DUE: FRIDAY, JUNE 12, 2020

1. One way of defining $\Lambda_{\rm QCD}$ (which does not depend on QCD perturbation theory) is as follows. The running coupling constant, $\overline{g}(Q)$, is the solution to the equation

$$\frac{d\overline{g}}{dt} = \beta(\overline{g})\,, \tag{1}$$

with boundary condition $\overline{g}(0) = g$, where $t \equiv \ln(Q/\mu)$, and μ is an arbitrary parameter with dimensions of mass introduced by the renormalization procedure. To solve eq. (1), introduce the indefinite integral

$$y(z) \equiv \int^z \frac{dz'}{\beta(z')}\,.$$

Then, the solution to eq. (1) is

$$t = y(\overline{g}) - y(g)\,.$$

Note that $y(g)$ is just the integration constant that is fixed by the boundary condition for the differential equation. We now define $\Lambda_{\rm QCD}$ through the following equation:

$$y(g) \equiv -\tfrac{1}{2}\ln\left(\frac{\Lambda^2_{\rm QCD}}{\mu^2}\right)\,. \tag{2}$$

(a) Working to lowest nontrivial order in QCD perturbation theory, show that $\Lambda_{\rm QCD}$ defined in eq. (2) coincides with the definition given in class.

(b) Show that $\Lambda_{\rm QCD}$ defined in eq. (2) is independent of the arbitrary mass parameter μ.

HINT: Show that $d\Lambda_{\rm QCD}/d\mu = 0$.

2. Consider an extension of QCD (called supersymmetric QCD), where we add to QCD a color octet neutral Majorana fermion called the gluino ($\tilde{g}$), and color triplet scalar particles, called squarks ($\tilde{q}$), which possess the same electroweak quantum numbers as the corresponding quarks. Take all particles of this model to be massless. The squarks and gluinos possess the following interactions and corresponding Feynman rules:

$g\tilde{q}\tilde{q}$	$-ig_s(p_1+p_2)_\mu\,\boldsymbol{T^a}$
$gg\tilde{q}\tilde{q}$	$ig_s^2\,g_{\mu\nu}(\boldsymbol{T^a T^b} + \boldsymbol{T^b T^a})$
$\tilde{g}\tilde{q}q$	$-ig_s\sqrt{\tfrac{1}{2}}(1\pm\gamma_5)\boldsymbol{T^a}$
$g\tilde{g}\tilde{g}$	$-g_s f^{abc}\gamma_\mu$

where in the $g\tilde{q}\tilde{q}$ vertex, a $\tilde{q}$ enters the vertex with momentum p_1 and leaves with momentum p_2. In the rule for the $g\tilde{g}\tilde{g}$ vertex, a is the adjoint color index of the gluon and b (c) is the adjoint color index of the gluino that leaves (enters) the vertex. In the rule for the $\tilde{g}\tilde{q}q$ vertex, use the positive (negative) sign if the outgoing $\tilde{q}$ is the partner of a right (left) handed quark, and vice versa for an incoming $\tilde{q}$. In particular, for every quark flavor, there are two corresponding squark partners (called $\tilde{q}_R$ and $\tilde{q}_L$). The $g\tilde{q}\tilde{q}$ Feynman rule applies to both $g\tilde{q}_L\tilde{q}_L$ and to $g\tilde{q}_R\tilde{q}_R$. However, there is no $g\tilde{q}_L\tilde{q}_R$ interaction since the gluon couples diagonally to pairs of scalars or fermions. In your calculation, take the gauge group to be SU(N) with structure constants f^{abc} and denote the generators in the defining (fundamental) representation of SU(N) by $\boldsymbol{T^a}$. (Of course, for QCD, one should take $N = 3$.)

(a) Using dimensional regularization and the $\overline{\text{MS}}$ renormalization scheme, compute the lowest order contribution to the QCD β-function in a non-abelian gauge theory based on SU(N) color coupled to n_f quark flavors, $2n_f$ squark partners and a gluino. This requires a number of steps:

(i) Start with the result for $Z_g = Z_{1F}Z_2^{-1}Z_3^{-1/2}$ derived in class for ordinary QCD. Draw Feynman diagrams corresponding to the new supersymmetric contributions to Z_{1F}, Z_2 and Z_3.

(ii) Argue that the one-loop supersymmetric contributions to $Z_{1F}Z_2^{-1}$ cancel exactly. (Recall that in QED, $Z_{1F}Z_2^{-1} = 1$.) As a result, one need only consider the supersymmetric contributions to Z_3.

(iii) Using the result for Z_3 in ordinary QCD obtained in class, the gluino contribution to Z_3 can be obtained by inspection. Keep in mind that the gluino transforms under the adjoint representation of SU(N) color. Moreover, the gluino is a Majorana fermion which possesses half the number of degrees of freedom of a Dirac fermion. This yields an extra factor of 1/2.

(iv) Thus, the only new computation required is the squark loop contribution to Z_3. Compute this contribution, and then combining this with the result of (iii), obtain the supersymmetric QCD one-loop β-function.

