shown below.



The color factor associated with this vertex is $\delta^{ab}\boldsymbol{T^a T^b} = C_F\mathbb{1}$. We have suppressed the factor of e associated with the photon vertex. We have also taken all quark masses to be zero. Hence, the invariant matrix element in the Feynman gauge is given by,

$$
\begin{aligned}
i\mathcal{M} &= -ie_q C_F(-ig_s\mu^\epsilon)^2\,\overline{u}(k_1)\int\frac{d^nq}{(2\pi)^n}\gamma^\alpha\left(\frac{i\not{q}}{q^2}\right)\gamma^\mu\left(\frac{i(\not{q}-\not{k}_1-\not{k}_2)}{(q-k_1-k_2)^2}\right)\gamma^\beta v(k_2)\left(\frac{-ig_{\alpha\beta}}{(q-k_1)^2}\right) \\
&= -e_q C_F g_s^2\mu^{2\epsilon}\,\overline{u}(k_1)\int\frac{d^nq}{(2\pi)^n}\frac{\gamma_\alpha\not{q}\gamma^\mu(\not{q}-\not{k}_1-\not{k}_2)\gamma^\alpha}{q^2(q-k_1)^2(q-k_1-k_2)^2}\,v(k_2)\,. \qquad (1)
\end{aligned}
$$

The Dirac algebra can be simplified by a judicious application of the anticommutation relations of the gamma matrices,

$$
\begin{aligned}
\gamma_\alpha\not{q}\gamma^\mu(\not{q}-\not{k}_1-\not{k}_2)\gamma^\alpha &= -2(\not{q}-\not{k}_1-\not{k}_2)\gamma^\mu\not{q} + (4-n)\not{q}\gamma^\mu(\not{q}-\not{k}_1-\not{k}_2) \\
&= (2-n)\not{q}\gamma^\mu\not{q} + 2(\not{k}_1+\not{k}_2)\gamma^\mu\not{q} - (4-n)\not{q}\gamma^\mu(\not{k}_1+\not{k}_2) \\
&= (2-n)(2\not{q}q^\mu - q^2\gamma^\mu) + 2(\not{k}_1+\not{k}_2)\gamma^\mu\not{q} - (4-n)\not{q}\gamma^\mu(\not{k}_1+\not{k}_2)\,. \qquad (2)
\end{aligned}
$$

After making use of the massless Dirac equation, $\overline{u}(k_1)\not{k}_1 = \not{k}_2 v(k_2) = 0$, it then follows that

$$
\mathcal{M} = ie_q C_F g_s^2\mu^{2\epsilon}\,\overline{u}(k_1)\int\frac{d^nq}{(2\pi)^n}\frac{(2-n)(2\not{q}q^\mu - q^2\gamma^\mu) + 2\not{k}_2\gamma^\mu\not{q} - (4-n)\not{q}\gamma^\mu\not{k}_1}{q^2(q-k_1)^2(q-k_1-k_2)^2}\,v(k_2)\,. \qquad (3)
$$

2. The Passarino-Veltman functions for the one-loop vertex

In the evaluation of eq. (3), we shall employ the Passarino-Veltman functions introduced in part (c) of problem 1 of Problem Set 3, along with some obvious generalizations:

$$\begin{aligned}
&C_0(p_1^2, p_2^2, p^2; m_1^2, m_2^2, m_3^2) \\
&\qquad \equiv -16\pi^2 i \int \frac{d^n q}{(2\pi)^n}\, \frac{1}{(q^2 - m_1^2)[(q+p_1)^2 - m_2^2][(q+p_1+p_2)^2 - m_3^2]}\,, \\
&C^\mu(p_1, p_2, p; m_1^2, m_2^2, m_3^2) \\
&\qquad \equiv -16\pi^2 i \int \frac{d^n q}{(2\pi)^n}\, \frac{q^\mu}{(q^2 - m_1^2)[(q+p_1)^2 - m_2^2][(q+p_1+p_2)^2 - m_3^2]}\,, \\
&C^{\mu\nu}(p_1, p_2, p; m_1^2, m_2^2, m_3^2) \\
&\qquad \equiv -16\pi^2 i \int \frac{d^n q}{(2\pi)^n}\, \frac{q^\mu q^\nu}{(q^2 - m_1^2)[(q+p_1)^2 - m_2^2][(q+p_1+p_2)^2 - m_3^2]}\,,
\end{aligned} \tag{4}$$

