1. Define the following functions:[1]

$$A_0(m^2) \equiv -16\pi^2 i \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2 - m^2 + i\varepsilon}\,, \tag{1}$$

$$B_0(p^2; m_1^2, m_2^2) \equiv -16\pi^2 i \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - m_1^2 + i\varepsilon)[(q+p)^2 - m_2^2 + i\varepsilon]}\,, \tag{2}$$

$$B^\mu(p; m_1^2, m_2^2) \equiv -16\pi^2 i \int \frac{d^n q}{(2\pi)^n} \frac{q^\mu}{(q^2 - m_1^2 + i\varepsilon)[(q+p)^2 - m_2^2 + i\varepsilon]}\,, \tag{3}$$

where ε is a positive infinitesimal quantity and m, m_1 and m_2 are real nonnegative parameters.

(a) Compute A_0 and B_0 explicitly using dimensional regularization. Expand your results about $n = 4$ and drop all terms that vanish as $n \to 4$. Using the notation

$$\Delta \equiv \frac{1}{\epsilon} - \gamma + \ln 4\pi\,, \tag{4}$$

where $n = 4 - 2\epsilon$ and γ is Euler's constant, express your result in each case as a the sum of two terms: one term involving Δ and a second term that is finite as $n \to 4$.

Using the result of the handout entitled, *Useful formulae for computing one-loop integrals*,

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 + 2q\cdot p - m^2 + i\varepsilon)^r} = i(-1)^r (p^2 + m^2)^{2-\epsilon-r} (4\pi)^{\epsilon-2} \frac{\Gamma(\epsilon + r - 2)}{\Gamma(r)}\,, \tag{5}$$

where $\epsilon \equiv 2 - \frac{1}{2}n$, it follows that[2]

$$A_0(m^2) = -\frac{(m^2)^{1-\epsilon}(4\pi\mu^2)^\epsilon \Gamma(\epsilon)}{\epsilon - 1}\,, \tag{6}$$

after using the relation $(\epsilon - 1)\Gamma(\epsilon - 1) = \Gamma(\epsilon)$. Expanding the about $\epsilon = 0$,

$$(4\pi)^\epsilon \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \ln(4\pi) + \mathcal{O}(\epsilon)\,, \tag{7}$$

in eq. (6), it follows that

$$A_0(m^2) = m^2 \left[\frac{1}{\epsilon} - \gamma + \ln(4\pi) + 1 - \ln\left(\frac{m^2}{\mu^2}\right) + \mathcal{O}(\epsilon)\right]\,, \tag{8}$$

where γ is Euler's constant. Defining Δ as in eq. (4), and taking the limit of $\epsilon \to 0$, we end up with

$$\boxed{A_0(m^2) = m^2 \left[\Delta + 1 - \ln\left(\frac{m^2}{\mu^2}\right)\right]\,.} \tag{9}$$

[1] A_0, B_0, B^μ (and C_0 of part (c) below) are examples of the Passarino-Veltman functions, first introduced in G. Passarino and M. Veltman, Nucl. Phys. B **160**, 151 (1979), albeit with a slightly different normalization and a different metric convention.

[2] For convenience, I have altered the definition of A_0, B_0 and B^μ by multiplying eqs. (1)–(3) by $(\mu^2)^\epsilon$, where μ^2 is a positive squared-mass parameter. This will ensure that the arguments of all logarithms are dimensionless.

In order to evaluate $B_0(p^2; m_1^2, m_2^2)$, we first employ Feynman's trick to write

$$\frac{1}{(q^2 - m_1^2 + i\varepsilon)[(q+p)^2 - m_2^2 + i\varepsilon]} = \int_0^1 \frac{dx}{\big[(1-x)(q^2 - m_1^2) + x[(q+p)^2 - m_2^2] + i\varepsilon\big]^2}$$
$$= \int_0^1 \frac{dx}{\big[q^2 + 2xq\cdot p + (p^2 + m_1^2 - m_2^2)x - m_1^2 + i\varepsilon\big]^2}\,. \quad (10)$$

Plugging this result into eq. (2), interchanging the order of integration and employing eq. (5), it follows that

$$B_0(p^2; m_1^2, m_2^2) = (4\pi\mu^2)^\epsilon\,\Gamma(\epsilon)\int_0^1 \big[p^2x^2 - (p^2 + m_1^2 - m_2^2)x + m_1^2 - i\varepsilon\big]^{-\epsilon} dx$$
$$= \left(\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right)\left[1 - \epsilon\int_0^1 \ln\left(\frac{p^2x^2 - (p^2 + m_1^2 - m_2^2)x + m_1^2 - i\varepsilon}{\mu^2}\right) dx + \mathcal{O}(\epsilon^2)\right]$$
$$= \Delta - \int_0^1 \ln\left(\frac{p^2x^2 - (p^2 + m_1^2 - m_2^2)x + m_1^2 - i\varepsilon}{\mu^2}\right) dx + \mathcal{O}(\epsilon)\,, \quad (11)$$

after expanding in ϵ and using eq. (4).

Eq. (11) motivates the computation of the following integral,

$$\mathcal{I} \equiv \int_0^1 \ln(Ax^2 + Bx + C - i\varepsilon)\, dx\,, \quad (12)$$

where A, B and C are real numbers and ε is a positive infinitesimal constant. Assuming $A \neq 0$, we integrate by parts with $u = \ln(A^2x + Bx + C - i\varepsilon)$ and $dv = dx$ to obtain

$$\mathcal{I} = \ln(A + B + C - i\varepsilon) - \int_0^1 \frac{(2Ax^2 + Bx)\, dx}{Ax^2 + Bx + C - i\varepsilon}$$
$$= \ln(A + B + C - i\varepsilon) - \int_0^1 \frac{2(Ax^2 + Bx + C - i\varepsilon) - (Bx + 2C - 2i\varepsilon)\, dx}{Ax^2 + Bx + C - i\varepsilon}$$
$$= \ln(A + B + C - i\varepsilon) - 2 + \int_0^1 \frac{(Bx + 2C)\, dx}{Ax^2 + Bx + C - i\varepsilon}\,, \quad (13)$$

where it is safe to drop the $i\varepsilon$ factor in the numerator. Defining $B = r_1A$ and $C = r_2A$,

$$\mathcal{I} = \ln(A + B + C - i\varepsilon) - 2 + \int_0^1 \frac{(r_1x + 2r_2)\, dx}{x^2 + r_1x + r_2 - i\varepsilon\,\mathrm{sgn}\,A}\,, \quad (14)$$

We consider three cases.

<u>Case 1</u>: $r_1^2 < 4r_2$

In this case, the roots of polynomial equation $x^2 + r_1x + r_2 = 0$ are complex. Consequently, the denominator of the integrand above never vanishes, and we are free to set $\varepsilon = 0$. After factoring the denominator,

$$x^2 + r_1x + r_2 = (x - x_+)(x - x_-)\,, \quad (15)$$

where

$$x_\pm \equiv \tfrac{1}{2}\big[-r_1 \pm i\sqrt{4r_2 - r_1^2}\big]\,, \tag{16}$$

we perform a partial fractioning,

$$\frac{r_1 x + 2r_2}{(x-x_+)(x-x_-)} = -\left(\frac{x_+}{x-x_+} + \frac{x_-}{x-x_-}\right)\,, \tag{17}$$

after noting that

$$x_+ + x_- = -r_1\,, \qquad\qquad x_+ x_- = r_2\,. \tag{18}$$

Hence, it follows that

$$\begin{aligned}
\mathcal{I} &= \ln(A+B+C-i\varepsilon) - 2 - \int_0^1 \left(\frac{x_+}{x-x_+} + \frac{x_-}{x-x_-}\right) dx \\
&= \ln(A+B+C-i\varepsilon) - 2 - x_+\big[\ln(1-x_+) - \ln(-x_+)\big] - x_-\big[\ln(1-x_-) - \ln(-x_-)\big] \\
&= \ln(A+B+C-i\varepsilon) - 2 - 2\,\mathrm{Re}\,\big\{x_+\big[\ln(1-x_+) - \ln(-x_+)\big]\big\} \\
&= \ln(A+B+C-i\varepsilon) - 2 + \mathrm{Re}\left\{\big(r_1 - i\sqrt{4r_2 - r_1^2}\big)\big[\ln(1-x_+) - \ln(-x_+)\big]\right\},
\end{aligned} \tag{19}$$

after using $(x_+)^* = x_1$ in the penultimate step above. The logarithms above are the principal values of the corresponding complex logarithms defined on the cut complex plane, where the branch cut runs along the real axis from $-\infty$ to the origin.

