In evaluating loop processes in QCD, infrared divergences due to the emission of soft gluons and mass singularities due to the emission of collinear gluons by massless quarks must be regulated. This is most easily accomplished by employing dimensional regularization. In practice, the regularization procedure requires that the phase space integration should be carried out in n spacetime dimensions when computing cross sections. In these notes, I will evaluate the two-body and three-body phase space integrals in $\boldsymbol{n}$ spacetime dimensions in the case where all final state particles are massless.

1. Massless two-body phase space integrals in n dimensions

The two-body phase space integral in $n-1$ space dimensions (and one time dimension) is defined by

$$\mathcal{I}_2 = (2\pi)^n \int \frac{d^{n-1}\boldsymbol{k}_1}{(2\pi)^{n-1}\,2E_1} \int \frac{d^{n-1}\boldsymbol{k}_2}{(2\pi)^{n-1}\,2E_2}\,\delta^n(q-k_1-k_2)\,, \tag{1}$$

corresponding to the production of two massless particles with n-dimensional spacetime momentum vectors, $k_1=(E_1;\boldsymbol{k}_1)$ and $k_2=(E_2;\boldsymbol{k}_2)$, with $E_1=|\boldsymbol{k}_1|$ and $E_2=|\boldsymbol{k}_2|$. Momentum conservation, $q=k_1+k_2$, is enforced by the delta function in eq. (1). In particular, we can identify $Q\equiv\sqrt{q^2}$ with the invariant mass of the two outgoing particles.

Employing the identity,[1]

$$\int \frac{d^{n-1}\boldsymbol{k}_1}{(2\pi)^{n-1}\,2E_1}\,\delta^4(q-k_1-k_2) = \int d^n k_1\,\delta(k_1^2)\Theta(k_{10})\delta^n(q-k_1-k_2) = \delta\big[(q-k_2)^2\big]\,, \tag{2}$$

after integrating over the n-dimensional vector $k_1=(k_{10};\boldsymbol{k}_1)$, Hence, it follows that

$$\mathcal{I}_2 = \tfrac{1}{2}(2\pi)^{-n}\int \frac{d^{n-1}\boldsymbol{k}_2}{E_2}\,\delta(q^2-2q\!\cdot\!k_2)\,, \tag{3}$$

after using $k_2^2=0$.

To evaluate eq. (3), it is convenient to work in the rest frame of the two-body system, corresponding to $\boldsymbol{q}=0$. The, $q\!\cdot\!k=q_0E_2=QE_2$, where $Q\equiv\sqrt{q_0^2-\boldsymbol{q}^2}=q_0$. Hence, after employing spherical coordinates,

$$\mathcal{I}_2 = \tfrac{1}{2}(2\pi)^{2-n}\int E_2^{n-3}\,dE_2\,d\Omega_{n-2}\,\delta(Q^2-2QE_2) = \frac{(2\pi)^{2-n}}{4Q}\left(\frac{Q}{2}\right)^{n-3}\int d\Omega_2^{(n-2)}\,. \tag{4}$$

The integral over the solid angle yields the $(n-2)$-dimensional surface area of the boundary of an $(n-1)$-dimensional solid ball, which is well known,

$$\int d\Omega_2^{(n-2)} = \frac{2\pi^{\frac{1}{2}n-1}}{\Gamma(\frac{1}{2}n-1)}\int_{-1}^{1}(\sin\theta_2)^{n-4}\,d\cos\theta_2 = (4\pi)^{\frac{1}{2}n-1}\frac{\Gamma(\frac{1}{2}n-1)}{\Gamma(n-2)} = \frac{2\pi^{(n-1)/2}}{\Gamma\big(\frac{n-1}{2}\big)}\,, \tag{5}$$

where θ_2 is the polar angle of $\boldsymbol{k}_2$ with respect to a fixed z-axis.

[1] To derive eq. (2), write $d^nk_1=dk_0\,d^{n-1}\boldsymbol{k}_1$ and $\delta(k_1^2)=\delta(k_{10}^2-\boldsymbol{k}^2)$. One can then integrate over k_{10} by making use of this delta function.