where $p + p_1 + p_2 = 0$, with all external four-momenta pointing into the triangle graph. Note that the usual factors of $i\varepsilon$ have been suppressed. It is straightforward to use Lorentz covariance to decompose C^μ and $C^{\mu\nu}$ as follows,

$$C^\mu(p_1, p_2, p; m_1^2, m_2^2, m_3^2) = p_1^\mu\, C_{11} + p_2^\mu\, C_{12}\,, \tag{5}$$

$$C^{\mu\nu}(p_1, p_2, p; m_1^2, m_2^2, m_3^2) = p_1^\mu p_1^\nu\, C_{21} + p_2^\mu p_2^\nu\, C_{22} + (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) C_{23} + g^{\mu\nu}\, C_{24}\,, \tag{6}$$

where the C_{ij} are Lorentz invariant functions of their arguments,

$$C_{ij} \equiv C_{ij}(p_1^2, p_2^2, p^2; m_1^2, m_2^2, m_3^2)\,. \tag{7}$$

To evaluate eq. (3), we will need to compute C^μ and $C^{\mu\nu}$ in the case of $m_1 = m_2 = m_3 = 0$ and $p_1^2 = p_2^2 = 0$. In this limit, these loop functions can be easily evaluated. Making good use of the class handout entitled *Useful formulae for computing one-loop integrals*, it is instructive to first evaluate,

$$\begin{aligned}
\frac{i}{16\pi^2} C_0(0,0;p^2;0,0,0) &= \int \frac{d^n q}{(2\pi)^n}\, \frac{1}{q^2 (q+p_1)^2 (q+p_1+p_2)^2} \\
&= 2\int_0^1 x\,dx \int_0^1 dy \int \frac{d^n q}{(2\pi)^n}\, \frac{1}{\big[xyq^2 + x(1-y)(q+p_1)^2 + (1-x)(q+p_1+p_2)^2\big]^3} \\
&= 2\int_0^1 x\,dx \int_0^1 dy \int \frac{d^n q}{(2\pi)^n}\, \frac{1}{\big[q^2 + 2q\cdot\big[p_1(1-xy) + p_2(1-x)\big] + p^2(1-x)\big]^3}\,,
\end{aligned}$$

after using $p_1^2 = p_2^2 = 0$. Integrating over q yields,

$$C_0(0,0;p^2;0,0,0) = -(4\pi)^\epsilon\, \Gamma(1+\epsilon) \int_0^1 x\,dx \int_0^1 dy\, \big[2p_1\cdot p_2(1-xy)(1-x) - p^2(1-x)\big]^{-1-\epsilon}\,, \tag{8}$$

where $n = 4 - 2\epsilon$. Using $p + p_1 + p_2 = 0$ along with $p_1^2 = p_2^2 = 0$, it follows that $p^2 = 2p_1 \cdot p_2$. Hence, we obtain

$$C_0(0, 0; p^2; 0, 0, 0) = \left(\frac{4\pi}{-p^2}\right)^\epsilon \frac{1}{p^2} \Gamma(1+\epsilon) \int_0^1 x^{-\epsilon}(1-x)^{-1-\epsilon}\, dx \int_0^1 y^{-1-\epsilon}\, dy\,. \tag{9}$$

Employing the definition of the Beta function,

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx\,, \tag{10}$$

we end up with,

$$C_0(0, 0; p^2; 0, 0, 0) = \left(\frac{4\pi}{-p^2}\right)^\epsilon \frac{1}{p^2} \frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{11}$$