Evaluating the real part of the expression above is straightforward.

$$\begin{aligned}
&\mathrm{Re}\left\{\big(r_1 - i\sqrt{4r_2 - r_1^2}\big)\big[\ln(1-x_+) - \ln(-x_+)\big]\right\} \\
&\quad = r_1\big\{\ln|1-x_+| - \ln|x_+|\big\} + \sqrt{4r_2 - r_1^2}\,\big[\arg(1-x_+) - \arg(-x_+)\big] \\
&\qquad = \tfrac{1}{2}r_1 \ln\left(\frac{r_1 + r_2 + 1}{r_2}\right) + \sqrt{4r_2 - r_1^2}\,\bigg[\arg\left(1 + \tfrac{1}{2}r_1 - \tfrac{1}{2}i\sqrt{4r_2 - r_1^2}\right) \\
&\qquad\qquad - \arg\left(\tfrac{1}{2}r_1 - \tfrac{1}{2}i\sqrt{4r_2 - r_1^2}\right)\bigg],
\end{aligned} \tag{20}$$

where the principal value of the argument lies in the range $-\pi < \arg z \leq \pi$ for any complex number z. Since the roots of polynomial equation $x^2 + r_1 x + r_2 = 0$ are complex, it follows that $x^2 + r_1 x + r_2 > 0$ for all values of x. Setting $x = 0$ and $x = 1$, respectively, in the inequality, one can conclude that $r_2 > 0$ and $r_1 + r_2 + 1 > 0$.

In order to evaluate the argument functions in eq. (20), we make use of the following result for the principal value of the argument function,

$$\arg(x - iy) = \begin{cases} \arctan(y/x)\,, & \text{for } x > 0 \text{ and } y > 0, \\ \pi + \arctan(y/x)\,, & \text{for } x < 0 \text{ and } y > 0, \end{cases} \tag{21}$$

where we have employed the principal value of the real arctangent function, which satisfies $|\arctan(y/x)| \leq \frac{1}{2}\pi$. Referring to p. 119 of F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark, editors, *NIST Handbook of Mathematical Functions* (Cambridge University Press, Cambridge, UK, 2010),

$$\arctan(y/x) = \begin{cases} \frac{1}{2}\pi - \arctan(x/y)\,, & \text{for } y/x > 0, \\ -\frac{1}{2}\pi - \arctan(x/y)\,, & \text{for } y/x < 0. \end{cases} \tag{22}$$

Noting that $\arctan(x/y) = -\arctan(-x/y)$, it follows that

$$\arg(x - iy) = \tfrac{1}{2}\pi + \arctan(-x/y)\,, \qquad \text{for } y > 0, \tag{23}$$

which holds for both signs of x.

Hence,

$$\begin{aligned} \arg\left(1 + \tfrac{1}{2}r_1 - \tfrac{1}{2}i\sqrt{4r_2 - r_1^2}\right) &- \arg\left(\tfrac{1}{2}r_1 - \tfrac{1}{2}i\sqrt{4r_2 - r_1^2}\right) \\ &= \arctan\left(\frac{2 + r_1}{\sqrt{4r_2 - r_1^2}}\right) - \arctan\left(\frac{r_1}{\sqrt{4r_2 - r_1^2}}\right). \end{aligned} \tag{24}$$

Combining the results of eqs. (19), (20) and (24),

$$\begin{aligned} \mathcal{I} = \ln(A + B + C - i\varepsilon) &+ \tfrac{1}{2}r_1 \ln\left(\frac{r_1 + r_2 + 1}{r_2}\right) - 2 \\ &+ \sqrt{4r_2 - r_1^2}\left[\arctan\left(\frac{2 + r_1}{\sqrt{4r_2 - r_1^2}}\right) - \arctan\left(\frac{r_1}{\sqrt{4r_2 - r_1^2}}\right)\right]. \end{aligned} \tag{25}$$

Plugging in $r_1 = B/A$ and $r_2 = C/A$, we obtain our final result,

$$\begin{aligned} \mathcal{I} = \ln(A + B + C - i\varepsilon) &+ \frac{B}{2A} \ln\left(\frac{A + B + C}{C}\right) - 2 \\ &+ \frac{\sqrt{4AC - B^2}}{A}\left[\arctan\left(\frac{2A + B}{\sqrt{4AC - B^2}}\right) - \arctan\left(\frac{B}{\sqrt{4AC - B^2}}\right)\right], \\ &\qquad \text{for } A \neq 0 \text{ and } B^2 - 4AC < 0. \end{aligned} \tag{26}$$

In particular,

$$\operatorname{Im}\mathcal{I} = -\pi\Theta(-A - B - C)\,, \qquad \text{for } A \neq 0 \text{ and } B^2 - 4AC < 0. \tag{27}$$

<u>Case 2</u>: $r_1^2 > 4r_2$

In this case, the roots of polynomial equation $x^2 + r_1 x + r_2 = 0$ are real, and

$$x^2 + r_1 x + r_2 = (x - x_+)(x - x_-)\,, \tag{28}$$

where

$$x_\pm \equiv \tfrac{1}{2}\left[-r_1 \pm \sqrt{r_1^2 - 4r_2}\right]. \tag{29}$$

The real roots satisfy,

$$x_+ + x_- = -r_1\,, \qquad x_+ - x_- = \sqrt{r_1^2 - 4r_2}\,, \qquad x_+ x_- = r_2\,, \tag{30}$$

If either (or both) x_+ or x_- lie in the integration region of $0 < x < 1$, then the denominator of the integrand of eq. (14) would vanish if we set $\varepsilon = 0$. Hence, we keep the $i\varepsilon$ term present and note that

$$x^2 + rx_1 + r_2 - i\varepsilon \operatorname{sgn} A = (x - x_+ + i\varepsilon \operatorname{sgn} A)(x - x_- - i\varepsilon \operatorname{sgn} A)\,. \tag{31}$$

Hence, performing a partial fractioning of the integrand of eq. (14) yields

$$\frac{(r_1 x + 2r_2)}{x^2 + r_1 x + r_2 - i\varepsilon \operatorname{sgn} A} = \frac{x_+}{x - x_+ - i\varepsilon \operatorname{sgn} A} + \frac{x_-}{x - x_- + i\varepsilon \operatorname{sgn} A}\,, \tag{32}$$

after omitting the term proportional to $i\varepsilon \operatorname{sgn} A$ in the numerator, which can be safely dropped in the limit of $\varepsilon \to 0$. Hence, it follows that

$$\begin{aligned}
\mathcal{I} &= \ln(A + B + C - i\varepsilon) - 2 - \int_0^1 \left(\frac{x_+}{x - x_+ - i\varepsilon \operatorname{sgn} A} + \frac{x_-}{x - x_- + i\varepsilon \operatorname{sgn} A}\right) dx \\
&= \ln(A + B + C - i\varepsilon) - 2 - x_+\big[\ln(1 - x_+ - i\varepsilon \operatorname{sgn} A) - \ln(-x_+ - i\varepsilon \operatorname{sgn} A)\big] \\
&\qquad\qquad - x_-\big[\ln(1 - x_- + i\varepsilon \operatorname{sgn} A) - \ln(-x_- + i\varepsilon \operatorname{sgn} A)\big]\,.
\end{aligned} \tag{33}$$

Note that

$$x_\pm = \frac{-B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2|A|}\,. \tag{34}$$

In the literature, it is more typical to define,

$$y_\pm = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\,, \tag{35}$$

under the assumption that $B^2 - 4AC > 0$. Note that

$$y_+ + y_- = -\frac{B}{A}\,, \qquad y_+ y_- = \frac{C}{A}\,, \qquad y_+ - y_- = \frac{\sqrt{B^2 - 4AC}}{A}\,. \tag{36}$$

and it follows that $y_+ > y_-$ if $\operatorname{sgn} A > 0$, but $y_+ < y_-$ if $\operatorname{sgn} A < 0$. In this notation, eq. (33) can be rewritten as,

$$\begin{aligned}
\mathcal{I} &= \ln(A + B + C - i\varepsilon) - 2 - y_+\big[\ln(1 - y_+ - i\varepsilon) - \ln(-y_+ - i\varepsilon)\big] \\
&\qquad - y_-\big[\ln(1 - y_- + i\varepsilon) - \ln(-y_- + i\varepsilon)\big]\,, \qquad \text{for } A \neq 0 \text{ and } B^2 - 4AC > 0.
\end{aligned} \tag{37}$$

