1. (a) Derive the result:

$$\int d^4z\, \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)} \frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)} = -\delta^4(x-y)\,,$$

and interpret diagrammatically in terms of momentum space Green functions, under the assumption that the quantum field $\phi(x)$ has no vacuum expectation value. Here, $W[J]$ is the generating functional for the connected Green functions, $\Phi(x)$ is the classical field, and $\Gamma[\Phi]$ is the generating functional for the one particle irreducible (1PI) Green functions.

We begin with the definition of the effective action,

$$\Gamma[\Phi] = W[J] - \int d^4x\, J(x)\Phi(x)\,, \tag{1}$$

where

$$\Phi(x) \equiv \frac{\delta W[J]}{\delta J(x)} \tag{2}$$

defines the classical field. From eq. (1), it follows that

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = -J(x)\,. \tag{3}$$

Taking a second functional derivative yields,

$$\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} = \frac{\delta\Phi(x)}{\delta J(y)}\,, \qquad\qquad \frac{\delta^2\Gamma[\Phi]}{\delta\Phi(x)\delta\Phi(y)} = -\frac{\delta J(x)}{\delta\Phi(y)}\,.$$

Hence,

$$\int d^4z\, \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\, \frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)} = -\int d^4z \frac{\delta\Phi(x)}{\delta J(z)}\, \frac{\delta J(z)}{\delta\Phi(y)} = -\frac{\delta\Phi(x)}{\delta\Phi(y)} = -\delta^4(x-y)\,, \tag{4}$$

where we have used the chain rule for functional derivatives at the second step above.

Recall that the two-point 1PI Green function, $\Gamma^{(2)}(x_1,x_2)$, and the two-point connected Green function $G_c^{(2)}(x_1,x_2)$, are defined as

$$\Gamma^{(2)}(x_1,x_2) = \left(\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(x)\delta\Phi(y)}\right)\bigg|_{\Phi=0}\,, \qquad\qquad G^{(2)}(x_1,x_2) = -i\left(\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}\right)\bigg|_{J=0}\,,$$

Under the assumption that the quantum field $\phi(x)$ has no vacuum expectation value,[1]

$$G_c^{(1)}(x) \equiv \langle\Omega|\phi(x)|\Omega\rangle = \frac{\delta W[J]}{\delta J(x)}\bigg|_{J=0} = \Phi(x)\big|_{J=0} = 0\,. \tag{5}$$

That is, setting $J=0$ implies that $\Phi=0$ and vice versa.

[1] If $\langle\Omega|\phi(x)|\Omega\rangle = v_0 \neq 0$, then one can redefine a new the quantum field, $\phi\to\phi+v_0$, whose vacuum expectation value is zero.

Using the above results, eq. (4) implies that

$$\int d^4z\, \Gamma^{(2)}(x,z) G_c^{(2)}(z,y) = i\delta^4(x-y)\,, \tag{6}$$

In momentum space,

$$\Gamma^{(2)}(p_1,p_2)(2\pi)^4\delta^4(p_1+p_2) = \int d^4x\, d^4z\, e^{i(p_1x+p_2z)}\Gamma^{(2)}(x,z)\,, \tag{7}$$

$$G_c^{(2)}(p_1,p_2)(2\pi)^4\delta^4(p_1+p_2) = \int d^4y\, d^4z\, e^{i(p_1z+p_2y)}G_c^{(2)}(z,y)\,, \tag{8}$$

where $px \equiv p_1\cdot x$ is the dot product of two four-vectors. Inverting the Fourier transforms yields

$$\Gamma^{(2)}(x,z) = \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\, e^{-i(p_1x+p_2z)}\Gamma^{(2)}(p_1,p_2)(2\pi)^4\delta^4(p_1+p_2) = \int \frac{d^4p}{(2\pi)^4}e^{-ip(x-z)}\Gamma^{(2)}(p,-p), \tag{9}$$

$$G_c^{(2)}(z,y) = \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\, e^{-i(p_1z+p_2y)}G_c^{(2)}(p_1,p_2)(2\pi)^4\delta^4(p_1+p_2) = \int \frac{d^4p'}{(2\pi)^4}e^{-ip'(z-y)}G_c^{(2)}(p',-p'). \tag{10}$$

It follows that

$$\begin{aligned}\int d^4z\, \Gamma^{(2)}(x,z)G_c^{(2)}(z,y) &= \int d^4z\, \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}e^{iz(p-p')}e^{-ipx}e^{ipy}\, \Gamma^{(2)}(p,-p)G_c^{(2)}(p',-p') \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}(2\pi)^4\delta^4(p-p')e^{-ipx}e^{ip'y}\, \Gamma^{(2)}(p,-p)G_c^{(2)}(p',-p') \\ &= \int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\Gamma^{(2)}(p,-p)G_c^{(2)}(p,-p)\end{aligned} \tag{11}$$

Employing the integral representation of the delta function, eq. (6) yields

$$\int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\left[\Gamma^{(2)}(p,-p)G_c^{(2)}(p,-p) - i\right] = 0\,. \tag{12}$$

Hence, it follows that

$$\Gamma^{(2)}(p,-p)G_c^{(2)}(p,-p) = i\,. \tag{13}$$

Since $G_c^{(2)}(p,-p)$ is the momentum space propagator, we conclude that $i\Gamma^{(2)}(p,-p)$ is the *negative* of the inverse propagator in momentum space.

(b) By taking one further functional derivative, show that Γ generates the amputated connected three-point function.

We shall take a functional derivative of eq. (4). On the right hand side of eq. (4), we have

$$\frac{\delta}{\delta J(w)}\,\delta^4(x-y) = 0\,,$$

since $\delta^4(x-y)$ is the analog of the Kronecker delta, δ_{ij}, for an infinite dimensional function space. On the left hand side of eq. (4),

$$\frac{\delta}{\delta J(w)}\left[\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\,\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)}\right]=\frac{\delta^3 W[J]}{\delta J(w)\delta J(x)\delta J(z)}\,\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)}+\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\,\frac{\delta}{\delta J(w)}\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)}\,.$$

In the second term on the right hand side above, we use the chain rule,

$$\frac{\delta}{\delta J(w)}=\int d^4v\,\frac{\delta\Phi(v)}{\delta J(w)}\,\frac{\delta}{\delta\Phi(v)}=\int d^4v\,\frac{\delta^2 W[J]}{\delta J(w)\delta J(v)}\,\frac{\delta}{\delta\Phi(v)}\,,$$

after using the definition of the classical field $\Phi(v)$ given in eq. (2). Hence, eq. (4) yields

$$\int d^4z\,\frac{\delta^3 W[J]}{\delta J(w)\delta J(x)\delta J(z)}\,\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)}+\int d^4z\,d^4v\,\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\,\frac{\delta^2 W[J]}{\delta J(w)\delta J(v)}\,\frac{\delta^3\Gamma[\Phi]}{\delta\Phi(v)\delta\Phi(z)\delta\Phi(y)}=0\,. \tag{14}$$

We now multiply eq. (14) by

$$\frac{\delta^2 W[J]}{\delta J(y)\delta J(u)}\,,$$

and integrate over d^4y. Using the result of eq. (4), we obtain

$$\begin{aligned}\int d^4z\,d^4y&\frac{\delta^3 W[J]}{\delta J(w)\delta J(x)\delta J(z)}\,\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(z)\delta\Phi(y)}\,\frac{\delta^2 W[J]}{\delta J(y)\delta J(u)}\\ &=-\int d^4z\frac{\delta^3 W[J]}{\delta J(w)\delta J(x)\delta J(z)}\,\delta^4(z-u)\\ &=-\frac{\delta^3 W[J]}{\delta J(w)\delta J(x)\delta J(u)}\,.\end{aligned}$$

Applying this result to eq. (14) yields

$$\frac{\delta^3 W[J]}{\delta J(w)\delta J(x)\delta J(u)}=\int d^4v\,d^4y\,d^4z\,\frac{\delta^3\Gamma[\Phi]}{\delta\Phi(v)\delta\Phi(z)\delta\Phi(y)}\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\,\frac{\delta^2 W[J]}{\delta J(w)\delta J(v)}\,\frac{\delta^2 W[J]}{\delta J(y)\delta J(u)}\,. \tag{15}$$

Recall the definition of the connected n-point Green function,

$$G_c^{(n)}(x_1,x_2,\ldots,x_n)=i^{1-n}\,\frac{\delta^n W[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}\bigg|_{J=0}\,, \tag{16}$$

and the n-point 1PI Green function,

$$\Gamma^{(n)}(x_1,x_2,\ldots,x_n)=\frac{\delta^n\Gamma[\phi]}{\delta\phi(x_1)\delta\phi(x_2)\cdots\delta\phi(x_n)}\bigg|_{\phi=0}\,. \tag{17}$$

In light of eq. (5), we can set $J=\Phi=0$ in eq. (15). Using eqs. (16) and (17), it then follows that

$$G_c^{(3)}(w,x,u)=i\int d^4v\,d^4y\,d^4z\,\Gamma^{(3)}(v,z,y)G_c^{(2)}(v,w)G_c^{(2)}(z,x)G_c^{(2)}(y,u)\,. \tag{18}$$

To invert this equation, we make use of the inverse propagator, which satisfies

$$\int d^4z\, G_c^{(2)}(x,z) G_c^{(2)\,-1}(z,y) = \delta^4(x-y)\,.$$

Then, we can rewrite eq. (18) as

$$i\Gamma^{(3)}(v,z,y) = \int d^4w\, d^4x\, d^4u\, G_c^{(2)\,-1}(w,v) G_c^{(2)\,-1}(x,z) G_c^{(2)\,-1}(u,y) G_c^{(3)}(w,x,u)\,.$$

The effect of the factors of $G_c^{(2)\,-1}$ is to remove the explicit propagators that appear on the external legs of the three-point Green function. That is, $i\Gamma^{(3)}$ is obtained from $G_c^{(3)}$ by amputating the full propagators on the three external legs.