Next, we evaluate,

$$\begin{aligned}\frac{i}{16\pi^2} C^\mu(p_1, p_2, p; 0, 0, 0) &= \int \frac{d^n q}{(2\pi)^n} \frac{q^\mu}{q^2(q+p_1)^2(q+p_1+p_2)^2} \\ &= 2\int_0^1 x\, dx \int_0^1 dy \int \frac{d^n q}{(2\pi)^n} \frac{q^\mu}{\big[q^2 + 2q\cdot\big[p_1(1-xy) + p_2(1-x)\big] + p^2(1-x)\big]^3}\,.\end{aligned}$$

Integrating over q yields

$$\begin{aligned}C^\mu(p_1, p_2, p; 0, 0, 0) = (4\pi)^\epsilon\, \Gamma(1+\epsilon) \int_0^1 x\, dx \int_0^1 dy \big[2p_1\cdot p_2(1-xy)(1-x) - p^2(1-x)\big]^{-1-\epsilon} \\ \times \big[p_1^\mu(1-xy) + p_2^\mu(1-x)\big]\,.\end{aligned} \tag{12}$$

Using eq. (5) it follows that

$$\begin{aligned}C_{11}(0, 0; p^2; 0, 0, 0) &= -\left(\frac{4\pi}{-p^2}\right)^\epsilon \frac{1}{p^2} \Gamma(1+\epsilon) \int_0^1 x^{-\epsilon}(1-x)^{-1-\epsilon}\, dx \int_0^1 y^{-1-\epsilon}(1-xy)\, dy \\ &= -\left(\frac{4\pi}{-p^2}\right)^\epsilon \frac{1}{p^2} \Gamma(1+\epsilon) \left[\frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} - \frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\right] \\ &= -\left(\frac{4\pi}{-p^2}\right)^\epsilon \frac{1}{p^2} \frac{(1-\epsilon)\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\,,\end{aligned} \tag{13}$$

and

$$\begin{aligned}C_{12}(0, 0; p^2; 0, 0, 0) &= -\left(\frac{4\pi}{-p^2}\right)^\epsilon \frac{1}{p^2} \Gamma(1+\epsilon) \int_0^1 x^{-\epsilon}(1-x)^{-\epsilon}\, dx \int_0^1 y^{-1-\epsilon}\, dy \\ &= -\left(\frac{4\pi}{-p^2}\right)^\epsilon \frac{1}{p^2} \frac{\Gamma(1+\epsilon)\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\,.\end{aligned} \tag{14}$$

Finally, the complete evaluation of

$$C^{\mu\nu}(p_1,p_2,p;0,0,0) = -16\pi^2 i\int\frac{d^nq}{(2\pi)^n}\,\frac{q^\mu q^\nu}{q^2(q+p_1)^2(q+p_1+p_2)^2}\,, \tag{15}$$

is provided in the Appendix. In addition, note that $g_{\mu\nu}C^{\mu\nu}(p_1,p_2,p;0,0,0)=0$, which is easily demonstrated as follows,

$$\begin{aligned}
g_{\mu\nu}C^{\mu\nu}(p_1,p_2,p;0,0,0) &= -16\pi^2 i\int\frac{d^nq}{(2\pi)^n}\frac{1}{(q+p_1)^2(q+p_1+p_2)^2}\\
&= -16\pi^2 i\int\frac{d^nq}{(2\pi)^n}\frac{1}{q^2(q+p_2)^2}\,.\\
&= -16\pi^2 i\int_0^1 dx\int\frac{d^nq}{(2\pi)^n}\frac{1}{(q^2+2q\cdot p_2 x)^2}\\
&= (4\pi)^\epsilon\,\Gamma(\epsilon)(p_2^2)^{-\epsilon}\int_0^1 x^{-2\epsilon}\,dx = \frac{(4\pi)^\epsilon\,\Gamma(\epsilon)}{1-2\epsilon}\,(p_2^2)^{-\epsilon} = 0\,,
\end{aligned} \tag{16}$$

since $p_2^2=0$ by assumption. This result is confirmed again in eq. (A.5).