Case 3: $r_1^2 = 4r_2$

In this limit, $B^2 - 4AC = 0$ and eq. (13) yields,

$$\mathcal{I} = \ln\left(A + B + \frac{B^2}{4A} - i\varepsilon\right) - 2 + \frac{B}{A}\int_0^1 \frac{\left(x + \frac{B}{2A}\right)\, dx}{\left(x + \frac{B}{2A}\right)^2 - i\varepsilon \operatorname{sgn} A}\,. \tag{38}$$

Employing the Sokhotski-Plemelj formula, with $x_0 \equiv B/(2A)$,

$$\lim_{\varepsilon\to 0^+} \frac{(x+x_0)}{(x+x_0)^2 - i\varepsilon \operatorname{sgn} A} = \mathrm{P}\frac{1}{x+x_0} + i\pi(x+x_0)\delta\big((x+x_0)^2\big) = \mathrm{P}\frac{1}{x+x_0}\,, \tag{39}$$

where we have used the well-known property of δ-functions that $f(x)\delta(x) = f(0)\delta(x)$. Hence,

$$\begin{aligned}
\mathcal{I} &= \ln\left(A + B + \frac{B^2}{4A} - i\varepsilon\right) - 2 + \frac{B}{A}\,\mathrm{P}\int_0^1 \frac{dx}{x+x_0} \\
&= \ln\left[A\left(1+\frac{B}{2A}\right)^2 - i\varepsilon\right] - 2 + \frac{B}{A}\lim_{\delta\to 0^+}\left\{\int_0^{-x_0-\delta}\frac{dx}{x+x_0} + \int_{-x_0+\delta}^1 \frac{dx}{x+x_0}\right\} \\
&= \ln\left[(A - i\varepsilon)\left(1+\frac{B}{2A}\right)^2\right] - 2 + \frac{B}{A}\lim_{\delta\to 0^+}\left\{\ln\left|\frac{\delta}{x_0}\right| + \ln\left|\frac{1+x_0}{\delta}\right|\right\} \\
&= \ln(A - i\varepsilon) + 2\ln\left|1+\frac{B}{2A}\right| - 2 + \frac{B}{A}\ln\left|\frac{1+x_0}{x_0}\right| \\
&= \ln(A - i\varepsilon) + 2\ln\left|1+\frac{B}{2A}\right| - 2 + \frac{B}{A}\left[\ln\left|1+\frac{B}{2A}\right| - \ln\left|\frac{B}{2A}\right|\right] \\
&= \ln(A - i\varepsilon) - 2 + \left(\frac{B+2A}{A}\right)\ln\left|1+\frac{B}{2A}\right| - \frac{B}{A}\ln\left|\frac{B}{2A}\right|\,.
\end{aligned} \tag{40}$$

One further simplification yields our final result,

$$\mathcal{I} = \ln(A - i\varepsilon) - 2 + 2\ln\left|1+\frac{B}{2A}\right| + \frac{B}{A}\ln\left|1+\frac{2A}{B}\right|\,, \qquad \text{for } A \neq 0 \text{ and } B^2 - 4AC = 0. \tag{41}$$

It is straightforward to check that in the limit of $B^2 = 4AC$, both eqs. (26) and (37) yield the result quoted in eq. (41). For example, in this limit, $y_+ = y_- = -B/(2A)$. Hence, we can take the limit of $y_+ = y_-$ of eq. (37) to obtain,

$$\mathcal{I} = \ln\left(A + B + \frac{B^2}{4A} - i\varepsilon\right) - 2 + \frac{B}{A}\left[\operatorname{Re}\ln\left(1 + \frac{B}{2A} + i\varepsilon\right) - \operatorname{Re}\ln\left(\frac{B}{2A} + i\varepsilon\right)\right]\,, \tag{42}$$

which is equivalent to eq. (40).

This completes the analysis of eq. (12) in the case of $A \neq 0$. The case of $A \neq 0$ requires a separate analysis, which is provided below. In this case, we assume that $B \neq 0$ and compute,

$$\mathcal{I}_0 \equiv \int_0^1 \ln(Bx + C - i\varepsilon)\, dx\,. \tag{43}$$

Integrating by parts by taking $u = \ln(Bx + C + i\varepsilon)$ and $dv = dx$, it follows that

$$\begin{aligned}
\mathcal{I}_0 &= \ln(B + C - i\varepsilon) - \int_0^1 \frac{Bx\,dx}{Bx + C - i\varepsilon} \\
&= \ln(B + C - i\varepsilon) - \int_0^1 \frac{Bx + C - i\varepsilon - (C - i\varepsilon)\,dx}{Bx + C - i\varepsilon} \\
&= \ln(B + C - i\varepsilon) - 1 + C \int_0^1 \frac{dx}{Bx + C - i\varepsilon}\,,
\end{aligned} \tag{44}$$

where it is safe to drop the $i\varepsilon$ factor in the numerator. Defining $C = rB$,

$$\begin{aligned}
\mathcal{I}_0 = &= \ln(B + C - i\varepsilon) - 1 + r\big[\ln(1 + r - i\varepsilon\,\mathrm{sgn}\,B) - \ln(r - i\varepsilon\,\mathrm{sgn}\,B)\big] \\
&= \ln(B + C - i\varepsilon) - 1 + \frac{C}{B}\left[\ln\left(\frac{B + C - i\varepsilon}{B}\right) - \ln\left(\frac{C - i\varepsilon}{B}\right)\right] \\
&= \ln(B + C - i\varepsilon) - 1 + \frac{C}{B}\ln\left(\frac{B + C}{C} + i\varepsilon\,\mathrm{sgn}\,B\right)\,, \qquad \text{for } B \neq 0.
\end{aligned} \tag{45}$$

This form is not so useful in the case of $B = -C$. However, one can rewrite eq. (45) as follows,

$$\mathcal{I}_0 = L - 1 + \left(1 + \frac{C}{B}\right)\ln\left(\frac{B + C - i\varepsilon}{B}\right) - \frac{C}{B}\ln\left(\frac{C - i\varepsilon}{B}\right)\,, \tag{46}$$

where

$$L \equiv \ln(B + C - i\varepsilon) - \ln\left(\frac{B + C - i\varepsilon}{B}\right)\,. \tag{47}$$

If $B > 0$, then $L = \ln B$. If $B < 0$ and $B + C > 0$,

$$L = \ln(B + C) - i\pi - \ln(B + C) + \ln(-B) = \ln(-B) - i\pi\,. \tag{48}$$

If $B < 0$ and $B + C < 0$,

$$L = \ln(-B - C) - i\pi - \ln(-B - C) + \ln(-B) = \ln(-B) - i\pi\,. \tag{49}$$

Thus, we conclude that in all three cases considered above,

$$L = \ln(B - i\varepsilon)\,. \tag{50}$$

Hence, another alternative form for eq. (45) is given by,

$$\mathcal{I}_0 = \ln(B - i\varepsilon) - 1 + \left(1 + \frac{C}{B}\right)\ln\left(\frac{B + C - i\varepsilon}{B}\right) - \frac{C}{B}\ln\left(\frac{C - i\varepsilon}{B}\right)\,, \qquad \text{for } B \neq 0. \tag{51}$$

We shall now apply the above results to $B_0(p^2; m_1^2, m_2^2)$. Comparing eqs. (11) and (12), we identify,

$$A = \frac{p^2}{\mu^2}\,, \qquad B = -\frac{p^2 + m_1^2 - m_2^2}{\mu^2}\,, \qquad C = \frac{m_1^2}{\mu^2}\,. \tag{52}$$

It follows that

$$r_1 = -1 + \frac{m_2^2 - m_1^2}{p^2}\,, \qquad r_2 = \frac{m_1^2}{p^2}\,, \qquad r_1^2 - 4r_2 = \frac{\lambda(p^2, m_1^2, m_2^2)}{p^4}\,, \tag{53}$$

where $\lambda(a, b, c)$ is the well-known kinematical triangle function,

$$\lambda(a, b, c) \equiv a^2 + b^2 + c^2 - 2ab - 2ac - 2bc = (a + b - c)^2 - 4ab\,. \tag{54}$$