2. Consider a quantum field theory of a real scalar field governed by the Lagrangian density,

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\,. \tag{19}$$

(a) Evaluate perturbatively the generating functional for the connected Green functions, $W[J]$, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$ as follows. First, show that the generating functional for the full Green functions, $Z[J] \equiv \exp\{iW[J]\}$, can be written in the following form,

$$Z[J] = \mathcal{N}\left[1 - \frac{i\lambda}{4!}\int d^4y \left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^4 + \mathcal{O}(\lambda^2)\right] \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1)\Delta_F(x_1-x_2)J(x_2)\right\}\,, \tag{20}$$

where $\mathcal{N}$ is the J-independent constant. Then, carry out the functional derivatives with respect to J, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$. Using the result just obtained for $Z[J]$, obtain an expression for $W[J]$ keeping all terms up to and including terms of $\mathcal{O}(\lambda)$.

The generating functional for the connected Green functions, $W[J]$, is determined by

$$Z[J] = \exp\{iW[J]\}\,, \tag{21}$$

where

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi\, \exp\left\{i\int d^4x \left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi\right]\right\}\,, \tag{22}$$

where $\mathcal{N}$ is a normalization factor that is determined by $Z[J=0]=1$. Expanding the functional integral given in eq. (22) to $\mathcal{O}(\lambda)$,

$$\int \mathcal{D}\phi\, \exp\left\{i\int d^4x \left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 + J\phi\right]\right\}\left[1 - i\int d^4y\frac{\lambda}{4!}\phi^4(y)\right]$$

$$= \left[1 - \frac{i\lambda}{4!}\int d^4y\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^4\right]\int \mathcal{D}\phi\, \exp\left\{i\int d^4x\left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 + J\phi\right]\right\}\,,$$

since each $i^{-1}\delta/\delta J(x)$ operator brings down a factor of $\phi(x)$.

The free field value of $Z[J]$ obtained by setting $\lambda = 0$ was given in class,

$$Z_0[J] = \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1)\Delta_F(x_1 - x_2)J(x_2)\right\}\,, \tag{23}$$

where $i\Delta_F(x) \equiv -i(\Box_x + m^2 - i\epsilon)^{-1}$ is the free-field propagator. It then follows that

$$Z[J] = \mathcal{N}\left[1 - \frac{i\lambda}{4!}\int d^4y\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^4\right]\exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1)\Delta_F(x_1 - x_2)J(x_2)\right\}\,. \tag{24}$$

There is no need to separately evaluate $\mathcal{N}$ since it can be determined at the end of our computation using $Z[J = 0] = 1$.

To evaluate eq. (24), we first compute

$$\frac{1}{i}\frac{\delta}{\delta J(y)}\exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1)\Delta_F(x_1 - x_2)J(x_2)\right\}$$
$$= -\int d^4x\, \Delta_F(x - y)J(x)\,\exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1)\Delta_F(x_1 - x_2)J(x_2)\right\}\,,$$

where we have used $\Delta_F(x - y) = \Delta_F(y - x)$ to combine two equivalent terms resulting from the functional derivative of $-\frac{1}{2}\int d^4x_1\, d^4x_2\, J(x_1)\Delta_F(x_1 - x_2)J(x_2)$. Taking a second functional derivative yields

$$\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^2\exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1 - x_2)J(x_2)\right\}$$
$$= \left\{i\Delta_F(0) + \left[\int d^4x\, \Delta_F(x - y)J(x)\right]^2\right\}\exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1 - x_2)J(x_2)\right\}\,.$$

Taking a third functional derivative yields

$$\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^3\exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1 - x_2)J(x_2)\right\}$$
$$= \left\{-3i\Delta_F(0)\int d^4x\, \Delta_F(x - y)J(x) - \left[\int d^4x\, \Delta_F(x - y)J(x)\right]^3\right\}$$
$$\times\exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1 - x_2)J(x_2)\right\}\,.$$

Finally, taking a fourth functional derivative yields

$$\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^4\exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1 - x_2)J(x_2)\right\}$$
$$= \left\{-3[\Delta_F(0)]^2 + 6i\Delta_F(0)\left[\int d^4x\, \Delta_F(x - y)J(x)\right]^2 + \left[\int d^4x\, \Delta_F(x - y)J(x)\right]^4\right\}$$
$$\times\exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x)\Delta_F(x_1 - x_2)J(x_2)\right\}\,.$$

The end result is

$$Z[J]=\mathcal{N}\Bigg\{1+\frac{i\lambda}{8}\Bigg[\int d^4y[\Delta_F(0)]^2-2i\Delta_F(0)\int d^4y\,d^4x_1\,d^4x_2\,\Delta_F(x_1-y)\Delta_F(x_2-y)J(x_1)J(x_2)$$

$$-\frac{1}{3}\int d^4y\,d^4x_1\,d^4x_2\,d^4x_3\,d^4x_4\,\Delta_F(x_1-y)\cdots\Delta_F(x_4-y)J(x_1)\cdots J(x_4)\Bigg]\Bigg\}$$

$$\times\exp\left\{-\frac{i}{2}\int d^4x_1\,d^4x_2\,J(x)\Delta_F(x_1-x_2)J(x_2)\right\}.$$

Using $Z[0]=1$, it follows that to $\mathcal{O}(\lambda)$,

$$\mathcal{N}=1-\frac{i\lambda}{8}\int d^4y[\Delta_F(0)]^2\,.$$

Thus,

$$Z[J]=\Bigg\{1-\frac{i\lambda}{4!}\Bigg[6i\Delta_F(0)\int d^4y\,d^4x_1\,d^4x_2\,\Delta_F(x_1-y)\Delta_F(x_2-y)J(x_1)J(x_2)$$

$$+\int d^4y\,d^4x_1\,d^4x_2\,d^4x_3\,d^4x_4\,\Delta_F(x_1-y)\cdots\Delta_F(x_4-y)J(x_1)\cdots J(x_4)\Bigg]\Bigg\}$$

$$\times\exp\left\{-\frac{i}{2}\int d^4x_1\,d^4x_2\,J(x)\Delta_F(x_1-x_2)J(x_2)\right\}.$$

Since we are only keeping terms of $\mathcal{O}(\lambda)$, we can also rewrite $Z[J]$ in the following form,

$$Z[J]=\exp\Bigg\{-\frac{i}{2}\int d^4x_1\,d^4x_2\,J(x)\Delta_F(x_1-x_2)J(x_2)$$

$$-\frac{i\lambda}{4!}\Bigg[6i\Delta_F(0)\int d^4y\,d^4x_1\,d^4x_2\,\Delta_F(x_1-y)\Delta_F(x_2-y)J(x_1)J(x_2)$$

$$+\int d^4y\,d^4x_1\,d^4x_2\,d^4x_3\,d^4x_4\,\Delta_F(x_1-y)\cdots\Delta_F(x_4-y)J(x_1)\cdots J(x_4)\Bigg]\Bigg\}.$$

Hence, using eq. (21) it follows that

$$W[J]=-\frac{1}{2}\int d^4x_1\,d^4x_2\,J(x)\Delta_F(x_1-x_2)J(x_2) \tag{25}$$

$$-\frac{i\lambda}{4}\Delta_F(0)\int d^4y\,d^4x_1\,d^4x_2\,\Delta_F(x_1-y)\Delta_F(x_2-y)J(x_1)J(x_2)$$

$$-\frac{\lambda}{4!}\int d^4y\,d^4x_1\,d^4x_2\,d^4x_3\,d^4x_4\,\Delta_F(x_1-y)\cdots\Delta_F(x_4-y)J(x_1)\cdots J(x_4)\,.$$

(b) Using the result of part (a) for $W[J]$, compute the four-point connected Green function. By taking the appropriate Fourier transform, derive the momentum space Feynman rule for the four-point scalar interaction.

Using eqs. (16) and (25), it immediately follows that

$$G_c^{(4)}(x_1, x_2, x_3, x_4) = -i\lambda \int d^4y \Delta_F(x_1 - y)\Delta_F(x_2 - y)\Delta_F(x_3 - y)\Delta_F(x_4 - y)\,. \tag{26}$$

In particular, note that the coefficient of 1/4! is canceled due to the fact that there are 4! ways to take the functional derivatives in eq. (16).

The connected Green function in momentum space is obtained by taking the following Fourier transform,

$$\begin{aligned}
&G_c^{(4)}(p_1, p_2, p_3, p_4)(2\pi)^4\delta^4(p_1 + p_2 + p_3 + p_4) \\
&\qquad = \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, e^{i(p_1x_1+\cdots+p_4x_4)} G_c^{(4)}(x_1, x_2, x_3, x_4) \\
&\qquad = -i\lambda \int d^4y\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, e^{i(p_1x_1+\cdots+p_4x_4)} \Delta_F(x_1 - y)\cdots\Delta_F(x_4 - y) \\
&\qquad = -i\lambda \int d^4y\, d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, e^{iy(p_1+\cdots+p_4)} e^{ip_1(x_1-y)}\Delta_F(x_1 - y)\cdots e^{ip_4(x_4-y)}\Delta_F(x_4 - y)\,.
\end{aligned}$$

We can now perform the integration over $x_1, \ldots, x_4$ using the expression for the free-field propagator in momentum space,

$$\frac{1}{p^2 - m^2 + i\epsilon} = \int d^4x\, e^{ipx}\Delta_F(x)\,, \tag{27}$$

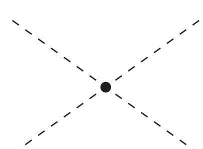
where m is the mass of the scalar field. Employing the integral representation of the momentum conserving delta function,

$$\int d^4y\, e^{iy(p_1+\cdots+p_4)} = (2\pi)^4\delta^4(p_1 + p_2 + p_3 + p_4)\,, \tag{28}$$

the end result is

$$G_c^{(4)}(p_1, p_2, p_3, p_4) = -i\lambda\, \frac{i}{p_1^2 - m^2 + i\epsilon} \cdots \frac{i}{p_4^2 - m^2 + i\epsilon}\,. \tag{29}$$

If we now amputate the four external propagators, we arrive at the Feynman rule for the four-point scalar interaction shown below.



$-i\lambda$

<u>*REMARK*</u>: In the class handout entitled *Wick Expansion in the functional integration formalism*, eq. (26) is derived by employing an alternative version of the Wick expansion based on Coleman's lemma.

(c) Evaluate perturbatively the classical field $\Phi(x)$ and the generating functional for the 1PI Green functions, $\Gamma[\Phi]$, keeping all terms up to and including terms of $\mathcal{O}(\lambda)$. Then, repeat part (b) for the four-point 1PI Green function.