Hence, we end up with,

$$\mathcal{I}_2 = \frac{1}{8\pi}\left(\frac{Q^2}{4\pi}\right)^{\frac{1}{2}n-2} \frac{\Gamma(\frac{1}{2}n-1)}{\Gamma(n-2)}\,. \tag{6}$$

In terms of $\epsilon \equiv 2 - \frac{1}{2}n$,

$$\boxed{\mathcal{I}_2 = \frac{1}{8\pi}\left(\frac{Q^2}{4\pi}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\,.} \tag{7}$$

2. Massless three-body phase space integrals in n dimensions

The three-body phase space integral in $n-1$ space dimensions (and one time dimension) is defined by

$$\begin{aligned}\mathcal{I}_3 &= (2\pi)^n \int \frac{d^{n-1}\boldsymbol{k}_1}{(2\pi)^{n-1}\,2E_1}\int \frac{d^{n-1}\boldsymbol{k}_2}{(2\pi)^{n-1}\,2E_2}\int \frac{d^{n-1}\boldsymbol{k}_3}{(2\pi)^{n-1}\,2E_3}\,\delta^n(q-k_1-k_2-k_3)\\ &= \frac{(2\pi)^{3-2n}}{8}\int \frac{d^{n-1}\boldsymbol{k}_1\,d^{n-1}\boldsymbol{k}_2\,d^{n-1}\boldsymbol{k}_3}{E_1E_2E_3}\,\delta^{(n-1)}(\boldsymbol{q}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\delta(q_0-E_1-E_2-E_3).\end{aligned} \tag{8}$$

Since the three outgoing particles are massless, we have $E_i = |\boldsymbol{k}_i|$ for $i = 1, 2, 3$. To evaluate this integral, it is convenient to work in the rest frame of the three-body system, corresponding to $\boldsymbol{q} = 0$, in which case $q_0 = \sqrt{q_0^2 - \boldsymbol{q}^2} = Q$. We can immediately integrate over $\boldsymbol{k}_3$ using one of the delta functions to obtain,

$$\mathcal{I}_3 = \frac{(2\pi)^{3-2n}}{8}\int \frac{E_1^{n-3}\,dE_1\,d\Omega_1^{(n-2)}\,E_2^{n-3}\,dE_2\,d\Omega_2^{(n-2)}}{E_3}\,\delta(Q-E_1-E_2-E_3)\,, \tag{9}$$

where we have employed spherical coordinates for the remaining integrations over $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. To make further progress, we shall choose the z-axis to lie along the direction of $\boldsymbol{k}_1$. Then, we can do the integration over the angles of $\boldsymbol{k}_1$ for free. Using eq. (5), it follows that

$$\begin{aligned}\mathcal{I}_3 &= \frac{(2\pi)^{3-2n}(4\pi)^{\frac{1}{2}n-1}\Gamma(\frac{1}{2}n-1)}{8\Gamma(n-2)}\int \frac{E_1^{n-3}\,dE_1\,E_2^{n-3}\,dE_2}{E_3}\\ &\qquad\qquad \times \frac{(2\pi)^{\frac{1}{2}n-1}}{\Gamma(\frac{1}{2}n-1)}\int(\sin\theta_2)^{n-4}\,d\cos\theta_2\,\delta(Q-E_1-E_2-E_3)\\ &= \frac{(2\pi)^{1-n}}{4\,\Gamma(n-2)}\int \frac{E_1^{n-3}\,dE_1\,E_2^{n-3}\,dE_2}{E_3}\int(\sin\theta_2)^{n-4}\,d\cos\theta_2\,\delta(Q-E_1-E_2-E_3)\,.\end{aligned} \tag{10}$$

where θ_2 is the is the polar angle of $\boldsymbol{k}_2$ with respect to the fixed z-axis (that lies along $\boldsymbol{k}_1$).

We can now make use of momentum conservation, $\boldsymbol{Q} = \boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3$ to obtain,

$$E_3^2 = |\boldsymbol{k}_3|^2 = |\boldsymbol{k}_1 + \boldsymbol{k}_2|^2 = E_1^2 + E_2^2 + 2E_1E_2\cos\theta_2\,. \tag{11}$$

Consequently,

$$\int d\cos\theta_2\,\delta(Q-E_1-E_2-E_3) = \int dE_3\,\frac{d\cos\theta_2}{dE_3}\,\delta(Q-E_1-E_2-E_3) = \frac{E_3}{E_1E_2}\,, \qquad (12)$$

after using eq. (11) to compute $d\cos\theta_2/dE_3$. Employing the result of eq. (12) in eq. (10) yields,

$$\mathcal{I}_3 = \frac{(2\pi)^{1-n}}{4\,\Gamma(n-2)}\int E_1^{n-4}\,dE_1\,E_2^{n-4}\,dE_2\left[1-\left(\frac{E_3^2-E_1^2-E_2^2}{2E_1E_2}\right)^2\right]^{\frac{1}{2}n-2}\,, \qquad (13)$$

after using eq. (11) to substitute for $\sin^2\theta_2 = 1-\cos^2\theta_2$.