3. Evaluating the one-loop QCD correction to the $\gamma^* q\bar{q}$ vertex

Starting from eq. (3), we first perform one additional simplification with the help of the anticommutation relations of the gamma matrices,

$$\not{k}_2\gamma^\mu\not{q}v(k_2) = (2k_2^\mu\not{q} - \gamma^\mu\not{k}_2\not{q})v(k_2) = (2k_2^\mu\not{q} - 2k_2\cdot q\gamma^\mu)v(k_2)\,,$$

$$\overline{u}(k_1)\not{q}\gamma^\mu\not{k}_1 = \overline{u}(k_1)\not{q}(2k_1^\mu - \not{k}_1\gamma^\mu) = \overline{u}(k_1)(2k_1^\mu\not{q} - 2q\cdot k_1\gamma^\mu)\,,$$

after again using $\overline{u}(k_1)\not{k}_1 = \not{k}_2 v(k_2) = 0$. Inserting these results into eq. (3) yields,

$$\begin{aligned}
\mathcal{M} &= ie_q C_F g_s^2\mu^{2\epsilon}\,\overline{u}(k_1)\\
&\quad\times\int\frac{d^nq}{(2\pi)^n}\frac{(2-n)(2\not{q}q^\mu - q^2\gamma^\mu) + 2(2k_2^\mu\not{q} - 2k_2\cdot q\gamma^\mu) - (4-n)(2k_1^\mu\not{q} - 2q\cdot k_1\gamma^\mu)}{q^2(q-k_1)^2(q-k_1-k_2)^2}\,v(k_2)\\
&= \frac{e_qC_Fg_s^2}{8\pi^2}\mu^{2\epsilon}\,\overline{u}(k_1)\gamma_\nu v(k_2)\Big\{(1-\epsilon)\big[2C^{\mu\nu} - (g_{\alpha\beta}C^{\alpha\beta})g^{\mu\nu}\big] - 2(k_2^\mu C^\nu - g^{\mu\nu}k_{2\alpha}C^\alpha)\\
&\qquad\qquad + 2\epsilon(k_1^\mu C^\nu - g_{\mu\nu}k_{1\alpha}C^\alpha)\Big\},\\
&= \frac{e_qC_Fg_s^2}{4\pi^2}\mu^{2\epsilon}\,\overline{u}(k_1)\gamma_\nu v(k_2)\Big\{(1-\epsilon)C^{\mu\nu} - (k_2^\mu C^\nu - g^{\mu\nu}k_{2\alpha}C^\alpha) + \epsilon(k_1^\mu C^\nu - g_{\mu\nu}k_{1\alpha}C^\alpha)\Big\},
\end{aligned} \tag{17}$$

after employing eq. (15), where

$$C^\mu \equiv C^\mu(-k_1,-k_2,p;0,0,0)\,, \qquad C^{\mu\nu} \equiv C^{\mu\nu}(-k_1,-k_2,p;0,0,0)\,. \tag{18}$$

Using eqs. (5) and (6),

$$C^{\mu} = -k_1^{\mu} C_{11} - k_2^{\mu} C_{12}\,, \tag{19}$$

$$C^{\mu\nu} = k_1^{\mu} k_1^{\nu} C_{21} + k_2^{\mu} k_2^{\nu} C_{22} + (k_1^{\mu} k_2^{\nu} + k_2^{\mu} k_1^{\nu}) C_{23} + g^{\mu\nu} C_{24}\,, \tag{20}$$

Inserting these results back into eq. (16) and using $p^2 = 2k_1 \cdot k_2$ along with the massless Dirac equation, we end up with

$$\mathcal{M} = \frac{e_q C_F g_s^2}{8\pi^2} \mu^{2\epsilon}\, \overline{u}(k_1)\gamma^{\mu} v(k_2) \Big\{ 2(1-\epsilon) C_{24} - p^2 (C_{11} - \epsilon\, C_{12}) \Big\}. \tag{21}$$