The effective action is given by eq. (1), where the classical field is defined by eq. (2). Using eq. (25), it follows that

$$\Phi(x) = -\int d^4x_1\,\Delta_F(x-x_1)J(x_1) - \tfrac{1}{2}i\lambda\Delta_F(0)\int d^4y\,d^4x_1\,\Delta_F(x-y)\Delta_F(x_1-y)J(x_1)$$
$$-\frac{\lambda}{6}\int d^4y\,d^4x_1\,d^4x_2\,d^4x_3\,\Delta_F(x-y)\Delta_F(x_1-y)\Delta_F(x_2-y)\Delta_F(x_3-y)J(x_1)J(x_2)J(x_3)\,. \tag{30}$$

We must invert this equation and solve for $J(x)$. This can be done using an iterative process. Operate on eq. (30) with the operator $\Box_x + m^2 - i\epsilon$. Using

$$(\Box_x + m^2 - i\epsilon)\Delta_F(x-y) = -\delta^4(x-y)\,, \tag{31}$$

it follows that

$$(\Box_x + m^2 - i\epsilon)\Phi(x) = J(x) + \tfrac{1}{2}i\lambda\Delta_F(0)\int d^4x_1\,\Delta_F(x_1-x)J(x_1)$$
$$+\frac{\lambda}{6}\int d^4x_1\,d^4x_2\,d^4x_3\,\Delta_F(x_1-x)\Delta_F(x_2-x)\Delta_F(x_3-x)J(x_1)J(x_2)J(x_3)\,. \tag{32}$$

At $\mathcal{O}(\lambda^0)$, we have $J(x) = (\Box_x + m^2 - i\epsilon)\Phi(x)$. Thus, in the $\mathcal{O}(\lambda)$ term in eq. (32), we can replace $J(x_k)$ with $(\Box_{x_k} + m^2 - i\epsilon)\Phi(x_k)$, for $k = 1, 2, 3$. We can then move the operators $(\Box_{x_k} + m^2 - i\epsilon)$ so that they operate on the $\Delta_F(x_k - x)$ by two successive integrations by parts. Using eq. (31), we produce three delta functions, after which the integrals over x_1, x_2 and x_3 are trivially done. The end result is

$$(\Box_x + m^2 - i\epsilon)\Phi(x) = J(x) - \tfrac{1}{2}i\lambda\Delta_F(0)\Phi(x) - \tfrac{1}{6}\lambda\big[\Phi(x)\big]^3\,.$$

Hence, to $\mathcal{O}(\lambda)$,

$$J(x) = (\Box_x + m^2 - i\epsilon)\Phi(x) + \tfrac{1}{2}i\lambda\Delta_F(0)\Phi(x) + \tfrac{1}{6}\lambda\big[\Phi(x)\big]^3\,. \tag{33}$$

We can use the same procedure to rewrite $W[J]$ in terms of the classical field $\Phi(x)$. We simply insert eq. (33) into eq. (25), and keep only terms up to $\mathcal{O}(\lambda)$. This yields

$$W[J] = \tfrac{1}{2}\int d^4x\,\Phi(x)\Big\{(\Box_x + m^2)\Phi(x) + \tfrac{1}{2}i\lambda\Delta_F(0)\Phi(x) + \tfrac{1}{6}[\Phi(x)]^3\Big\}$$
$$-\tfrac{1}{4}i\lambda\Delta_F(0)\int d^4x\,\big[\Phi(x)\big]^2 - \frac{\lambda}{4!}\int d^4x\,\big[\Phi(x)\big]^4\,,$$

after taking the $\epsilon \to 0$ limit. Using eq. (1) to obtain the effective action, we note that

$$\int d^4x\, J(x)\Phi(x) = \int d^4x\, \Phi(x)\Big\{(\Box_x + m^2)\Phi(x) + \tfrac{1}{2}i\lambda\Delta_F(0)\Phi(x) + \tfrac{1}{6}\lambda[\Phi(x)]^3\Big\}\,,$$

where we have again used eq. (33) and have kept only terms up to $\mathcal{O}(\lambda)$. Hence, we end up with

$$\Gamma[\Phi] = -\tfrac{1}{2}\int d^4x\, \Phi(x)(\Box_x + m^2)\Phi(x) - \tfrac{1}{4}i\lambda\Delta_F(0)\int d^4x\, \big[\Phi(x)\big]^2 - \frac{\lambda}{4!}\int d^4x\, \big[\Phi(x)\big]^4\,. \qquad (34)$$

Finally, we make use of eq. (17) to compute the 1PI four-point function,

$$\Gamma^{(4)}(x_1,\ldots,x_4) = \frac{\delta^n\Gamma[\Phi]}{\delta\Phi(x_1)\cdots\delta\Phi(x_4)}\bigg|_{\Phi=0}\,.$$

Using eq. (34),

$$\Gamma^{(4)}(x_1,\ldots,x_4) = -\lambda\int d^4x\, \delta^4(x-x_1)\delta^4(x-x_2)\delta^4(x-x_3)\delta^4(x-x_4)\,.$$

In momentum space,

$$\begin{aligned}\Gamma^{(4)}(p_1,p_2,p_3,p_4)&(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\\ &= \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, e^{i(p_1x_1+\cdots+p_4x_4)}\Gamma^{(4)}(x_1,x_2,x_3,x_4)\\ &= -\lambda\int d^4x\, e^{ix(p_1+\cdots+p_4)}\\ &= -(2\pi)^4\lambda\,\delta^4(p_1+p_2+p_3+p_4)\,.\end{aligned}$$

That is,

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = -\lambda\,.$$

The momentum space Feynman rule for the four-point scalar interaction corresponds to the leading order contribution to $i\Gamma^{(4)}(p_1,p_2,p_3,p_4)$.

3. Consider a quantum field theory of a real scalar field governed by the Lagrangian density,

$$\mathscr{L} = \tfrac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\,, \qquad (35)$$

and the corresponding equation of motion,

$$\Box\,\phi(x) + V'(\phi) = 0\,,$$

where $\Box \equiv \partial^\mu\partial_\mu$ and $V' \equiv dV/d\phi$. The goal of this exercise is to derive the equation of motion for the Green function $\langle\Omega|T\big\{\phi(x)\phi(y)\big\}|\Omega\rangle$,

$$\Box_x\langle\Omega|T\big\{\phi(x)\phi(y)\big\}|\Omega\rangle = -\langle\Omega|T\big\{V'(\phi(x))\phi(y)\big\}|\Omega\rangle - i\delta^4(x-y)\,. \qquad (36)$$

In order to obtain eq. (36), we shall employ the following technique. Start from the path integral definition of the generating functional,

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \left[\mathscr{L} + J(x)\phi(x) \right] \right\} , \tag{37}$$

where $\mathcal{N}$ is chosen such that $Z[J=0]=1$. That is,

$$\mathcal{N}^{-1} = \int \mathscr{D}\phi \, \exp\left\{ i \int d^4x \mathscr{L} \right\} . \tag{38}$$

Consider a change of variables in the path integral, $\phi(x) \to \phi(x) + \varepsilon(x)$, where $\varepsilon(x)$ is an arbitrary infinitesimal function of x. Note that a change of variables does not change the value of of $Z[J]$.[2] The Jacobian corresponding to the change of field variables, $\phi(x) \to \phi(x) + \varepsilon(x)$ is unity.

Applying this change of variables to eq. (37) yields

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \left[\mathscr{L} + J(x)\phi(x) \right] \right\} \exp\left\{ i \int d^4x \left[\partial^\mu \phi \partial_\mu \epsilon - \epsilon(x) V'(\phi) + \epsilon(x) J(x) \right] \right\} ,$$

where we have used $V(\phi+\epsilon) = V(\phi) + \epsilon V'(\phi) + \mathcal{O}(\epsilon^2)$, and we have dropped all terms of $\mathcal{O}(\epsilon^2)$. We can further expand the second exponential above, keeping only those terms up to of $\mathcal{O}(\epsilon)$. Subtracting the resulting expression from eq. (37) yields

$$i\mathcal{N} \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \left[\mathscr{L} + J(x)\phi(x) \right] \right\} \int d^4x \, \epsilon(x) \big[-\Box_x \Phi - V'(\phi) + J(x) \big] = 0 ,$$

after an integration by parts. Since this expression is valid for any infinitesimal function $\epsilon(x)$, we may choose $\epsilon(x) = \epsilon \, \delta^4(x-y)$, where ϵ is an infinitesimal constant. We can then carry out the second integration above to obtain,

$$\mathcal{N} \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \left[\mathscr{L} + J(x)\phi(x) \right] \right\} \big[-\Box_y \Phi - V'(\phi) + J(y) \big] = 0 . \tag{39}$$

We now take the functional derivative of eq. (39) with respect to $J(x)$ and employ

$$\frac{\delta J(x)}{\delta J(y)} = \delta^4(x-y) . \tag{40}$$

Setting $J=0$ at the end of the calculation, we end up with

$$-i\mathcal{N} \int \mathcal{D}\phi \big[\phi(x) \Box_y \phi(y) + \phi(x) V'\big(\phi(y)\big) \big] \exp\{iS[\phi]\} + \mathcal{N} \int \mathscr{D}\phi \, \delta^4(x-y) \exp\{iS[\phi]\} = 0 , \tag{41}$$

where

$$S[\phi] = \int d^4x \, \mathscr{L} , \tag{42}$$

is the action functional. Note that one can pull the $\Box_y$ outside of the path integral in eq. (41) since it is independent of the field configurations that are being integrated over. Thus,

$$\Box_y \mathcal{N} \int \mathcal{D}\phi \, \phi(x)\phi(y) \exp\{iS[\phi]\} = -\mathcal{N} \int \mathcal{D}\phi \, \exp\{iS[\phi]\} \big[\phi(x) V'\big(\phi(y)\big) + i\delta^4(x-y) \big] . \tag{43}$$

[2]Just as in the case of ordinary integration, a change of integration variables does not change the value of the functional integral.