One can obtain a more useful form for $\mathcal{I}_3$ by employing the Lorentz invariant quantities,

$$s_1 = (k_2+k_3)^2 = (q-k_1)^2\,, \qquad (14)$$

$$s_2 = (k_1+k_3)^2 = (q-k_2)^2\,, \qquad (15)$$

$$s_3 = (k_1+k_2)^2 = (q-k_3)^2\,, \qquad (16)$$

where

$$s_1+s_2+s_3 = q^2\,, \qquad (17)$$

holds in light of $k_1^2 = k_2^2 = k_3^2 = 0$. Noting that

$$s_i = (q-k_i)^2 = q^2 - 2k_i{\cdot}q = Q^2 - 2E_iQ\,, \qquad (18)$$

where we have evaluated the s_i in the frame where $\boldsymbol{q}=0$, it follows that

$$E_i = \frac{Q^2-s_i}{2Q}\,. \qquad (19)$$

Hence we can change integration variables from $\{E_1, E_2\}$ to $\{s_1, s_2\}$. The Jacobian of the transformation yields $ds_1ds_2 = 4Q^2dE_1dE_2$. Moreover,

$$1-\left(\frac{E_3^2-E_1^2-E_2^2}{2E_1E_2}\right)^2 = \frac{4Q^2s_1s_2(Q^2-s_1-s_2)}{(Q-s_1)^2(Q-s_2)^2}\,. \qquad (20)$$

Hence, eq. (13) yields

$$\mathcal{I}_3 = \frac{(4\pi)^{1-n}(Q^2)^{1-\frac{1}{2}n}}{2\,\Gamma(n-2)}\int ds_1\,ds_2\left[s_1s_2(Q^2-s_1-s_2)\right]^{\frac{1}{2}n-2}\,. \qquad (21)$$

To determine the limits of the integration region, it is convenient to define dimensionless variables x_1 and x_2 as follows,

$$s_1 = Q^2(1-x_1)\,, \qquad s_2 = Q^2(1-x_2)\,. \qquad (22)$$

Note that $E_i \geq 0$ imply that $x_i \geq 0$ for $i=1,2$. In addition, $s_i \geq 0$ imply that $x_i \leq 1$ for $i=1,2$. Finally, using eq. (17) in the frame where $\boldsymbol{q}=0$, it follows that $s_1+s_2+s_3 = Q^2$.

Hence, $s_3 \geq 0$ implies that $Q^2 - s_1 - s_2 = Q^2(x_1 + x_2 - 1) \geq 0$, and we conclude that $x_1 + x_2 \geq 1$.

Thus, the integration region for the massless three-body phase space integral is determined by,

$$0 \leq x_1 \leq 1\,, \quad 0 \leq x_2 \leq 1\,, \quad \text{subject to the constraint, } x_1 + x_2 \geq 1. \tag{23}$$

Hence,

$$\mathcal{I}_3 = \frac{(4\pi)^{1-n}(Q^2)^{n-3}}{2\,\Gamma(n-2)} \int_0^1 dx_2 \int_{1-x_2}^1 dx_1 \left[(1-x_1)(1-x_2)(x_1+x_2-1)\right]^{\frac{1}{2}n-2} . \tag{24}$$

It is desirable to uncouple the two integrals. This can be done by defining new dimensionless variables,

$$x = x_2\,, \qquad y = \frac{1-x_1}{x_2}\,. \tag{25}$$

It follows that $dx_1 dx_2 = x\,dxdy$. Moreover, $0 \leq x, y \leq 1$ without any additional constraints. Hence, we arrive at our final expression,

$$\boxed{\mathcal{I}_3 = \frac{(Q^2)^{1-2\epsilon}}{2(4\pi)^{3-2\epsilon}\,\Gamma(2-2\epsilon)} \int_0^1 x^{1-2\epsilon}(1-x)^{-\epsilon}\,dx \int_0^1 y^{-\epsilon}(1-y)^{-\epsilon}\,dy\,,} \tag{26}$$

where $\epsilon \equiv 2 - \frac{1}{2}n$ and

$$s_1 = Q^2 xy\,, \qquad s_2 = Q^2(1-x)\,, \qquad s_3 = Q^2 x(1-y)\,. \tag{27}$$