We recognize the tree-level (Born) matrix element,

$$i\mathcal{M}_B = -ie_q \overline{u}(k_1)\gamma^{\mu} v(k_2)\,. \tag{22}$$

Hence,

$$\mathcal{M} = -\frac{C_F g_s^2}{8\pi^2} \mu^{2\epsilon} \Big\{ 2(1-\epsilon) C_{24} - p^2 (C_{11} - \epsilon C_{12}) \Big\} \mathcal{M}_B. \tag{23}$$

Finally, we make use of eqs. (12), (13) and (A.4) to obtain,

$$\begin{aligned} 2(1-\epsilon) C_{24} - p^2 (C_{11} - \epsilon\, C_{12}) &= \left(\frac{4\pi}{-p^2}\right)^{\epsilon} \Bigg\{ \frac{(1-\epsilon)\Gamma(\epsilon)\,\Gamma^2(1-\epsilon)}{\Gamma(3-2\epsilon)} + \frac{(1-\epsilon)\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)} \\ &\qquad - \frac{\epsilon\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \Bigg\} \\ &= \left(\frac{4\pi}{-p^2}\right)^{\epsilon} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(3-2\epsilon)} \left[\frac{1-\epsilon}{\epsilon} + \frac{2(1-\epsilon)^2}{\epsilon^2} + 2(1-\epsilon) \right] \\ &= \left(\frac{4\pi}{-p^2}\right)^{\epsilon} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(3-2\epsilon)} \left[\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + 3 + \mathcal{O}(\epsilon) \right]. \end{aligned} \tag{24}$$

Thus, we arrive at our final result,

$$\boxed{\mathcal{M} = -\frac{C_F g_s^2}{8\pi^2} \left(\frac{-p^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(3-2\epsilon)} \left[\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + 3 + \mathcal{O}(\epsilon) \right] \mathcal{M}_B\,.} \tag{25}$$

Note that in class, the expression inside the square brackets of eq. (23) was given as,

$$\frac{1 + (1-\epsilon)^2}{\epsilon^2} - \frac{(1-\epsilon)^2}{\epsilon}\,, \tag{26}$$

which differs from the result obtained above only in the $\mathcal{O}(\epsilon)$ term, which is irrelevant since one will take the limit of $\epsilon \to 0$ once the effects of the real emission process, $\gamma^* \to q\bar{q}g$, are included in the calculation of the $\mathcal{O}(\alpha_s)$ corrections to

$$\sigma(\gamma^* \to \text{hadrons}) = \sigma(\gamma^* \to q\bar{q}) + \sigma(\gamma^* \to q\bar{q}g)\,. \tag{27}$$

APPENDIX: Explicit computation of $C^{\mu\nu}(p_1, p_2, p; 0, 0, 0)$

In this Appendix, we provide an explicit computation of

$$\frac{i}{16\pi^2}C^{\mu\nu}(0,0;p;0,0,0) = \int\frac{d^nq}{(2\pi)^n}\,\frac{q^\mu q^\nu}{q^2(q+p_1)^2(q+p_1+p_2)^2}$$

$$= 2\int_0^1 x\,dx\int_0^1 dy\int\frac{d^nq}{(2\pi)^n}\,\frac{q^\mu q^\nu}{\left[q^2+2q\cdot\left[p_1(1-xy)+p_2(1-x)\right]+p^2(1-x)\right]^3}\,.$$

Integrating over q yields,

$$C^{\mu\nu}(p_1,p_2;p;0,0,0) = -(4\pi)^\epsilon\,\Gamma(\epsilon)\int_0^1 x\,dx\int_0^1 dy\left[2p_1\cdot p_2(1-xy)(1-x)-p^2(1-x)\right]^{-1-\epsilon}$$
$$\times\left\{\epsilon\left[p_1^\mu(1-xy)+p_2^\mu(1-x)\right]\left[p_1^\nu(1-xy)+p_2^\nu(1-x)\right]\right.$$
$$\left.-\tfrac{1}{2}g^{\mu\nu}\left[2p_1\cdot p_2(1-xy)(1-x)-p^2(1-x)\right]\right\}.$$

