DUE: THURSDAY, APRIL 16, 2020

1. (a) Derive the result:

$$\int d^4z \, \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)} \frac{\delta^2 \Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)} = -\delta^4(x-y)\,,$$

and interpret diagrammatically in terms of momentum space Green functions, under the assumption that the quantum field $\phi(x)$ has no vacuum expectation value. Here, $W[J]$ is the generating functional for the connected Green functions, $\Phi(x)$ is the classical field, and $\Gamma[\Phi]$ is the generating functional for the one particle irreducible (1PI) Green functions.

(b) By taking one further functional derivative, show that Γ generates the amputated connected three-point function.

2. Consider a quantum field theory of a real scalar field governed by the Lagrangian density,

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\,.$$

(a) Evaluate perturbatively the generating functional for the connected Green functions, $W[J]$, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$ as follows. First, show that $Z[J] \equiv \exp\{iW[J]\}$ can be written in the following form,

$$Z[J] = \mathcal{N}\left[1 - \frac{i\lambda}{4!}\int d^4y \left(\frac{1}{i}\,\frac{\delta}{\delta J(y)}\right)^4 + \mathcal{O}(\lambda^2)\right] \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1)\Delta_F(x_1-x_2)J(x_2)\right\}\,, \tag{1}$$

where $\mathcal{N}$ is the J-independent constant. Then, carry out the functional derivatives with respect to J, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$. Using the result just obtained for $Z[J]$, obtain an expression for $W[J]$ keeping all terms up to and including terms of $\mathcal{O}(\lambda)$.

(b) Using the result of part (a) for $W[J]$, compute the four-point connected Green function. By taking the appropriate Fourier transform, derive the momentum space Feynman rule for the four-point scalar interaction.

(c) Evaluate perturbatively the classical field $\Phi(x)$ and the generating functional for the 1PI Green functions, $\Gamma[\Phi]$, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$. Then, repeat part (b) for the four-point 1PI Green function.

3. Consider a quantum field theory of a real scalar field governed by the Lagrangian density,

$$\mathscr{L} = \tfrac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\,, \tag{2}$$

and the corresponding equation of motion,

$$\Box\,\phi(x) + V'(\phi) = 0\,,$$

where $\Box \equiv \partial^\mu\partial_\mu$ and $V' \equiv dV/d\phi$. The goal of this exercise is to derive the equation of motion for the Green function $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$,

$$\Box_x\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = -\langle\Omega|T\{V'(\phi(x))\phi(y)\}|\Omega\rangle - i\delta^4(x-y)\,. \tag{3}$$

In order to obtain eq. (3), you should employ the following technique. Start from the path integral definition of the generating functional,

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi\ \exp\left\{i\int d^4x\left[\mathscr{L} + J(x)\phi(x)\right]\right\}\,, \tag{4}$$

where $\mathcal{N}$ is chosen such that $Z[0] = 1$. Perform a change of variables in the path integral, $\phi(x) \to \phi(x) + \varepsilon(x)$, where $\varepsilon(x)$ is an arbitrary infinitesimal function of x. Noting that a change of variables[1] does not change the value of of $Z[J]$, show that to first order in $\varepsilon(x)$,

$$\int \mathcal{D}\phi\ \exp\left\{i\int d^4x\left[\mathscr{L} + J(x)\phi(x)\right]\right\}\int d^4x\,\varepsilon(x)\big[-\Box\Phi - V'(\phi) + J(x)\big] = 0\,. \tag{5}$$

Since $\varepsilon(x)$ is arbitrary, we may choose $\varepsilon(x) = \epsilon\,\delta^4(x-y)$, where ϵ is an infinitesimal constant. With this choice for $\varepsilon(x)$, show that by taking the functional derivative of the eq. (5) with respect to $J(x)$ and then setting $J = 0$, one ends up with eq. (3).

HINT: What is the Jacobian corresponding to the change of variables, $\phi(x) \to \phi(x) + \varepsilon(x)$?

4. Consider a theory of a real scalar field governed by eq. (2) with $V(\phi) = \frac{1}{2}m^2\phi^2$.

(a) Compute exactly the free-field Feynman propagator, $\Delta_F(x)$, in coordinate space.

HINT: You will need to consult a good table of integrals such as I.S. Gradshteyn & I.M. Ryzhik, *Table of Integrals, Series and Products*. You should be able to find integrals that when differentiated with respect to a parameter yield the integral of interest for this problem.

(b) Evaluate the leading singularities of $\Delta_F(x)$ near the light cone, $x^2 = 0$.

(c) Using the Källen–Lehmann representation, comment on the leading singularity of the exact two-point function near the light cone.

[1] Just as in the case of ordinary integration, a change of integration variables does not change the value of the functional integral.