Employing eq. (38), it follows that

$$\Box_y \mathcal{N}\int \mathcal{D}\phi\,\phi(x)\phi(y)\exp\{iS[\phi]\} = -i\delta^4(x-y)\ -\mathcal{N}\int \mathcal{D}\phi\,\phi(x)V'\big(\phi(y)\big)\,\exp\{iS[\phi]\}\,. \tag{44}$$

In order to interpret eq. (44), note that the n-point Green functions are given by

$$\langle\Omega|T\big[\phi(x_1)\phi(x_2)\cdots\phi(x_n)\big]|\Omega\rangle = i^{-n}\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots J(x_n)}\bigg|_{J=0}\,.$$

Using eq. (37), it follows that

$$\langle\Omega|T\big[\phi(x_1)\phi(x_2)\cdots\phi(x_n)\big]|\Omega\rangle = \mathcal{N}\int \mathcal{D}\phi\,\phi(x_1)\phi(x_2)\cdots\phi(x_n)\exp\big\{iS[\phi]\big\}\,. \tag{45}$$

Hence, eq. (44) can be rewritten as

$$\Box_y\langle\Omega|T\big\{\phi(x)\phi(y)\big\}|\Omega\rangle = -\langle\Omega|T\big\{\phi(x)V'(\phi(y))\big\}|\Omega\rangle - i\delta^4(x-y)\,. \tag{46}$$

We now redefine the the variables x and y by interchanging $x\leftrightarrow y$ in eq. (46). Because the ordering of the fields that appear inside a time ordered product is irrelevant (since it is the time ordering prescription that dictates the order of the fields in a time-ordered product), and using the fact that $\delta^4(x-y)$ is an even function of its argument, we obtain eq. (36) as desired.

An alternative method for obtaining eq. (36)

One can also derive eq. (46) from the Schwinger-Dyson differential equation, which is given in eq. (14.122) on p. 276 of Matthew D. Schwartz, *Quantum Filed Theory and the Standard Model* (Cambridge University Press, Cambridge, UK, 2014),[3]

$$-i\Box_x\frac{\delta Z[J]}{\delta J(x)} = \left\{\mathscr{L}'_{\rm int}\left[-i\frac{\delta}{\delta J(x)}\right] + J(x)\right\}Z[J]\,. \tag{47}$$

Taking a functional derivative of eq. (47) with respect to $J(y)$,

$$-i\Box_x\frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)} = \left\{\mathscr{L}'_{\rm int}\left[-i\frac{\delta}{\delta J(x)}\right] + J(x)\right\}\frac{\delta Z[J]}{\delta J(y)} + \delta^4(x-y)Z[J]\,, \tag{48}$$

after using the product rule for differentiating and

$$\frac{\delta J(x)}{\delta J(y)} = \delta^4(x-y)\,. \tag{49}$$

We now make use of the definition of the generating functional,

$$Z[J] = \frac{\displaystyle\int \mathcal{D}\phi\exp\left\{iS[\phi] + i\int d^4x\,J(x)\phi(x)\right\}}{\displaystyle\int \mathcal{D}\phi\exp\big\{iS[\phi]\big\}}\,, \tag{50}$$

[3]More accurately, one should employ functional derivatives in eq. (47) rather than the partial derivatives used by Schwartz.

where

$$S[\phi] = \int d^4x\, \mathscr{L}[\phi] = \int d^4x\, \left\{-\tfrac{1}{2}\phi(x)\Box_x\phi(x) + \mathscr{L}_{\rm int}[\phi]\right\}, \tag{51}$$

and $\mathscr{L}_{\rm int} = -V(\phi)$. It follows that

$$\left(\frac{1}{i}\right)^2 \frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)} = \frac{\displaystyle\int \mathcal{D}\phi\, \phi(x)\phi(y) \exp\left\{iS[\phi] + i\int d^4x\, J(x)\phi(x)\right\}}{\displaystyle\int \mathcal{D}\phi \exp\bigl\{iS[\phi]\bigr\}}, \tag{52}$$

and

$$\mathscr{L}'_{\rm int}\left[-i\frac{\delta}{\delta J(y)}\right] Z[J] = \frac{\displaystyle\int \mathcal{D}\phi\, \mathscr{L}'_{\rm int}\bigl(\phi(x)\bigr) \exp\left\{iS[\phi] + i\int d^4x\, J(x)\phi(x)\right\}}{\displaystyle\int \mathcal{D}\phi \exp\bigl\{iS[\phi]\bigr\}}.$$

Taking another functional derivative with respect to $J(y)$ then yields,

$$\mathscr{L}'_{\rm int}\left[-i\frac{\delta}{\delta J(y)}\right] \frac{1}{i}\frac{\delta Z[J]}{\delta J(x)} = \frac{\displaystyle\int \mathcal{D}\phi\, \mathscr{L}'_{\rm int}\bigl(\phi(x)\bigr)\phi(y) \exp\left\{iS[\phi] + i\int d^4x\, J(x)\phi(x)\right\}}{\displaystyle\int \mathcal{D}\phi \exp\bigl\{iS[\phi]\bigr\}}. \tag{53}$$

Employing eqs. (52) and (53) in eq. (48) and then setting $J = 0$ at the end of the computation, we end up with

$$\Box_x \frac{\displaystyle\int \mathcal{D}\phi\, \phi(x)\phi(y) \exp\bigl\{iS[\phi]\bigr\}}{\displaystyle\int \mathcal{D}\phi \exp\bigl\{iS[\phi]\bigr\}} = \frac{\displaystyle\int \mathcal{D}\phi\, \mathscr{L}'_{\rm int}\bigl(\phi(x)\bigr)\phi(y) \exp\bigl\{iS[\phi]\bigr\}}{\displaystyle\int \mathcal{D}\phi \exp\bigl\{iS[\phi]\bigr\}} - i\delta^4(x-y), \tag{54}$$

where we have used $Z[0] = 1$.

The n-point Green functions are given by

$$\langle\Omega|T\bigl[\phi(x_1)\phi(x_2)\cdots\phi(x_n)\bigr]|\Omega\rangle = i^{-n}\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots J(x_n)}\bigg|_{J=0}.$$

Using eq. (50), it follows that

$$\langle\Omega|T\bigl[\phi(x_1)\phi(x_2)\cdots\phi(x_n)\bigr]|\Omega\rangle = \frac{\displaystyle\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\cdots\phi(x_n) \exp\bigl\{iS[\phi]\bigr\}}{\displaystyle\int \mathcal{D}\phi \exp\bigl\{iS[\phi]\bigr\}}. \tag{55}$$

Since $\mathscr{L}_{\rm int} = -V(\phi)$, we see that for a potential that is polynomial in ϕ (or more generally, by expanding $V(\phi)$ as a functional Taylor series in ϕ), eq. (54) is equivalent to,

$$\Box_x\langle\Omega|T\bigl\{\phi(x)\phi(y)\bigr\}|\Omega\rangle = -\langle\Omega|T\bigl\{V'(\phi(x))\phi(y)\bigr\}|\Omega\rangle - i\delta^4(x-y).$$

That is, eq. (36) is proven.

4. Consider a theory of a real scalar field governed by eq. (35) with $V(\phi) = \frac{1}{2}m^2\phi^2$.

(a) Compute exactly the free-field Feynman propagator, $\Delta_F(x)$, in coordinate space.

This is an exercise in Bessel functions, so get out your copy of *Table of Integrals, Series, and Products Eighth Edition* by I. S. Gradshteyn and I. M. Ryzhik, edited by Daniel Zwillinger and Victor Moll (Academic Press, Waltham, MA, 2015), henceforth denoted as G&R. We begin with the integral representation of the free-field Feynman propagator [cf. eq. (27)],

$$\begin{aligned}\Delta_F(x) &= \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx} \frac{1}{p^2 - m^2 + i\epsilon} \\ &= \frac{1}{(2\pi)^4} \int d^3\boldsymbol{p} \int_{-\infty}^{\infty} dp_0\, e^{-ip_0x_0 + i\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}} \frac{1}{p_0^2 - \vec{\boldsymbol{p}}^{\,2} - m^2 + i\epsilon}\,. \end{aligned} \tag{56}$$

where ϵ is a real positive infinitesimal quantity.

Consider the integral

$$\mathcal{I} = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} dp_0\, \frac{e^{-ip_0x_0}}{p_0^2 - \vec{\boldsymbol{p}}^{\,2} - m^2 + i\epsilon}\,. \tag{57}$$

First, we consider the case of $x_0 > 0$. In the limit of $\epsilon \to 0$, the integrand has poles at $p_0 = p_\pm$, where

$$p_\pm \equiv \pm\sqrt{\vec{\boldsymbol{p}}^{\,2} + m^2}\,.$$

To evaluate $\mathcal{I}$, we shall close the contour in the lower half of the complex p_0-plane since $e^{-ip_0x_0}$ is exponentially small along the semicircle at infinity when $x_0 > 0$. Only the pole p_+ lies inside the closed contour. Hence by the residue theorem,

$$\mathcal{I} = -\frac{\pi i \exp\left\{-ix_0\sqrt{\vec{\boldsymbol{p}}^{\,2} + m^2}\right\}}{\sqrt{\vec{\boldsymbol{p}}^{\,2} + m^2}}\,, \qquad \text{for } x_0 > 0, \tag{58}$$

where the minus sign is due to the fact that the closed contour is clockwise.

Second, we consider the case of $x_0 < 0$. In this case, we shall evaluate $\mathcal{I}$ by closing the contour in the upper half of the complex p_0-plane so that $e^{-ip_0x_0}$ is exponentially small along the semicircle at infinity when $x_0 < 0$. Only the pole p_- lies inside the closed contour. Hence by the residue theorem,

$$\mathcal{I} = -\frac{\pi i \exp\left\{ix_0\sqrt{\vec{\boldsymbol{p}}^{\,2} + m^2}\right\}}{\sqrt{\vec{\boldsymbol{p}}^{\,2} + m^2}}\,, \qquad \text{for } x_0 < 0. \tag{59}$$

In this case, the closed contour is counterclockwise and the minus sign arises due to the fact that $p_- - p_+$ is negative.