Thus,

$$\boxed{B_0(p^2; m_1^2, m_2^2) = \Delta - F(p^2; m_1^2, m_2^2)\,,} \tag{55}$$

where

$$\begin{aligned} F(p^2; m_1^2, m_2^2) = {} & \ln\left(\frac{m_2^2}{\mu^2}\right) - \left(\frac{p^2 + m_1^2 - m_2^2}{2p^2}\right)\ln\left(\frac{m_2^2}{m_1^2}\right) - 2 \\ & + \frac{(-\lambda)^{1/2}}{p^2}\left[\arctan\left(\frac{p^2 - m_1^2 + m_2^2}{(-\lambda)^{1/2}}\right) + \arctan\left(\frac{p^2 + m_1^2 - m_2^2}{(-\lambda)^{1/2}}\right)\right], \\ & \text{for } p^2 \neq 0 \text{ and } \lambda \equiv \lambda(p^2, m_1^2, m_2^2) < 0\,, \end{aligned} \tag{56}$$

$$\begin{aligned} F(p^2; m_1^2, m_2^2) = {} & \ln\left(\frac{m_2^2}{\mu^2}\right) - 2 \\ & - \left(\frac{p^2 + m_1^2 - m_2^2 + \lambda^{1/2}}{2p^2}\right)\left[\ln\left(\frac{p^2 - m_1^2 + m_2^2 - \lambda^{1/2}}{2p^2} - i\varepsilon\right)\right. \\ & \qquad \left. - \ln\left(\frac{-p^2 - m_1^2 + m_2^2 - \lambda^{1/2}}{2p^2} - i\varepsilon\right)\right] \\ & - \left(\frac{p^2 + m_1^2 - m_2^2 - \lambda^{1/2}}{2p^2}\right)\left[\ln\left(\frac{p^2 - m_1^2 + m_2^2 + \lambda^{1/2}}{2p^2} - i\varepsilon\right)\right. \\ & \qquad \left. - \ln\left(\frac{-p^2 - m_1^2 + m_2^2 + \lambda^{1/2}}{2p^2} - i\varepsilon\right)\right] \\ & \text{for } p^2 \neq 0 \text{ and } \lambda \equiv \lambda(p^2, m_1^2, m_2^2) > 0\,, \end{aligned} \tag{57}$$

$$\begin{aligned} F(p^2; m_1^2, m_2^2) = {} & \ln\left(\frac{p^2}{\mu^2}\right) - 2 + \left(\frac{p^2 - m_1^2 + m_2^2}{p^2}\right)\ln\left|\frac{p^2 - m_1^2 + m_2^2}{2p^2}\right| \\ & + \left(\frac{p^2 + m_1^2 - m_2^2}{p^2}\right)\ln\left|\frac{p^2 + m_1^2 - m_2^2}{2p^2}\right|\,, \\ & \text{for } p^2 \neq 0 \text{ and } \lambda(p^2, m_1^2, m_2^2) = 0, \end{aligned} \tag{58}$$

$$F(0; m_1^2, m_2^2) = \frac{1}{m_1^2 - m_2^2}\left[m_1^2 \ln\left(\frac{m_1^2}{\mu^2}\right) - m_2^2 \ln\left(\frac{m_2^2}{\mu^2}\right)\right] - 1\,, \qquad \text{for } m_1^2 \neq m_2^2, \tag{59}$$

$$F(0; m^2, m^2) = \ln\left(\frac{m^2}{\mu^2}\right)\,, \tag{60}$$

under the assumption that m_1, m_2 and μ are real quantities.

Note that an alternate expression for λ is given by,

$$\lambda(p^2, m_1^2, m_2^2) = \left[p^2 - (m_1+m_2)^2\right]\left[p^2 - (m_1-m_2)^2\right]. \tag{61}$$

It follows that,

$$\begin{aligned}
\lambda(p^2, m_1^2, m_2^2) < 0 \quad &\Longrightarrow \quad (m_1-m_2)^2 < p^2 < (m_1+m_2)^2\,, \\
\lambda(p^2, m_1^2, m_2^2) > 0 \quad &\Longrightarrow \quad p^2 < (m_1-m_2)^2 \quad \text{or} \quad p^2 > (m_1+m_2)^2\,.
\end{aligned} \tag{62}$$

In light of Cutkosky's cutting rules, $\operatorname{Im} F(p^2; m_1^2, m_2^2) \neq 0$ if and only if $p^2 > (m_1+m_2)^2$, in which case the internal lines of the one-loop self-energy graph can go on-shell.

Thus, we can simplify the expression given by eq. (57) as follows. Using the definition of the principal value of the complex logarithm,

$$\ln(x - i\varepsilon) = \ln|x| - i\pi\Theta(-x)\,, \quad \text{for } x \in \mathbb{R},\ x \neq 0 \text{ and positive infinitesimal } \varepsilon, \tag{63}$$

it follows that $\operatorname{Re}\ln x = \ln|x|$. Hence, after combining logarithms, eq. (57) yields,

$$\begin{aligned}
\operatorname{Re} F(p^2; m_1^2, m_2^2) = \ln\left(\frac{m_2^2}{\mu^2}\right) - 2 - \left(\frac{p^2 + m_1^2 - m_2^2}{2p^2}\right)\ln\left(\frac{m_2^2}{m_1^2}\right) \\
+ \frac{\lambda^{1/2}(p^2, m_1^2, m_2^2)}{2p^2}\ln\left(\frac{p^2 - m_1^2 - m_2^2 + \lambda^{1/2}(p^2, m_1^2, m_2^2)}{p^2 - m_1^2 - m_2^2 - \lambda^{1/2}(p^2, m_1^2, m_2^2)}\right),
\end{aligned} \tag{64}$$

In the case of $p^2 > (m_1+m_2)^2$, one can check that

$$p^2 - m_1^2 + m_2^2 \pm \lambda^{1/2} > 0 \quad \text{and} \quad -p^2 - m_1^2 + m_2^2 \pm \lambda^{1/2} < 0.$$

Hence,

$$\operatorname{Im} F(p^2; m_1^2, m_2^2) = -\frac{\pi\lambda^{1/2}(p^2, m_1^2, m_2^2)}{p^2}\,\Theta\big(p^2 - (m_1+m_2)^2\big)\,. \tag{65}$$

It then follows that an alternate expression for eq. (57) is

$$\begin{aligned}
F(p^2; m_1^2, m_2^2) = \ln\left(\frac{m_2^2}{\mu^2}\right) - 2 - \left(\frac{p^2 + m_1^2 - m_2^2}{2p^2}\right)\ln\left(\frac{m_2^2}{m_1^2}\right) \\
+ \frac{\lambda^{1/2}(p^2, m_1^2, m_2^2)}{2p^2}\left[\ln\left(\frac{p^2 - m_1^2 - m_2^2 + \lambda^{1/2}(p^2, m_1^2, m_2^2)}{p^2 - m_1^2 - m_2^2 - \lambda^{1/2}(p^2, m_1^2, m_2^2)}\right) - 2i\pi\,\Theta\big(p^2 - (m_1+m_2)^2\big)\right], \\
\text{for } p^2 \neq 0 \text{ and } \lambda \equiv \lambda(p^2, m_1^2, m_2^2) > 0\,.
\end{aligned} \tag{66}$$

One can perform one further simplification by noting that

$$\frac{p^2 - m_1^2 - m_2^2 + \lambda^{1/2}(p^2, m_1^2, m_2^2)}{p^2 - m_1^2 - m_2^2 - \lambda^{1/2}(p^2, m_1^2, m_2^2)} = \left(\frac{\left[p^2 - (m_1-m_2)^2\right]^{1/2} + \left[p^2 - (m_1+m_2)^2\right]^{1/2}}{\left[p^2 - (m_1-m_2)^2\right]^{1/2} - \left[p^2 - (m_1+m_2)^2\right]^{1/2}}\right)^2, \tag{67}$$

which is useful in the case of $p^2 > (m_1+m_2)^2$. Hence,

$$\boxed{\begin{aligned}F(p^2;m_1^2,m_2^2) &= \ln\left(\frac{m_2^2}{\mu^2}\right) - 2 - \left(\frac{p^2+m_1^2-m_2^2}{2p^2}\right)\ln\left(\frac{m_2^2}{m_1^2}\right) \\ &\quad + \frac{\lambda^{1/2}(p^2,m_1^2,m_2^2)}{p^2}\left[\ln\left(\frac{\left[p^2-(m_1-m_2)^2\right]^{1/2}+\left[p^2-(m_1+m_2)^2\right]^{1/2}}{\left[p^2-(m_1-m_2)^2\right]^{1/2}-\left[p^2-(m_1+m_2)^2\right]^{1/2}}\right)-i\pi\right], \\ &\qquad \text{for } p^2>(m_1+m_2)^2.\end{aligned}} \tag{68}$$