(b) Does the QCD running coupling constant run faster or slower at large momentum scales in a supersymmetric theory as compared to the non-supersymmetric one?

(c) Compute the one-loop $\mathcal{O}(\alpha_s)$ relation between the $\overline{\text{MS}}$ running top-quark mass, $m_t(m_t)$, and the "pole mass" (denoted by M_t) in ordinary QCD. Ignore all electroweak contributions.

(d) Repeat part (c) for supersymmetric QCD. Which new Feynman graphs contribute? How is the one-loop relation of part (c) modified? For simplicity, you may take the gluino to be massless and the top-squarks to be degenerate in mass with the top-quark.

3. Consider a theory of a single massless scalar real field:

$$\mathscr{L} = \partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda}{4!}\phi^4\,.$$

In class, we computed the effective potential (V_{eff}) in the one-loop approximation. The renormalized V_{eff} depends on the parameter μ (which is either the mass scale of dimensional regularization or the off-shell subtraction point used in the definition of λ). The unrenormalized V_{eff} is independent of μ.

(a) Deduce the renormalization group equation (RGE) satisfied by the renormalized $V_{\rm eff}$. Your equation should involve the beta-function $\beta(\lambda_R)$ and the anomalous dimension $\gamma_d(\lambda_R)$, where λ_R is the renormalized coupling.

(b) By dimensional analysis, the renormalized $V_{\rm eff}$ can be written as:

$$V_{\rm eff}(\phi_R) = \frac{Y(\lambda_R, t)\phi_R^4}{4!}\,,$$

where $t = \log(\phi_R/\mu)$ and ϕ_R is the renormalized scalar field. Assume that $V_{\rm eff}$ is defined in the physical scheme where,

$$\left.\frac{d^2V_{\rm eff}}{d\phi_R^2}\right|_{\phi_R=0} = 0, \qquad\qquad \left.\frac{d^4V_{\rm eff}}{d\phi_R^4}\right|_{\phi_R=\mu} = \lambda_R\,. \tag{3}$$

Rewrite the RGE of part (a) as an equation for $Y(\lambda_R, t)$. Solve the resulting equation for Y as a function of a suitably defined running coupling constant $\overline{\lambda}(t)$.

(c) Assuming that β is constant (independent of λ_R) and $\gamma_d = 0$, use the result of part (b) to obtain a formula for the renormalized $V_{\rm eff}$. Compare this result to the one-loop effective potential computed in class.

(d) Repeat part (c), but now use the one-loop approximations for β and γ_d. (*HINT:* γ_d is still zero in this approximation. Why?) The resulting $V_{\rm eff}$ is now the renormalization group improved effective potential. Recall that $V_{\rm eff}$ in the one-loop approximation had a local maximum at $\phi_R = 0$ and a local minimum for a nonzero value of ϕ_R. Is the extremum of the renormalization group improved $V_{\rm eff}$ at $\phi_R = 0$ a minimum or a maximum? Is the discrete $\phi_R \to -\phi_R$ symmetry spontaneously broken?

4. Consider scalar electrodynamics where the bare tree-level scalar mass parameter is zero,

$$\mathscr{L} = (D_\mu\phi)(D^\mu\phi)^* - \lambda(\phi\phi^*)^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\,, \tag{4}$$

where $D_\mu \equiv \partial_\mu + ieA_\mu$.

(a) Compute the one-loop effective potential in the Landau gauge ($\xi = 0$) in two different schemes: the $\overline{\rm MS}$ scheme and the physical scheme analogous to the one defined in eq. (3). Assume that the renormalized couplings have the property that λ_R is of $\mathcal{O}(e_R^4)$, and keep only terms of the same order in $V_{\rm eff}$.

(b) Show that the $U(1)$ gauge symmetry is spontaneously broken and compute the mass of the resulting Higgs boson (m_H) in terms of the mass of the vector boson (m_V). Show that in the one-loop approximation considered here, the Higgs boson mass is scheme independent by showing that you get the same result in both schemes of part (a).

(c) [*EXTRA CREDIT*]: Consider the dependence of the one-loop effective potential on the gauge parameter ξ. If one employs the gauge fixing term exhibited in eq. (4), the calculation of the effective potential using the tadpole method is unwieldy due to the mixing of the photon field and the derivative of the scalar field in the shifted Lagrangian. This problem is ameliorated by employing the alternative gauge fixing term,

$$\mathscr{L}_{\rm GF} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi e \phi_1 \phi_2)^2 \,. \tag{5}$$

Employing this new gauge fixing term, repeat the computations of part (a). Show that in the one-loop approximation considered here, the Higgs boson mass is independent of ξ.

HINT: In the computation of $V_{\rm eff}$ in the case of $\xi \neq 0$, show that one cannot neglect the contributions of the Faddeev-Popov ghosts (which contribute in the case of $\xi \neq 0$ even in the abelian gauge theory).