Without loss of generality, we may choose the z-axis to lie along the vector $\vec{\boldsymbol{x}}$, in which case $e^{i\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}} = e^{ipr\cos\theta}$, where $p \equiv |\vec{\boldsymbol{p}}|$ and $r \equiv |\vec{\boldsymbol{x}}|$. Hence, it follows that

$$\int d\cos\theta\, d\phi\, e^{ipr\cos\theta} = \frac{2\pi}{ipr}\left(e^{ipr} - e^{-ipr}\right) = \frac{4\pi\sin(pr)}{pr}\,. \tag{60}$$

Collecting all of our results above, if follows that

$$i\Delta_F(x_0;\vec{\boldsymbol{x}}) = \frac{1}{4\pi^2 r}\int_0^\infty \frac{p\,\sin(pr)\exp\bigl\{-i|x_0|\sqrt{p^2+m^2}\bigr\}}{\sqrt{p^2+m^2}}\,dp\,. \tag{61}$$

Note that the integral above is not convergent due to the oscillatory behavior of the integrand as $p\to\infty$. This is not surprising since $\Delta_F(x)$ is not an ordinary function. In fact, $\Delta_F(x)$ is a tempered distribution, which is an example of a generalized function. Thus, one must regard the integral representation given in eq. (61) in the same way as the integral representation of a delta function given in eq. (28).

To perform the integral exhibited in eq. (61), we observe the following formula 3.914 no. 9 on p. 495 of G&R which states that[4]

$$\frac{1}{r}\int_0^\infty \frac{p\exp(-z\sqrt{p^2+m^2}\,)}{\sqrt{p^2+m^2}}\sin(pr)\,dp = \frac{m}{\sqrt{r^2+z^2}}\,K_1\bigl(m\sqrt{r^2+z^2}\bigr)\,,\qquad \text{for } \operatorname{Re} m>0 \text{ and } \operatorname{Re} z>0. \tag{62}$$

Since $z=i|x_0|$ in eq. (61), one cannot immediately employ eq. (62) to evaluate the integral of interest. However, one can define a generalized function that is represented by eq. (61) by replacing $|x_0|\to|x_0|-i\epsilon$, where ϵ is a positive infinitesimal quantity [which is unrelated to the ϵ that appears in eqs. (56) and (57)]. Hence we set $z=i(|x_0|-i\epsilon)$ in eq. (62), which satisfies the condition that $\operatorname{Re} z>0$. Indeed, this ensures the necessary damping of the integrand as $p\to\infty$ in order to guarantee that eq. (62) is convergent. Thus, we shall assign the following result to the otherwise divergent integral,

$$\frac{1}{r}\int_0^\infty \frac{y\exp(-i|x_0|\sqrt{p^2+m^2}\,)}{\sqrt{p^2+m^2}}\sin(py)\,dp = \lim_{\epsilon\to 0^+}\frac{m}{\sqrt{r^2-x_0^2+i\epsilon}}\,K_1\bigl(a\sqrt{r^2-x_0^2+i\epsilon}\bigr)\,. \tag{63}$$

We can identify $r^2-x_0^2=-x^\mu x_\mu=-x^2$. Hence, it follows that

$$\frac{1}{r}\int_0^\infty \frac{y\exp(-i|x_0|\sqrt{p^2+m^2}\,)}{\sqrt{p^2+m^2}}\sin(pr)\,dp = \frac{m}{\sqrt{-x^2+i\epsilon}}\,K_1\bigl(m\sqrt{-x^2+i\epsilon}\bigr)\,, \tag{64}$$

where the $\epsilon\to 0^+$ limit is henceforth implicitly assumed. Consequently, eq. (61) yields,

$$i\Delta_F(x) = \frac{m}{4\pi^2}\frac{K_1\bigl(m\sqrt{-x^2+i\epsilon}\bigr)}{\sqrt{-x^2+i\epsilon}}\,. \tag{65}$$

Note that in the case of $x^2>0$, one must be careful in interpreting both the square root and the Bessel function K_1 of an imaginary argument. Here, I shall employ eq. (5.7.6) of N.N. Lebedev, *Special Functions and Their Applications* (Dover Publications, Mineola, New York, 1972),

$$K_\nu(z) = -\tfrac{1}{2}i\pi e^{-i\pi\nu/2}H_\nu^{(2)}(ze^{-i\pi/2})\,,\qquad \text{for } -\tfrac{1}{2}\pi<\arg z<\pi. \tag{66}$$

One can apply eq. (66) to eq. (65) with $z\equiv m\sqrt{-x^2+i\epsilon}$ in the cases of $x^2>0$ and $x^2<0$, respectively, since in either case the condition on $\arg z$ is satisfied.

[4]The constraints on the parameters m and z were omitted by G&R, but they can be found in the corresponding reference cited by G&R. See formula (36) on p. 75 of *Table of Integral Transforms*, Volume 1, edited by A. Erdélyi (McGraw-Hill, New York, 1954).

We now compute,

$$\sqrt{-x^2+i\epsilon} = \sqrt{x^2\,e^{i\pi}+i\epsilon} = \sqrt{(x^2-i\epsilon)e^{i\pi}} = e^{i\pi/2}\sqrt{x^2-i\epsilon}\,. \qquad (67)$$

In the case of $x^2 < 0$, note that $x^2 - i\epsilon$ lies just below the branch cut that runs along the negative real axis, which implies that $\arg\sqrt{x^2-i\epsilon} \simeq -\frac{1}{2}\pi$. In contrast, $-x^2+i\epsilon$ lies just above the branch cut in the case of $x^2 > 0$, which implies that $\arg\sqrt{-x^2+i\epsilon} \simeq \frac{1}{2}\pi$. Thus, in both cases, it follows that the last step of eq. (67) is valid.

Hence, independently of the sign of x^2,

$$\frac{K_1\big(m\sqrt{-x^2+i\epsilon}\big)}{\sqrt{-x^2+i\epsilon}} = \frac{-\frac{1}{2}i\pi e^{-i\pi/2}H_1^{(2)}(m\sqrt{x^2-i\epsilon})}{e^{i\pi/2}\sqrt{x^2-i\epsilon}} = \tfrac{1}{2}i\pi\frac{H_1^{(2)}(m\sqrt{x^2-i\epsilon})}{\sqrt{x^2-i\epsilon}}\,. \qquad (68)$$

It then follows that an equivalent form of eq. (65) is given by,

$$i\Delta_F(x) = \frac{im}{8\pi}\frac{H_1^{(2)}(m\sqrt{x^2-i\epsilon})}{\sqrt{x^2-i\epsilon}}\,. \qquad (69)$$

Admittedly, eq. (65) is more convenient in the case of $x^2 < 0$, whereas eq. (69) is more convenient in the case of $x^2 > 0$. Hence, one can replace eqs. (65) and (69) with the more convenient expression,[5]

$$\boxed{i\Delta_F(x) = \frac{m}{4\pi^2}\left[\frac{K_1\big(m\sqrt{-x^2+i\epsilon}\big)}{\sqrt{-x^2+i\epsilon}}\,\Theta(-x^2) + \frac{i\pi H_1^{(2)}\big(m\sqrt{x^2-i\epsilon}\big)}{2\sqrt{x^2-i\epsilon}}\,\Theta(x^2)\right]\,,} \qquad (70)$$

where we have employed the step function, $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$, subject to the condition that $\Theta(x)+\Theta(-x) = 1$.[6] Indeed, the form of $\Delta_F(x)$ is consistent with our previous assertion that $\Delta_F(x)$ is a generalized function.

An alternative derivation of eq. (70) is provided in Appendix A.

(b) Evaluate the leading singularities of $\Delta_F(x)$ near the light cone, $x^2 = 0$.

In order to examine the leading singularities near the light cone, we shall employ the expansions for $H_1^{(2)}(z) \equiv J_1(z) - iY_1(z)$ and for $K_1(z)$ given on pp. 927–928 of G&R,

$$\tfrac{1}{2}i\pi H_1^{(2)}(z) = -\frac{1}{z} + \frac{z}{2}\left[\ln\left(\frac{z}{2}\right) + \gamma - \frac{1}{2} + \frac{i\pi}{2}\right] + \mathcal{O}(z^3)\,, \qquad (71)$$

$$K_1(z) = \frac{1}{z} + \frac{z}{2}\left[\ln\left(\frac{z}{2}\right) + \gamma - \frac{1}{2}\right] + \mathcal{O}(z^3)\,. \qquad (72)$$

Since $H_1^{(2)}(z)$ and $K_1(z)$ possess branch cuts along the negative real axis, eqs. (71) and (72) are valid for $|\arg z| < \pi$.

[5]A different technique for evaluating the integrals of this section is presented in H.-H. Zhang, K.-X. Feng, S.-W. Qiu, A. Zhao and X.-S. Li, Chinese Physics C **34**, 1576 (2010). In this work, the authors also demonstrate that both eqs. (65) and (69) are separately valid, independently of the sign of x^2.

[6]One need not specify the values of $\Theta(0^+)$ and $\Theta(0^-)$. Indeed, when $\Theta(x)$ is regarded as a generalized function, the specification of the value of $\Theta(x)$ at the origin has no significance [e.g., see p. 63 of D.S. Jones, *The theory of generalised functions* (Cambridge University Press, Cambridge, UK, 1982)].