Likewise,

$$\frac{p^2-m_1^2-m_2^2+\lambda^{1/2}(p^2,m_1^2,m_2^2)}{p^2-m_1^2-m_2^2-\lambda^{1/2}(p^2,m_1^2,m_2^2)} = \left(\frac{\left[(m_1+m_2)^2-p^2\right]^{1/2}+\left[(m_1-m_2)^2-p^2\right]^{1/2}}{\left[(m_1+m_2)^2-p^2\right]^{1/2}-\left[(m_1-m_2)^2-p^2\right]^{1/2}}\right)^2, \tag{69}$$

which is useful in the case of $p^2<(m_1-m_2)^2$. Hence,

$$\boxed{\begin{aligned}F(p^2;m_1^2,m_2^2) &= \ln\left(\frac{m_2^2}{\mu^2}\right) - 2 - \left(\frac{p^2+m_1^2-m_2^2}{2p^2}\right)\ln\left(\frac{m_2^2}{m_1^2}\right) \\ &\quad + \frac{\lambda^{1/2}(p^2,m_1^2,m_2^2)}{p^2}\ln\left(\frac{\left[(m_1+m_2)^2-p^2\right]^{1/2}+\left[(m_1-m_2)^2-p^2\right]^{1/2}}{\left[(m_1+m_2)^2-p^2\right]^{1/2}-\left[(m_1-m_2)^2-p^2\right]^{1/2}}\right), \\ &\qquad \text{for } p^2<(m_1-m_2)^2 \text{ and } p^2\neq 0.\end{aligned}} \tag{70}$$

One can also obtain expressions that are valid for $p^2=(m_1\pm m_2)^2$ by taking the appropriate limits in eqs. (68) and (70). Since $\lambda(p^2,m_1^2,m_2^2)\to 0$ in both limiting cases, we find

$$F(p^2;m_1^2,m_2^2) = \begin{cases} \dfrac{1}{m_1+m_2}\left[m_1\ln\left(\dfrac{m_1^2}{\mu^2}\right)+m_2\ln\left(\dfrac{m_2^2}{\mu^2}\right)\right]-2\,, & \text{for } p^2=(m_1+m_2)^2\,, \\ \dfrac{1}{m_1-m_2}\left[m_1\ln\left(\dfrac{m_1^2}{\mu^2}\right)-m_2\ln\left(\dfrac{m_2^2}{\mu^2}\right)\right]-2\,, & \text{for } p^2=(m_1-m_2)^2\,. \end{cases} \tag{71}$$

It is straightforward to check that these expressions match the expected result of eq. (58) for $p^2=(m_1\pm m_2)^2$, respectively.

It is instructive to verify the results of eqs. (68) and (70) in the limit of $m_1=m_2=m$,

$$F(p^2;m^2,m^2) = \begin{cases} \ln\left(\dfrac{m^2}{\mu^2}\right)-2+\sqrt{1-\dfrac{4m^2}{p^2}}\ln\left(\dfrac{\sqrt{1-\frac{4m^2}{p^2}}+1}{\sqrt{1-\frac{4m^2}{p^2}}-1}\right), & \text{for } p^2<0 \\ \ln\left(\dfrac{m^2}{\mu^2}\right)-2+\sqrt{1-\dfrac{4m^2}{p^2}}\left[\ln\left(\dfrac{1+\sqrt{1-\frac{4m^2}{p^2}}}{1-\sqrt{1-\frac{4m^2}{p^2}}}\right)-i\pi\right], & \text{for } p^2>4m^2, \end{cases} \tag{72}$$

as expected in light of eqs. (90) and (98) of the Solutions to Problem Set 2.

The limit of $p^2 \to 0$ of eq. (70) exists, and can be evaluated by expanding out the argument of the logarithm in powers of p^2. We leave this as an exercise for the reader. The end result of this computation is

$$\boxed{\begin{aligned} &F(0;m_1^2,m_2^2) = \frac{1}{m_1^2-m_2^2}\left[m_1^2\ln\left(\frac{m_1^2}{\mu^2}\right) - m_2^2\ln\left(\frac{m_2^2}{\mu^2}\right)\right] - 1\,, \quad \text{for } m_1 \neq m_2, \\ &F(0;m^2,m^2) = \ln\left(\frac{m^2}{\mu^2}\right)\,, \end{aligned}} \tag{73}$$

in agreement with eqs. (59) and (60).

Finally, we can simplify the expression given by eq. (56) by employing the following relation given on p. 58 of G&R,[3]

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi\,\text{sgn}(x)\Theta(xy-1)\,, \qquad \text{for } x, y \in \mathbb{R}. \tag{74}$$

Using this identity, eq. (56) yields,

$$\begin{aligned} F(p^2;m_1^2,m_2^2) &= \ln\left(\frac{m_2^2}{\mu^2}\right) - \left(\frac{p^2+m_1^2-m_2^2}{2p^2}\right)\ln\left(\frac{m_2^2}{m_1^2}\right) - 2 \\ &\quad + \frac{\left[-\lambda(p^2,m_1^2,m_2^2)\right]^{1/2}}{p^2}\left[\arctan\left(\frac{\left[-\lambda(p^2,m_1^2,m_2^2)\right]^{1/2}}{m_1^2+m_2^2-p^2}\right) + \pi\,\Theta(p^2-m_1^2-m_2^2)\right]\,, \\ &\qquad \text{for } (m_1-m_2)^2 < p^2 < (m_1+m_2)^2. \end{aligned} \tag{75}$$

One further simplification is possible by employing the following relation given on p. 59 of G&R,

$$2\arctan x = \arctan\left(\frac{2x}{1-x^2}\right) + \pi\,\text{sgn}(x)\Theta(|x|-1)\,, \qquad \text{for } x \in \mathbb{R}. \tag{76}$$

Using this identity, it follows that

$$2\arctan\left(\frac{\sqrt{p^2-(m_1-m_2)^2}}{\sqrt{(m_1+m_2)^2-p^2}}\right) = \arctan\left(\frac{\left[-\lambda(p^2,m_1^2,m_2^2)\right]^{1/2}}{m_1^2+m_2^2-p^2}\right) + \pi\,\Theta(p^2-m_1^2-m_2^2)\,, \tag{77}$$

after making use of eq. (61). Hence, we end up with

$$\boxed{\begin{aligned} F(p^2;m_1^2,m_2^2) &= \ln\left(\frac{m_2^2}{\mu^2}\right) - \left(\frac{p^2+m_1^2-m_2^2}{2p^2}\right)\ln\left(\frac{m_2^2}{m_1^2}\right) - 2 \\ &\quad + \frac{2\left[-\lambda(p^2,m_1^2,m_2^2)\right]^{1/2}}{p^2}\arctan\left(\frac{\sqrt{p^2-(m_1-m_2)^2}}{\sqrt{(m_1+m_2)^2-p^2}}\right)\,, \\ &\qquad \text{for } (m_1-m_2)^2 < p^2 < (m_1+m_2)^2. \end{aligned}} \tag{78}$$

[3]We use the notation G&R to refer to I.S. Gradshteyn and I.M. Ryzhikm *Table of Integrals, Series, and Products*, Eighth Edition, edited by Daniel Zwillinger and Victor Moll (Academic Press, Waltham, MA, 2015).

It is straightforward to check that the limiting behavior of eq. (78) as $p^2 \to (m_1 \pm m_2)^2$ reproduces the results of eq. (71). As an aside, one could choose to present an equivalent expression for eq. (78) by replacing the arctangent function with an arcsine function by using the identity,

$$\arctan\left(\frac{\sqrt{p^2-(m_1-m_2)^2}}{\sqrt{(m_1+m_2)^2-p^2}}\right) = \arcsin\left[\left(\frac{p^2-(m_1-m_2)^2}{4m_1m_2}\right)^{1/2}\right], \tag{79}$$

which is valid for $(m_1-m_2)^2 < p^2 < (m_1+m_2)^2$ [cf. formula 2. on p. 57 of G&R].

It is instructive to verify the result of eq. (78) in the limit of $m_1 = m_2 = m$,

$$F(p^2; m^2, m^2) = \ln\left(\frac{m_2^2}{\mu^2}\right) - 2 + 2\left(\frac{4m^2}{p^2}-1\right)^{1/2} \arctan\left(\frac{1}{\sqrt{\frac{4m^2}{p^2}-1}}\right), \quad \text{for } 0 < p^2 < 4m^2, \tag{80}$$

as expected in light of eq. (81) of the Solutions to Problem Set 2.