In light of eq. (6) it follows that,

$$C_{21}(0,0;p^2;0,0,0) = \left(\frac{4\pi}{-p^2}\right)^\epsilon\frac{1}{p^2}\,\Gamma(1+\epsilon)\int_0^1 x^{-\epsilon}(1-x)^{-1-\epsilon}\,dx\int_0^1 y^{-1-\epsilon}(1-xy)^2\,dy$$
$$= \left(\frac{4\pi}{-p^2}\right)^\epsilon\frac{1}{p^2}\,\Gamma(1+\epsilon)\left[\frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)}-\frac{2\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}+\frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\right]$$
$$= \left(\frac{4\pi}{-p^2}\right)^\epsilon\frac{1}{p^2}\,\frac{(1-\frac{1}{2}\epsilon)\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\,, \tag{A.1}$$

$$C_{22}(0,0;p^2;0,0,0) = \left(\frac{4\pi}{-p^2}\right)^\epsilon\frac{1}{p^2}\,\Gamma(1+\epsilon)\int_0^1 x^{-\epsilon}(1-x)^{1-\epsilon}\,dx\int_0^1 y^{-1-\epsilon}\,dy$$
$$= \left(\frac{4\pi}{-p^2}\right)^\epsilon\frac{1}{p^2}\,\frac{\Gamma(1+\epsilon)\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\,, \tag{A.2}$$

$$C_{23}(0,0;p^2;0,0,0) = \left(\frac{4\pi}{-p^2}\right)^\epsilon\frac{1}{p^2}\,\Gamma(1+\epsilon)\int_0^1 x^{-\epsilon}(1-x)^{-\epsilon}\,dx\int_0^1 y^{-1-\epsilon}(1-xy)\,dy$$
$$= \left(\frac{4\pi}{-p^2}\right)^\epsilon\frac{1}{p^2}\Gamma(1+\epsilon)\left[\frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)}-\frac{\Gamma^2(1-\epsilon)}{\Gamma(3-2\epsilon)}\right]$$
$$= \left(\frac{4\pi}{-p^2}\right)^\epsilon\frac{1}{p^2}\,\frac{(2-\epsilon)\Gamma(1+\epsilon)\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-2\epsilon)}\,, \tag{A.3}$$

$$C_{24}(0,0;p^2;0,0,0) = \frac{1}{2}\left(\frac{4\pi}{-p^2}\right)^\epsilon\Gamma(\epsilon)\int_0^1 x^{1-\epsilon}(1-x)^{-\epsilon}\,dx\int_0^1 y^{-\epsilon}\,dy$$
$$= \left(\frac{4\pi}{-p^2}\right)^\epsilon\frac{\Gamma(\epsilon)\,\Gamma^2(1-\epsilon)}{2\,\Gamma(3-2\epsilon)}\,. \tag{A.4}$$

One useful check of eqs. (A.3) and (A.4) can be performed by verifying eq. (15). In particular, using eq. (6) and putting $p_1^2 = p_2^2 = 0$, it follows that

$$\begin{aligned} g_{\mu\nu} C^{\mu\nu} &= 2 p_1 \cdot p_2 C_{23} + n C_{24} = p^2 C_{23} + (4 - 2\epsilon) C_{24} \\ &= \left(\frac{4\pi}{-p^2} \right)^\epsilon \frac{(2-\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-2\epsilon)} \Big[\Gamma(1+\epsilon)\Gamma(-\epsilon) + \Gamma(1-\epsilon)\Gamma(\epsilon) \Big] = 0 \, , \end{aligned} \tag{A.5}$$

after noting that

$$\Gamma(1-\epsilon)\Gamma(\epsilon) = -\epsilon\Gamma(-\epsilon)\Gamma(\epsilon) = -\Gamma(1+\epsilon)\Gamma(-\epsilon) \, . \tag{A.6}$$