To obtain the leading singularities of $i\Delta_F(x)$, one can either insert the expansion given in eq. (72) into eq. (65) or the expansion given in eq. (71) into eq. (69) to obtain,

$$i\Delta_F(x) \sim \begin{cases} \dfrac{1}{4\pi^2(-x^2+i\epsilon)} + \dfrac{m^2}{8\pi^2}\left[\ln\left(\tfrac{1}{2}m\sqrt{-x^2+i\epsilon}\right)+\gamma-\tfrac{1}{2}\right], & \text{as } x^2\to 0^-, \\ -\dfrac{1}{4\pi^2(x^2-i\epsilon)} + \dfrac{m^2}{8\pi^2}\left[\ln\left(\tfrac{1}{2}m\sqrt{x^2-i\epsilon}\right)+\gamma-\tfrac{1}{2}+\tfrac{1}{2}i\pi\right], & \text{as } x^2\to 0^+. \end{cases} \tag{73}$$

In light of eq. (67), we see that the two limiting cases above are analytic continuations of each other. Indeed, it is a simple matter to check that if $x^2>0$ then

$$\begin{aligned} \tfrac{1}{2}i\pi + \lim_{\epsilon\to 0}\ln\sqrt{x^2-i\epsilon} &= \tfrac{1}{2}\left[i\pi + \lim_{\epsilon\to 0}\ln(x^2-i\epsilon)\right] = \tfrac{1}{2}\left[i\pi+\ln|x^2| - i\pi\Theta(-x^2)\right] \\ &= \ln\sqrt{|x^2|}+\tfrac{1}{2}i\pi\left[1-\Theta(-x^2)\right] = \ln\sqrt{|x^2|}+\tfrac{1}{2}i\pi\,\Theta(x^2)\,, \end{aligned} \tag{74}$$

where we have employed the identity, $\Theta(x^2)+\Theta(-x^2)=1$. The same end result is obtained if $x^2<0$, since

$$\lim_{\epsilon\to 0}\ln\sqrt{-x^2+i\epsilon} = \tfrac{1}{2}\lim_{\epsilon\to 0}\ln(-x^2+i\epsilon) = \tfrac{1}{2}\left[\ln|x^2|+i\pi\Theta(x^2)\right] = \ln\sqrt{|x^2|}+\tfrac{1}{2}i\pi\,\Theta(x^2)\,. \tag{75}$$

Thus, we may combine both limits in eq. (73) into a single equation,

$$i\Delta_F(x) \sim -\frac{1}{4\pi^2(x^2-i\epsilon)} + \frac{m^2}{8\pi^2}\left[\ln\left(\tfrac{1}{2}m\sqrt{|x^2|}\right)+\gamma-\tfrac{1}{2}+\tfrac{1}{2}i\pi\,\Theta(x^2)\right], \qquad \text{as } x^2\to 0. \tag{76}$$

Finally, we can make use of the Sokhotski-Plemelj formula,[7]

$$\frac{1}{z\pm i\epsilon} = \mathrm{P}\,\frac{1}{z}\mp i\pi\delta(z)\,, \tag{77}$$

where P is the Cauchy principal value prescription, which is employed when evaluating the integral of the product of a generalized function and a smooth test function according to the following rule,

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)}{x}\,dx \equiv \lim_{\delta\to 0^+}\left\{\int_{-\infty}^{-\delta}\frac{f(x)}{x}\,dx + \int_{\delta}^{\infty}\frac{f(x)}{x}\,dx\right\}. \tag{78}$$

Hence, eqs. (76) and (77) yield,

$$\boxed{i\Delta_F(x) = -\frac{i}{4\pi}\,\delta(x^2) - \frac{1}{4\pi^2}\,\mathrm{P}\,\frac{1}{x^2} + \frac{m^2}{8\pi^2}\left[\ln\left(\frac{m\sqrt{|x^2|}}{2}\right)+\gamma-\frac{1}{2}+\frac{i\pi}{2}\Theta(x^2)+\mathcal{O}(m^2x^2)\right]}\,, \tag{79}$$

where the terms of $\mathcal{O}(m^2x^2)$ vanish on the light cone.

The limit of eq. (79) as $m\to 0$ is noteworthy,

$$\lim_{m\to 0} i\Delta_F(x) = -\frac{i}{4\pi}\,\delta(x^2) - \frac{1}{4\pi^2}\,\mathrm{P}\,\frac{1}{x^2}\,. \tag{80}$$

[7] See the class handout entitled *The Sokhotski-Plemelj formula*.

In obtaining eq. (80), we have used dimensional analysis to conclude that all terms in eq. (79) that vanish on the light cone must be proportional to a positive power of m^2.

To check the result of eq. (80), it is instructive to perform an exact calculation of $i\Delta_F(x)$ in the case of $m = 0$ by returning to eq. (61)

$$i\Delta_F(x_0; \vec{\boldsymbol{x}})_{m=0} = \frac{1}{4\pi^2 r}\int_0^\infty \sin(pr)e^{-ip|x_0|}\,dp\,. \tag{81}$$

To evaluate this integral, I will make use of the integral representation of the step function,[8]

$$\Theta(k) = \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{e^{ikx}}{x - i\epsilon}\,dx\,. \tag{82}$$

Multiplying eq. (82) by i and then employing the inverse Fourier transform yields an expression for the generalized function $(x - i\epsilon)^{-1}$,

$$\frac{1}{x - i\epsilon} = i\int_{-\infty}^\infty \Theta(k)e^{-ikx}\,dk = i\int_0^\infty e^{-ikx}\,dk\,. \tag{83}$$

Employing eq. (83) in evaluating eq. (81),

$$\begin{aligned}
i\Delta_F(x_0; \vec{\boldsymbol{x}})_{m=0} &= -\frac{i}{8\pi^2 r}\int_0^\infty \left[e^{-ip(|x_0|-r)} - e^{-ip(|x_0|+r)}\right]dp \\
&= -\frac{1}{8\pi^2 r}\left[\frac{1}{|x_0| - r - i\epsilon} - \frac{1}{|x_0| + r - i\epsilon}\right] = -\frac{1}{4\pi^2\bigl(x_0^2 - r^2 - i\epsilon\bigr)} \\
&= -\frac{1}{4\pi^2(x^2 - i\epsilon)}\,,
\end{aligned} \tag{84}$$

where we have identified $x^2 = x_0^2 - r^2$. Thus we have recovered the $m \to 0$ limit of eq. (79). Equivalently, one can again employ the Sokhotski-Plemelj formula [cf. eq. (77)] to obtain

$$i\Delta_F(x)_{m=0} = -\frac{1}{4\pi^2}\left[\mathrm{P}\,\frac{1}{x^2} + i\pi\delta(x^2)\right] = -\frac{i}{4\pi}\delta(x^2) - \frac{1}{4\pi^2}\,\mathrm{P}\,\frac{1}{x^2}\,, \tag{85}$$

thereby confirming the result of eq. (80).

(c) Using the Källen–Lehmann representation, comment on the leading singularity of the exact two-point function near the light cone.

The Källen–Lehmann representation for the exact unrenormalized two-point Green function of scalar field theory is given by,

$$G_c^{(2)}(x) = \int_0^\infty dm^2\,\rho(m^2)\,i\Delta_F(x\,;\,m^2)\,, \tag{86}$$

where

$$\rho(m^2) \equiv Z_\phi\,\delta(m^2 - m_R^2) + \sigma(m^2)\,, \tag{87}$$

[8]See the class handout entitled *Integral representation of the Heavyside step function.*

is the spectral function, m_R is the renormalized mass and Z_ϕ is the wave function renormalization constant. The exact renormalized two-point Green function of scalar field theory is then given by,

$$G^{(2)}_{Rc}(x) = Z_\phi^{-1} G^{(2)}_c(x) = i\Delta_F(x\,;\,m_R^2) + Z_\phi^{-1}\int_{4m_R^2}^\infty dm^2\,\sigma(m^2)\,i\Delta_F(x\,;\,m^2),. \tag{88}$$

Near the light cone, we shall employ the two most singular terms of eq. (79) to obtain,

$$G^{(2)}_{Rc}(x) \simeq -\frac{1}{4\pi^2}\left[i\pi\delta(x^2) + P\,\frac{1}{x^2}\right]\left(1 + Z_\phi^{-1}\int_0^\infty dm^2\,\sigma(m^2)\right)\,, \qquad \text{as } x^2 \to 0. \tag{89}$$

In class, we proved that

$$1 = \int_0^\infty \rho(m^2)\,dm^2 = Z_\phi + \int_0^\infty \sigma(m^2)\,dm^2\,, \tag{90}$$

after making use of eq. (87). Inserting this result back into eq. (88) yields,

$$G^{(2)}_{Rc}(x) \simeq -\frac{Z_\phi^{-1}}{4\pi^2}\left[i\pi\delta(x^2) + P\,\frac{1}{x^2}\right]\,, \qquad \text{as } x^2 \to 0. \tag{91}$$

Since $0 \leq Z_\phi \leq 1$, we can conclude that the leading singularity on the light cone of the renormalized two-point Green function is at least as singular as the corresponding free field two-point Green function. In the special case of $Z_\phi = 0$, the leading singularity would be stronger than that of free field theory.

APPENDIX A: Alternative methods for evaluating the integral in eq. (61)

Our method for identifying the explicit form for the generalized function $\Delta_F(x)$ was to insert a convergence factor in the integrand of eq. (61), $\exp\bigl[-\epsilon\sqrt{p^2+m^2}\bigr]$, and then take $\epsilon \to 0^+$ at the end of the calculation. Indeed, this method is often employed to interpret the integral representation of the delta function,

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx}\,dk = \lim_{\epsilon\to 0^+}\frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx-\epsilon k^2}\,dk\,, \tag{92}$$

after inserting the convergence factor $e^{-\epsilon k^2}$.

An alternative method is to rewrite the integral given in eq. (61) as the derivative of a conditionally convergent integral, which then can be computed explicitly. We can also employ this alternative method to identify the integral representation of the delta function as follows. Consider the generalized function,

$$\mathcal{J}(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^0 e^{ikx}\,dk + \frac{1}{2\pi}\int_0^\infty e^{ikx}\,dk = \frac{1}{\pi}\int_0^\infty \cos(kx)\,dk\,, \tag{93}$$

after performing a variable change, $k \to -k$, in the second integral above. Note that the integral of $\cos(kx)$ is not convergent due to the oscillatory behavior of the integrand as $k \to \infty$ [as in the case of eq. (61)]. Nevertheless, one can employ the well known conditionally convergent integral,

$$\int_0^\infty \frac{\sin(kx)}{k}\,dk = \tfrac{1}{2}\pi\,\text{sgn}(x)\,, \tag{94}$$

where sgn(x) is the sign of the real number x.[9] Noting that

$$\mathrm{sgn}(x) = 2\Theta(x) - 1\,, \tag{95}$$

it follows that

$$\int_0^\infty \frac{\sin(kx)}{k}\,dk = \pi\bigl[\Theta(x) - \tfrac{1}{2}\bigr]\,. \tag{96}$$

Using eq. (93), we can identify $\mathcal{J}(x)$ as the derivative of a conditionally convergent integral,

$$\mathcal{J}(x) = \frac{\partial}{\partial x}\frac{1}{\pi}\int_0^\infty \frac{\sin(kx)}{k}\,dk = \frac{d}{dx}\Theta(x) = \delta(x)\,, \tag{97}$$

in agreement with the well-known integral representation of the delta function.