(b) Show that B^μ takes the following form

$$B^\mu(p; m_1^2, m_2^2) = p^\mu\, B_1(p^2, m_1^2, m_2^2)\,. \tag{81}$$

Find an expression for the scalar function B_1 in terms of B_0 and A_0 evaluated at the appropriate arguments.

In order to evaluate $B^\mu(p^2; m_1^2, m_2^2)$, we again employ Feynman's trick. Plugging the result of eq. (10) into eq. (3), interchanging the order of integration and employing the result of the handout entitled, *Useful formulae for computing one-loop integrals*,

$$\int \frac{d^nq}{(2\pi)^n} \frac{q^\mu}{(q^2+2q\cdot p - m^2 + i\varepsilon)^r} = -i(-1)^r(p^2+m^2)^{2-\epsilon-r}(4\pi)^{\epsilon-2}\,\frac{\Gamma(\epsilon+r-2)}{\Gamma(r)}\,p^\mu\,, \tag{82}$$

it follows that

$$\begin{aligned}
B^\mu(p^2; m_1^2, m_2^2) &= -p^\mu(4\pi\mu^2)^\epsilon\,\Gamma(\epsilon)\int_0^1 \left[p^2x^2 - (p^2+m_1^2-m_2^2)x + m_1^2 - i\varepsilon\right]^{-\epsilon} x\,dx \\
&= -p^\mu\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right)\left[\frac{1}{2} - \epsilon\int_0^1 \ln\left(\frac{p^2x^2-(p^2+m_1^2-m_2^2)x+m_1^2-i\varepsilon}{\mu^2}\right)x\,dx\right] \\
&= -p^\mu\left\{\tfrac{1}{2}\Delta - \int_0^1 \ln\left(\frac{p^2x^2-(p^2+m_1^2-m_2^2)x+m_1^2-i\varepsilon}{\mu^2}\right)x\,dx\right\}, \\
&= p^\mu B_1(p^2; m_1^2, m_2^2)\,,
\end{aligned} \tag{83}$$

where

$$B_1(p^2; m_1^2, m_2^2) = -\tfrac{1}{2}\Delta + \int_0^1 \ln\left(\frac{p^2x^2-(p^2+m_1^2-m_2^2)x+m_1^2-i\varepsilon}{\mu^2}\right)x\,dx\,. \tag{84}$$

after expanding in ϵ and dropping all terms of $\mathcal{O}(\epsilon)$.

In order to express B_1 in terms of B_0 and A_0, we return to the definition of B^μ given in eq. (3). By multiplying both sides of eq. (83) by p_μ, it follows that

$$B_1(p^2;m_1^2,m_2^2) = \frac{1}{p^2}\,p\cdot B(p^2;m_1^2,m_2^2) = -\frac{16\pi^2 i}{p^2}\int\frac{d^nq}{(2\pi)^n}\,\frac{p\cdot q}{(q^2-m_1^2+i\varepsilon)[(q+p)^2-m_2^2+i\varepsilon]}\,. \tag{85}$$

To simplify this result, we shall employ the following algebraic identity,

$$p\cdot q = \tfrac{1}{2}\bigl[(q+p)^2-q^2-p^2\bigr] = \tfrac{1}{2}\bigl[(q+p)^2-m_2^2-(q^2-m_1^2)-p^2+m_2^2-m_1^2\bigr]\,. \tag{86}$$

Plugging this result into eq. (85) yields,

$$\begin{aligned}B_1(p^2;m_1^2,m_2^2) &= -\frac{8\pi^2 i}{p^2}\int\frac{d^nq}{(2\pi)^n}\,\frac{1}{q^2-m_1^2+i\varepsilon}+\frac{8\pi^2 i}{p^2}\int\frac{d^nq}{(2\pi)^n}\,\frac{1}{(q+p)^2-m_2^2+i\varepsilon}\\ &\quad+\frac{8\pi^2 i}{p^2}(p^2+m_1^2-m_2^2)\int\frac{d^nq}{(2\pi)^n}\,\frac{1}{(q^2-m_1^2+i\varepsilon)[(q+p)^2-m_2^2+i\varepsilon]}\,.\end{aligned} \tag{87}$$

In the second integral above, we employ a new integration variable, $q'\equiv q+p$. The end result is,

$$\boxed{B_1(p^2;m_1^2,m_2^2) = \frac{1}{2p^2}\bigl[A_0(m_1^2)-A_0(m_2^2)-(p^2+m_1^2-m_2^2)B_0(p^2;m_1^2,m_2^2)\bigr]\,.} \tag{88}$$

(c) In analyzing a one-loop triangle graph, the following loop integral arises,

$$\begin{aligned}&C_0(p_1^2,p_2^2,p^2;m_1^2,m_2^2,m_3^2)\\ &\qquad\equiv -16\pi^2 i\int\frac{d^nq}{(2\pi)^n}\,\frac{1}{(q^2-m_1^2+i\varepsilon)[(q+p_1)^2-m_2^2+i\varepsilon][(q+p_1+p_2)^2-m_3^2+i\varepsilon]}\,,\end{aligned} \tag{89}$$

where $p+p_1+p_2=0$, with all external four-momenta pointing into the triangle.

Find an explicit expression for $C_0(0,0,0;m_1^2,m_2^2,m_3^2)$ under the assumption that all masses m_i are distinct. Repeat your analysis in two special cases: (i) $m_1=m_2\neq m_3$ and (ii) $m_1=m_2=m_3$.

We employ a version of Feynman's trick that is given in the handout, *Useful formulae for computing one-loop integrals*, to write

$$\begin{aligned}&\frac{1}{(q^2-m_1^2+i\varepsilon)[(q+p_1)^2-m_2^2+i\varepsilon][(q+p_1+p_2)^2-m_3^2+i\varepsilon]}\\ &\quad=2\int_0^1 x\,dx\int_0^1 dy\,\frac{1}{\bigl[xy(q^2-m_1^2)+x(1-y)[(q+p_1)^2-m_2^2]+(1-x)[(q+p_1+p_2)^2-m_3^2]+i\varepsilon\bigr]^3}.\end{aligned} \tag{90}$$

The denominator of the integrand simplifies to,

$$D = q^2+2q\cdot\bigl[p_1(1-xy)+p_2(1-x)\bigr]+x\bigl[m_3^2-m_2^2-p_2^2-2p_1\cdot p_2\bigr]-xy(p_1^2+m_1^2-m_2^2)+(p_1+p_2)^2-m_3^2+i\varepsilon\,. \tag{91}$$

Hence, it follows that

$$C_0(p_1^2, p_2^2, p^2; m_1^2, m_2^2, m_3^2) = -32\pi^2 i \int_0^1 x\, dx \int_0^1 dy \int \frac{d^n q}{(2\pi)^n} \frac{1}{D^3}\,, \tag{92}$$

where D is given explicitly by eq. (91). In particular,

$$\begin{aligned} C_0(0,0,0;m_1^2,m_2^2,m_3^2) &= -32\pi^2 i \int_0^1 x\, dx \int_0^1 dy \int \frac{d^n q}{(2\pi)^n} \frac{1}{\left[q^2 - x(m_2^2 - m_3^2) - xy(m_1^2 - m_2^2) - m_3^2 + i\varepsilon\right]^3} \\ &= -\int_0^1 x\, dx \int_0^1 \frac{dy}{x(m_2^2 - m_3^2) + xy(m_1^2 - m_2^2) + m_3^2}\,, \end{aligned} \tag{93}$$

after employing eq. (5) and setting $\epsilon = 0$. The integration over y is straightforward,

$$C_0(0,0,0;m_1^2,m_2^2,m_3^2) = -\frac{1}{m_1^2 - m_2^2} \int_0^1 \ln\left(\frac{xm_1^2 + (1-x)m_3^2}{xm_2^2 + (1-x)m_3^2}\right)\,. \tag{94}$$