Let us employ this alternative strategy in evaluating eq. (61). To perform the integral exhibited in eq. (61), we observe the following two formulae, 3.876 nos. 1 and 2 on p. 486 of G&R,

$$\int_0^\infty \frac{\sin(|x_0|\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}}\cos(pr)\,dp = \tfrac{1}{2}\pi J_0\bigl(m\sqrt{x_0^2-r^2}\bigr)\Theta(x_0^2-r^2), \tag{98}$$

$$\begin{aligned}\int_0^\infty \frac{\cos(|x_0|\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}}\cos(pr)\,dp = &-\tfrac{1}{2}\pi Y_0\bigl(m\sqrt{x_0^2-r^2}\bigr)\Theta(x_0^2-r^2) \\ &+K_0\bigl(m\sqrt{r^2-x_0^2}\bigr)\Theta(r^2-x_0^2),\end{aligned} \tag{99}$$

which satisfy the conditions specified by G&R since m, r and $|x_0|$ are all positive. Combining the two integrals above, it follows that

$$\begin{aligned}\int_0^\infty \frac{\exp(-i|x_0|\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}}\cos(pr)\,dy = &-\tfrac{1}{2}i\pi\bigl[J_0\bigl(m\sqrt{x_0^2-r^2}\bigr) - iY_0\bigl(m\sqrt{x_0^2-r^2}\bigr)\bigr]\Theta(x_0^2-r^2) \\ &+K_0\bigl(m\sqrt{r^2-x_0^2}\bigr)\Theta(r^2-x_0^2)\,.\end{aligned} \tag{100}$$

It then follows that

$$\begin{aligned}\frac{1}{r}\int_0^\infty \frac{p\exp(-i|x_0|\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}}\sin(pr)\,dp &= -\frac{1}{r}\frac{\partial}{\partial r}\int_0^\infty \frac{\exp(-i|x_0|\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}}\cos(pr)\,dp \\ &= 2\frac{d}{dx^2}\Bigl[K_0\bigl(m\sqrt{-x^2}\bigr)\,\Theta(-x^2) - \tfrac{1}{2}\pi Y_0\bigl(m\sqrt{x^2}\bigr)\Theta(x^2)\Bigr] - i\pi\frac{d}{dx^2}\Bigl[J_0\bigl(m\sqrt{x^2}\bigr)\Theta(x^2)\Bigr] \\ &= 2\frac{d}{dx^2}\Bigl[K_0\bigl(m\sqrt{-x^2}\bigr)\,\Theta(-x^2) - \tfrac{1}{2}i\pi H_0^{(2)}\bigl(m\sqrt{x^2}\bigr)\Theta(x^2)\Bigr]\,,\end{aligned} \tag{101}$$

after introducing the Hankel function of the second kind, $H_0^{(2)}(z) \equiv J_z(z) - iY_1(z)$ and identifying the square of the position four-vector, $x_0^2 - r^2 = x^\mu x_\mu = x^2$.

[9]Some books define sgn$(0) = 0$, in which case, eq. (94) would be valid at $x = 0$. However, when sgn(x) is regarded as a generalized function, the specification of the value at the origin has no significance (cf. footnote 6).

Thus, we focus on the quantity,

$$F(x^2) \equiv K_0\big(m\sqrt{-x^2}\big)\,\Theta(-x^2) - \tfrac{1}{2}i\pi H_0^{(2)}\big(m\sqrt{x^2}\big)\Theta(x^2)\,. \tag{102}$$

In order to compute dF/dx^2, we must pay attention to the behavior of $F(x^2)$ in the vicinity of $x^2 = 0$. To facilitate this analysis, we shall employ the small argument expansions of the Bessel functions, which can be deduced from results given on pp. 927–928 of G&R,

$$K_0(z) = -\ln\left(\frac{z}{2}\right) - \gamma + \mathcal{O}(z^2)\,, \tag{103}$$

$$\tfrac{1}{2}i\pi H_0^{(2)}(z) = \tfrac{1}{2}i\pi + \ln\left(\frac{z}{2}\right) + \gamma + \mathcal{O}(z^2)\,, \tag{104}$$

where γ is Euler's constant. It is therefore convenient to define,

$$\widetilde{K}_0(z) = K_0(z) + \ln\left(\frac{z}{2}\right)\,, \tag{105}$$

$$\widetilde{H}_0^{(2)}(z) = H_0^{(2)}(z) + \frac{2i}{\pi}\ln\left(\frac{z}{2}\right)\,, \tag{106}$$

each of which has a finite limit as $z \to 0$. Hence, we can rewrite eq. (102) as,

$$\begin{aligned} F(x^2) = {}& \widetilde{K}_0\big(m\sqrt{-x^2}\big)\,\Theta(-x^2) - \tfrac{1}{2}i\pi\widetilde{H}_0^{(2)}\big(m\sqrt{x^2}\big)\Theta(x^2) \\ & - \ln\big(\tfrac{1}{2}m\sqrt{-x^2}\big)\Theta(-x^2) - \ln\big(\tfrac{1}{2}m\sqrt{x^2}\big)\Theta(x^2)\,. \end{aligned} \tag{107}$$

We can simplify the second line of above expression by employing the following relation,

$$f(x^2)\Theta(x^2) + f(-x^2)\Theta(-x^2) = f(|x^2|)\,. \tag{108}$$

Hence,

$$F(x^2) = \widetilde{K}_0\big(m\sqrt{-x^2}\big)\,\Theta(-x^2) - \tfrac{1}{2}i\pi\widetilde{H}_0^{(2)}\big(m\sqrt{x^2}\big)\Theta(x^2) - \ln\big(\tfrac{1}{2}m|x^2|^{1/2}\big)\,. \tag{109}$$

We can now differentiate $F(x^2)$ with respect to x^2. Noting that,

$$\frac{d}{dz}\widetilde{K}_0(z) = -\widetilde{K}_1(z) \equiv -K_1(z) + \frac{1}{z}\,, \tag{110}$$

$$\frac{d}{dz}\widetilde{H}_0^{(2)}(z) = -\widetilde{H}_1^{(2)}(z) \equiv -H_1^{(2)}(z) + \frac{2i}{\pi z}\,, \tag{111}$$

where we have defined $\widetilde{K}_1(z)$ and $\widetilde{H}_1^{(2)}(z)$ such that the leading singular pieces of $K_1(z)$ and $H_1^{(2)}(z)$ as $z \to 0$ are removed, it follows that,

$$\begin{aligned} \frac{d}{dx^2}F(x^2) = {}& \frac{m}{2}\left[\frac{\widetilde{K}_1(m\sqrt{-x^2})}{\sqrt{-x^2}}\,\Theta(-x^2) + \frac{i\pi\widetilde{H}_1^{(2)}(m\sqrt{x^2})}{2\sqrt{x^2}}\,\Theta(x^2)\right] - \frac{1}{2}\,\frac{d}{dx^2}\ln|x^2| \\ & -\big[\widetilde{K}_0\big(m\sqrt{-x^2}\big) + \tfrac{1}{2}i\pi\widetilde{H}_0^{(2)}\big(m\sqrt{x^2}\big)\big]\delta(x^2)\,, \end{aligned} \tag{112}$$

after employing $\delta(x^2) = d\Theta(x^2)/dx^2$ and noting that $\delta(x^2) = \delta(-x^2)$. The second line of eq. (112) is evaluated by employing $f(x^2)\delta(x^2) = f(0)\delta(x^2)$, where $f(x^2)$ is a smooth function. Hence,

$$-\big[\widetilde{K}_0\big(m\sqrt{-x^2}\big) + \tfrac{1}{2}i\pi\widetilde{H}_0^{(2)}\big(m\sqrt{x^2}\big)\big]\delta(x^2) = -\big[\widetilde{K}_0(0) + \tfrac{1}{2}i\pi\widetilde{H}_0^{(2)}(0)\big]\delta(x^2) = -\tfrac{1}{2}i\pi\delta(x^2)\,, \tag{113}$$

in light of eqs. (103) and (104).

Finally, we make use of the following result that is derived in Appendix B,

$$\frac{d}{dx^2}\ln|x^2| = \mathrm{P}\,\frac{1}{x^2}\,, \tag{114}$$

where the symbol P stands for the principal value prescription. Hence, after using eqs. (112)–(114) the end result is,

$$\frac{d}{dx^2}F(x^2) = \frac{m}{2}\left[\frac{\widetilde{K}_1(m\sqrt{-x^2})}{\sqrt{-x^2}}\,\Theta(-x^2) + \frac{i\pi\widetilde{H}_1^{(2)}(m\sqrt{x^2})}{2\sqrt{x^2}}\,\Theta(x^2)\right] - \frac{1}{2}\left[\mathrm{P}\,\frac{1}{x^2} + i\pi\delta(x^2)\right]. \tag{115}$$

Consequently, eq. (101) yields,

$$\frac{1}{r}\int_0^\infty \frac{y\exp(-i|x_0|\sqrt{p^2+m^2}\,)}{\sqrt{p^2+m^2}}\sin(py)\,dp = -\mathrm{P}\,\frac{1}{x^2} - i\pi\,\delta(x^2) \tag{116}$$

$$+m\left[\frac{\widetilde{K}_1(m\sqrt{-x^2})}{\sqrt{-x^2}}\,\Theta(-x^2) + \frac{i\pi\widetilde{H}_1^{(2)}(m\sqrt{x^2})}{2\sqrt{x^2}}\,\Theta(x^2)\right]. \tag{117}$$

Hence, it follows from eqs. (61) and (117) that

$$\boxed{i\Delta_F(x) = -\frac{i}{4\pi}\,\delta(x^2) + \frac{m}{4\pi^2}\,\mathrm{P}\left\{\frac{K_1\big(m\sqrt{-x^2}\big)}{\sqrt{-x^2}}\,\Theta(-x^2) + \frac{i\pi H_1^{(2)}\big(m\sqrt{x^2}\big)}{2\sqrt{x^2}}\,\Theta(x^2)\right\},} \tag{118}$$

after re-expressing $\widetilde{K}_1$ and $\widetilde{H}_1^{(2)}$ in terms of K_1 and $H_1^{(2)}$, respectively, and making use of the identity, $\Theta(x^2)+\Theta(-x^2)=1$. The principal value prescription affects only those terms in eq. (118) inside the braces that behave as $1/x^2$ as $x^2\to 0$.

Inserting the expansions given by eqs. (71) and (72) into eq. (118) and making use of eq. (108), it follows that

$$i\Delta_F(x) = -\frac{i}{4\pi}\,\delta(x^2) - \frac{1}{4\pi^2}\,\mathrm{P}\,\frac{1}{x^2} + \frac{m^2}{8\pi^2}\left[\ln\left(\frac{m\sqrt{|x^2|}}{2}\right) + \gamma - \frac{1}{2} + \frac{i\pi}{2}\Theta(x^2) + \mathcal{O}(m^2x^2)\right], \tag{119}$$

in agreement with eq. (79). Note that the principal value prescription is not needed for the logarithmic term in eq. (79), since the integral of $\ln(\frac{1}{2}m\sqrt{|x^2|})$ multiplied by a well behaved test function, performed over an integration range that includes the point $x^2=0$, is convergent.

In particular, after employing eq. (77), the leading singular behavior of $i\Delta_F(x)$ is

$$i\Delta_F(x) = -\frac{i}{4\pi}\,\delta(x^2) - \frac{1}{4\pi^2}\,\mathrm{P}\,\frac{1}{x^2} + \ldots = \lim_{\epsilon\to 0^+}\frac{-1}{4\pi^2(x^2-i\epsilon)} + \ldots\,, \tag{120}$$

where $\ldots$ represents subleading terms as $x^2\to 0$, and the $\epsilon\to 0$ limit is taken only after integrating the product of $\Delta_F(x)$ and a well-behaved test function. Consequently, eq. (118) is equivalent to

$$i\Delta_F(x) = \frac{m}{4\pi^2}\left[\frac{K_1\big(m\sqrt{-x^2+i\epsilon}\big)}{\sqrt{-x^2+i\epsilon}}\,\Theta(-x^2) + \frac{i\pi H_1^{(2)}\big(m\sqrt{x^2-i\epsilon}\big)}{2\sqrt{x^2-i\epsilon}}\,\Theta(x^2)\right], \tag{121}$$

in agreement with eq. (70).

Yet another alternative method for evaluating the integral given by eq. (61)

Denoting $E_p \equiv \sqrt{p^2+m^2}$, we can rewrite eq. (61) as,

$$\begin{aligned}\Delta_F(x_0;\vec{\boldsymbol{x}}) &= \frac{-i}{4\pi^2 r}\int_0^\infty \frac{p\sin(pr)}{E_p}e^{-i|x_0|E_p}\,dp = \frac{-1}{8\pi^2 r}\int_0^\infty \frac{p(e^{ipr}-e^{-ipr})}{E_p}e^{-i|x_0|E_p}\,dp \\ &= \frac{-1}{8\pi^2 r}\int_{-\infty}^\infty \frac{p}{E_p}e^{ipr}e^{-i|x_0|E_p}\,dp = \frac{i}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^\infty \frac{dp}{E_p}e^{ipr}e^{-i|x_0|E_p}\,. \qquad (122)\end{aligned}$$

Introducing the rapidity ζ,

$$E_p = m\cosh\zeta\,, \qquad\qquad p = m\sinh\zeta\,, \qquad (123)$$

it follows that $dp = E_p d\zeta$. Hence,

$$i\Delta_F(x_0;\vec{\boldsymbol{x}}) = -\frac{1}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^\infty d\zeta\,\exp\bigl[-im\bigl(|x_0|\cosh\zeta - r\sinh\zeta\bigr)\bigr]\,. \qquad (124)$$

We consider two cases.

Case 1: $x^2 \equiv x_0^2 - r^2 > 0$ (or equivalently, $|x_0| > r$)

In this case, it is convenient to define a new variable η such that the condition $x^2 \equiv x_0^2 - r^2$ is satisfied,

$$|x_0| = \sqrt{x^2}\,\cosh\eta\,, \qquad r = \sqrt{x^2}\,\sinh\eta\,. \qquad (125)$$

It then follows that $|x_0|\cosh\zeta - r\sinh\zeta = \sqrt{x^2}\,\cosh(\zeta-\eta)$.

Hence,

$$\begin{aligned}i\Delta_F(x_0;\vec{\boldsymbol{x}}) &= -\frac{1}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^\infty d\zeta\,\exp\bigl[-im\sqrt{x^2}\,\cosh(\zeta-\eta)\bigr] \\ &= -\frac{1}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^\infty d\zeta\,\exp\bigl[-im\sqrt{x^2}\,\cosh\zeta\bigr] \\ &= \frac{1}{2\pi^2}\frac{d}{dx^2}\int_0^\infty d\zeta\,\bigl[\cos(m\sqrt{x^2}\cosh\zeta) - i\sin(m\sqrt{x^2}\cosh\zeta)\bigr]\,, \qquad (126)\end{aligned}$$

after making use of the symmetry of the integrand under $\zeta \to -\zeta$.

We now make use of G&R, formulae 3.714 nos. 2 and 3:

$$\int_0^\infty \sin(z\cosh x)dx = \tfrac{1}{2}\pi J_0(z)\,, \qquad \int_0^\infty \cos(z\cosh x)dx = -\tfrac{1}{2}\pi Y_0(z)\,, \qquad \text{for } \operatorname{Re} z > 0\,. \qquad (127)$$

It then follows that

$$i\Delta_F(x) = -\frac{i}{4\pi}\frac{d}{dx^2}H_0^{(2)}(m\sqrt{x^2})\,, \qquad \text{for } x^2 > 0\,, \qquad (128)$$

where $H_0^{(2)}(z) \equiv J_0(z) - iY_0(z)$.

Case 2: $x^2 \equiv x_0^2 - r^2 < 0$ (or equivalently, $|x_0| < r$)

In this case, it is convenient to define a new variable η such that the condition $x^2 \equiv x_0^2 - r^2$ is satisfied,

$$|x_0| = \sqrt{-x^2}\,\sinh\eta\,, \qquad r = \sqrt{-x^2}\,\cosh\eta\,. \tag{129}$$

It then follows that $|x_0|\cosh\zeta - r\sinh\zeta = -\sqrt{-x^2}\,\sinh(\zeta-\eta)$. Hence,

$$\begin{aligned} i\Delta_F(x_0;\vec{\boldsymbol{x}}) &= -\frac{1}{8\pi^2 r}\,\frac{\partial}{\partial r}\int_{-\infty}^{\infty} d\zeta\,\exp\bigl[im\sqrt{-x^2}\,\sinh(\zeta-\eta)\bigr] \\ &= -\frac{1}{8\pi^2 r}\,\frac{\partial}{\partial r}\int_{-\infty}^{\infty} d\zeta\,\exp\bigl[im\sqrt{-x^2}\,\sinh\zeta\bigr] \\ &= \frac{1}{2\pi^2}\,\frac{d}{dx^2}\int_0^{\infty} d\zeta\,\cos(m\sqrt{-x^2}\sinh\zeta)\,. \end{aligned} \tag{130}$$

We now make use of G&R, formulae 3.714 no. 1:

$$\int_0^{\infty}\cos(z\sinh x)dx = K_0(z)\,, \qquad \text{for } \operatorname{Re} z > 0\,. \tag{131}$$

It then follows that

$$i\Delta_F(x) = \frac{1}{2\pi^2}\,\frac{d}{dx^2}K_0(m\sqrt{-x^2})\,, \qquad \text{for } x^2 < 0. \tag{132}$$

Combining the results of eqs. (128) and (132),

$$i\Delta_F(x) = \frac{1}{2\pi^2}\,\frac{d}{dx^2}\left[K_0(m\sqrt{-x^2})\,\Theta(-x^2) - \tfrac{1}{2}i\pi H_0^{(2)}(m\sqrt{x^2})\,\Theta(x^2)\right]\,, \tag{133}$$

which reproduces eq. (101). The computation of the derivative with respect to x^2 then follows the same steps previously employed in deriving the final result given in eq. (118).

APPENDIX B: Proof of $d\ln|x|/dx = \mathrm{P}(1/x)$

Consider $\ln|x|$ as a generalized function. Noting that

$$\frac{d}{dx}\ln|x| = \frac{1}{x}\,, \qquad \text{for } x \neq 0\,, \tag{134}$$

one can extend this result to $x = 0$ by treating $d\ln|x|/dx$ as a generalized function. For any well-behaved test function $f(x)$ that vanishes sufficiently fast as $x \to \pm\infty$, it follows from an integration by parts that

$$\int_{-\infty}^{\infty} f(x)\frac{d}{dx}\ln|x|\,dx = -\int_{-\infty}^{\infty}\ln|x|f'(x)\,dx = -\lim_{\epsilon\to 0^+}\int_{|x|\geq\epsilon}\ln|x|f'(x)\,dx\,. \tag{135}$$

where $f'(x) \equiv df/dx$, and the boundary terms vanish due to the behavior of $f(x)$ at $\pm\infty$. Note that the limiting process above is smooth, since the integral above exists for all values of $\epsilon \geq 0$. To complete the calculation, we integrate by parts once more to obtain,

$$\int_{-\infty}^{\infty} f(x)\frac{d}{dx}\ln|x|\,dx = \lim_{\epsilon\to 0^+}\left[\int_{|x|\geq\epsilon}\frac{f(x)}{x}\,dx - \bigl[f(\epsilon) - f(-\epsilon)\bigr]\ln\epsilon\right]\,. \tag{136}$$

However, $\big[f(\epsilon) - f(-\epsilon)\big] \ln\epsilon = \mathcal{O}(\epsilon \ln\epsilon)$ which vanishes as $\epsilon \to 0$. Thus, we end up with

$$\int_{-\infty}^{\infty} f(x) \frac{d}{dx} \ln|x|\, dx = \lim_{\epsilon\to 0^+} \int_{|x|\geq\epsilon} \frac{f(x)}{x}\, dx\,. \tag{137}$$

We recognize the right hand side of eq. (137) as the principal value prescription,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)}{x}\, dx \equiv \lim_{\epsilon\to 0^+} \left\{ \int_{-\infty}^{-\epsilon} \frac{f(x)}{x}\, dx + \int_{\epsilon}^{\infty} \frac{f(x)}{x}\, dx \right\}, \tag{138}$$

Hence, we can identify the generalized function,[10]

$$\frac{d}{dx} \ln|x| = \mathrm{P}\,\frac{1}{x}\,, \tag{139}$$

which is meaningful at $x = 0$ via eq. (137).

[10]Another derivation of eq. (139) can be found in the class handout entitled *Examples of Generalized Functions*.