Employing eq. (46),

$$\begin{aligned} C_0(0,0,0;m_1^2,m_2^2,m_3^2) &= -\frac{1}{m_1^2 - m_2^2}\left\{\ln\left(\frac{m_1^2}{m_2^2}\right) + \frac{m_3^2}{m_1^2 - m_3^2}\ln\left(\frac{m_1^2}{m_3^2}\right) - \frac{m_3^2}{m_2^2 - m_3^2}\ln\left(\frac{m_2^2}{m_3^2}\right)\right\} \\ &= \frac{(m_1^2 - m_3^2)(m_2^2 - m_3^2)\ln\left(\frac{m_1^2}{m_2^2}\right) + m_3^2(m_2^2 - m_3^2)\ln\left(\frac{m_1^2}{m_3^2}\right) - m_3^2(m_1^2 - m_3^2)\ln\left(\frac{m_2^2}{m_3^2}\right)}{(m_1^2 - m_2^2)(m_2^2 - m_3^2)(m_3^2 - m_1^2)} \\ &= \frac{m_1^2(m_2^2 - m_3^2)\ln m_1^2 + m_2^2(m_3^2 - m_1^2)\ln m_2^2 + m_3^2(m_1^2 - m_2^2)\ln m_3^2}{(m_1^2 - m_2^2)(m_2^2 - m_3^2)(m_3^2 - m_1^2)}\,. \end{aligned} \tag{95}$$

An equivalent form for the above result appears often in the physics literature,

$$\boxed{C_0(0,0,0;m_1^2,m_2^2,m_3^2) = \frac{m_1^2 m_2^2 \ln(m_1^2/m_2^2) + m_2^2 m_3^2 \ln(m_2^2/m_3^2) + m_3^2 m_1^2 \ln(m_3^2/m_1^2)}{(m_1^2 - m_2^2)(m_2^2 - m_3^2)(m_3^2 - m_1^2)}\,.} \tag{96}$$

Note that $C_0(0,0,0;m_1^2,m_2^2,m_3^2)$ given above is invariant under an arbitrary permutation of its arguments, as expected after setting $p_1 = p_2 = p = 0$ in the definition of C_0 given in eq. (89).

If $m \equiv m_1 = m_2 \neq m_3$, we return to eq. (93),

$$\begin{aligned} C_0(0,0,0;m^2,m^2,m_3^2) &= -\int_0^1 \frac{x\, dx}{x(m^2 - m_3^2) + m_3^2} \\ &= -\frac{1}{m^2 - m_3^2}\int_0^1 \frac{x(m^2 - m_3^2) + m_3^2 - m_3^2}{x(m^2 - m_3^2) + m_3^2}\, dx \\ &= -\frac{1}{m^2 - m_3^2}\left[1 - m_3^2 \int_0^1 \frac{dx}{x(m^2 - m_3^2) + m_3^2}\right] \\ &= -\frac{1}{m^2 - m_3^2}\left[1 - \frac{m_3^2}{m^2 - m_3^2}\ln\left(\frac{m^2}{m_3^2}\right)\right]\,. \end{aligned} \tag{97}$$

Finally, if $m \equiv m_1 = m_2 = m_3$, then eq. (93) yields,

$$C_0(0,0,0;m^2,m^2,m^2) = -\frac{1}{2m^2}\,. \tag{98}$$

2. In QED, the renormalization group functions are:

$$\beta(e) = \mu \frac{de_R}{d\mu}\,,$$

$$\delta(e) = \mu \frac{da_R}{d\mu}\,,$$

$$m_R \gamma_m(e) = \mu \frac{dm_R}{d\mu}\,,$$

$$\gamma_i(e) = \tfrac{1}{2}\mu \frac{\partial}{\partial \mu} \ln Z_i \qquad (i = 2, 3)\,.$$

(a) Compute $\beta(e)$, $\delta(e)$, $\gamma_m(e)$, and $\gamma_i(e)$ in the one-loop approximation, using the MS-renormalization scheme.

In class, we showed that the bare and renormalized QED couplings are related by,

$$e = \mu^\epsilon Z_1 Z_2^{-1} Z_3^{-1/2} e_R\,,$$

where in this problem we shall use the subscript R to denote renormalized parameters, whereas quantities without subscripts will denote bare parameters. Using the Ward-Takahashi identity of QED which yields $Z_1 = Z_2$, it follows that

$$e = Z_3^{-1/2} \mu^\epsilon e_R\,.$$

The bare parameters are independent of μ. Hence,

$$0 = \mu \frac{de}{d\mu} = \mu \frac{d}{d\mu}\big(Z_3^{-1/2}\mu^\epsilon e_R\big)\,.$$

In the MS renormalization scheme,

$$Z_3 = 1 + \sum_{k=1}^{\infty} \frac{a_k(e_R)}{\epsilon^k}\,. \tag{99}$$

Using the chain rule of differentiation,

$$\epsilon e_R Z_3^{-1/2} + \beta(e_R, \epsilon)\left(e_R \frac{dZ_3^{-1/2}}{de_R} + Z_3^{-1/2}\right) = 0\,,$$

where

$$\beta(e_R, \epsilon) \equiv \mu \frac{de_R}{d\mu}\,. \tag{100}$$

Noting that

$$\frac{dZ_3^{-1/2}}{de_R} = -\tfrac{1}{2} Z_3^{-3/2} \frac{dZ_3}{de_R}\,,$$

it follows that

$$\left[\beta(e_R, \epsilon) + \epsilon e_R - \tfrac{1}{2} e_R \beta(e_R, \epsilon) Z_3^{-1} \frac{d}{de_R}\right] Z_3 = 0\,. \tag{101}$$

Inserting the expansion of Z_3 given in eq. (99), it follows that a solution that is consistent with the $1/\epsilon$ expansion of Z_3 is

$$\beta(e_R,\epsilon) = -\epsilon e_R + \beta(e_R)\,, \tag{102}$$

where $\beta(e_R)$ is independent of ϵ. In particular,

$$\beta(e_R) = \lim_{\epsilon\to 0}\beta(e_R,\epsilon)\,. \tag{103}$$

Eq. (101) can therefore be written as

$$\left[\beta(e_R) - \tfrac{1}{2}e_R\beta(e_R)Z_3^{-1}\frac{d}{de_R} + \tfrac{1}{2}\epsilon e_R^2 Z_3^{-1}\frac{d}{de_R}\right] Z_3 = 0\,.$$

Inserting eq. (99), and performing a formal expansion in $1/\epsilon$, we deduce that all coefficients of $1/\epsilon^k$ should vanish. Of particular interest to us here is the coefficient corresponding to $k=0$. In particular, we may take $Z_3^{-1}=1$ in the coefficient of the $k=0$ equation, in which case,

$$\beta(e_R) + \tfrac{1}{2}e_R^2\frac{da_1}{de_R} = 0\,. \tag{104}$$

In class, we obtained the following one-loop result for Z_3 in the MS renormalization scheme,

$$Z_3 = 1 - \frac{e_R^2}{12\pi^2\epsilon}\,. \tag{105}$$

That is, we can identify $a_1 = -e_R^2/(12\pi^2)$, and eq. (104) yields

$$\beta(e_R) = \frac{e_R^3}{12\pi^2}\,, \tag{106}$$

in the one-loop approximation.

Next we compute γ_m. The starting point is

$$m = Z_m m_R\,.$$

Again, we note that the bare mass is independent of μ. Hence,

$$0 = \mu\frac{dm}{d\mu} = \mu\frac{d}{d\mu}\big(Z_m m_R\big) = m_R\mu\frac{dZ_m}{d\mu} + Z_m\mu\frac{dm_R}{d\mu}\,.$$

By definition,

$$m_R\gamma_m(e_R) = \mu\frac{dm_R}{d\mu}\,.$$

Thus, using the chain rule, we can write

$$\mu\frac{de_R}{d\mu}\,\frac{dZ_m}{de_R} + \gamma_m(e_R)Z_m = 0\,.$$

Using eqs. (100) and (102),

$$\big[\beta(e_R) - \epsilon e_R\big]\frac{dZ_m}{de_R} + \gamma_m(e_R)Z_m = 0\,. \tag{107}$$

In the MS renormalization scheme,

$$Z_m = 1 + \sum_{k=1}^{\infty} \frac{b_k(e_R)}{\epsilon^k} \,. \tag{108}$$

Inserting this expansion into eq. (107), we can extract the equation corresponding to $k = 0$,

$$\gamma_m(e_R) - e_R \frac{db_1}{de_R} = 0 \,. \tag{109}$$

In class, we computed Z_m in the one-loop approximation in the MS scheme,

$$Z_m = 1 - \frac{3e_R^2}{16\pi^2\epsilon} \,.$$

That is we can identify $b_1 = -3e_R^2/(16\pi^2)$, in which case eq. (109) yields

$$\gamma_m(e_R) = -\frac{3e_R^2}{8\pi^2} \,.$$

Next, we present the one-loop computation of

$$\gamma_i(e_R) \equiv \tfrac{1}{2}\mu \frac{d}{d\mu} \ln Z_i = \tfrac{1}{2} Z_i^{-1} \mu \frac{dZ_i}{d\mu} \,, \qquad \text{for } i = 2, 3 \,. \tag{110}$$

In the MS renormalization scheme,

$$Z_2 = 1 + \sum_{k=1}^{\infty} \frac{c_k(e_R)}{\epsilon^k} \,. \tag{111}$$

Thus, using the chain rule,

$$\gamma_2(e_R) = \tfrac{1}{2} Z_2 \mu \frac{de_R}{d\mu} \frac{dZ_2}{de_R} \,.$$

Using eqs. (100) and (102),

$$\gamma_2(e_R) = \tfrac{1}{2} Z_2 \big[\beta(e_R) - \epsilon e_R\big] \frac{dZ_2}{de_R} \,. \tag{112}$$

Inserting eq. (111) into eq. (112), we can extract the equation corresponding to $k = 0$ by setting $Z_2 = 1$,

$$\gamma_2(e_R) = -\tfrac{1}{2} e_R \frac{dc_1}{de_R} \,. \tag{113}$$

A similar analysis yields

$$\gamma_3(e_R) = -\tfrac{1}{2} e_R \frac{da_1}{de_R} \,. \tag{114}$$

In class, we computed Z_2 in the one-loop approximation in the MS scheme,

$$Z_2 = 1 - \frac{e_R^2}{16\pi^2\epsilon} \,.$$

That is, we can identify $c_1 = -e_R^2/(16\pi^2)$, and eq. (109) yields

$$\gamma_2(e_R) = \frac{e_R^2}{16\pi^2}\,.$$

Likewise, using $a_1 = -e_R^2/(12\pi^2)$ [as noted below eq. (105)],

$$\gamma_3(e_R) = \frac{e_R^2}{12\pi^2}\,. \tag{115}$$

Finally, we compute δ. The starting point is

$$a = Z_a a_R = Z_3 a_R\,,$$

where we have employed the Ward identity $Z_a = Z_3$ derived in class. Hence, following the well known procedure,

$$0 = \mu\frac{da}{d\mu} = \mu\frac{d}{d\mu}\big(Z_3 a_R\big) = \mu a_R\frac{dZ_3}{d\mu} + Z_3\mu\frac{da_R}{d\mu}\,.$$

By definition,

$$\delta(e_R) = \mu\frac{da_R}{d\mu}\,.$$

Thus, it follows that

$$\mu a_R\frac{dZ_3}{d\mu} + \delta(e_R)Z_3 = 0\,.$$

Solving for $\delta(e_R)$,

$$\delta(e_R) = -\mu a_R\frac{d}{d\mu}\ln Z_3 = -2a_R\gamma_3(e_R)\,,$$

after using eq. (110) for $\gamma_3(e_R)$. Using eq. (115), it follows that in the one-loop approximation in the MS renormalization scheme,

$$\delta(e_R) = -\frac{a_R e_R^2}{6\pi^2}\,.$$

This completes the one-loop calculation of the renormalization group functions of QED in the MS renormalization scheme.

(b) The running coupling constant in QED can be written as:

$$\overline{\alpha}(Q) = \frac{3\pi}{\ln(\Lambda^2/Q^2)}\,, \tag{116}$$

in the one loop approximation. Using the boundary condition $\overline{\alpha}(\mu) \equiv e_R^2/4\pi$, express Λ in terms of μ and e_R. Show that Λ is a renormalization group invariant, that is:

$$\mu\frac{d\Lambda}{d\mu} = 0\,.$$

Evaluate Λ numerically. What is the physical significance of Λ?

In class, we showed that the running coupling constant of QED in the one-loop approximation was given by

$$\overline{\alpha}(Q) = \frac{\alpha_R}{1 - \frac{2\alpha_R}{3\pi}\ln\left(\frac{Q}{\mu}\right)}\,,$$

where $\alpha_R \equiv \overline{\alpha}(\mu)$. Comparing this with eq. (116), it follows that

$$\frac{2}{3\pi}\ln\left(\frac{\Lambda}{Q}\right) = \frac{1}{\alpha_R} - \frac{2}{3\pi}\ln\left(\frac{Q}{\mu}\right)\,.$$

Simplify this expression yields

$$\frac{2}{3\pi}\ln\left(\frac{\Lambda}{\mu}\right) = \frac{1}{\alpha_R}\,.$$

Hence,

$$\Lambda = \mu\exp\left(\frac{3\pi}{2\alpha_R}\right)\,.$$

To show that λ is formally independent of μ, we evaluate,

$$\mu\frac{d\Lambda}{d\mu} = \mu\exp\left(\frac{3\pi}{2\alpha_R}\right)\left[1 + \tfrac{3}{2}\pi\mu\frac{d\alpha_R^{-1}}{d\mu}\right]\,. \tag{117}$$

However, note that

$$\mu\frac{d\alpha_R^{-1}}{d\mu} = 4\pi\mu\frac{d}{d\mu}\left(\frac{1}{e_R^2}\right) = -\frac{8\pi\mu}{e_R^3}\frac{de_R}{d\mu} = -\frac{8\pi\beta(e_R)}{e_R^3} = -\frac{2}{3\pi}\,,$$

where we have used eqs. (101) and (103), and have employed the one-loop approximation for the β-function given in eq. (106). Inserting this result back in eq. (117), we end up with

$$\mu\frac{d\Lambda}{d\mu} = 0\,.$$

Since Λ is independent of μ, we conclude that it is a physically measurable observable of QED. To see what its numerical value, recall that eq. (143) of the Solutions to Problem Set 2 implies that $\overline{\alpha}(m_e) = \alpha_{\rm OS} \simeq 1/137$. It follows that

$$\Lambda = m_e\exp\left(\tfrac{3}{2}\pi\cdot 137\right) \simeq 10^{277}~\text{GeV}\,.$$

This is the same Λ that was obtained in eq. (147) of the Solutions to Problem Set 2, which is called the Landau pole and corresponds to the energy scale at which the one-loop approximation to the QED running coupling blows up. Indeed, Λ is a physical quantity that indicates the energy scale at which the description of QED as a weakly coupled theory breaks down.

ADDITIONAL REMARKS:

The definition of Λ above is based on the one-loop approximation. In fact, it is not difficult to define a μ-independent Λ to all orders in perturbation theory. We begin with the formal definition of the running coupling,

$$s\frac{\partial\overline{e}(s)}{\partial s} = \beta\big(\overline{e}(s)\big)\,, \qquad \text{where } \overline{e}(s=1) = e_R\,.$$

Integrating this equation and putting $s = Q/\mu$,

$$\ln\left(\frac{Q}{\mu}\right) = \int_{e_R}^{\overline{e}(Q)} \frac{de}{\beta(e)}\,. \tag{118}$$

Let us define the indefinite integral

$$G(e) \equiv \int^{e} \frac{de'}{\beta(e')}\,.$$

Then, eq. (118) can be rewritten as

$$\ln\left(\frac{Q}{\mu}\right) = G\big(\overline{e}(Q)\big) - G(e_R)\,. \tag{119}$$

We now define Λ via the equation

$$\ln\left(\frac{\Lambda}{\mu}\right) = -G(e_R)\,.$$

Inserting this result back into eq. (119) yields

$$\ln\left(\frac{Q}{\Lambda}\right) = G\big(\overline{e}(Q)\big)\,. \tag{120}$$

No perturbative approximation has been made here. Moreover, Λ defined via eq. (120) is explicitly independent of μ. Finally, it is straightforward to check that in the one-loop approximation, we recover our previous results.

(c) Find the relation between the $\overline{\rm MS}$ mass parameter, m_R, and the physical electron mass m_e (i.e., the pole mass) in the one-loop approximation.

In class, we derived

$$\Sigma(p) = -\not{p}\left\{Z_2 - 1 + \frac{\alpha_R}{2\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)(1-\epsilon)\int_0^1 dx\,(1-x)x^{-\epsilon}\left[\frac{m_R^2 - p^2(1-x)}{\mu^2}\right]^{-\epsilon}\right\}$$
$$+m_R\left\{Z_m Z_2 - 1 + \frac{\alpha_R}{2\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)(2-\epsilon)\int_0^1 dx\,x^{-\epsilon}\left[\frac{m_R^2 - p^2(1-x)}{\mu^2}\right]^{-\epsilon}\right\}. \tag{121}$$

In the $\overline{\rm MS}$ renormalization scheme,

$$Z_2 = 1 - \frac{\alpha_R}{4\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\,,$$

$$Z_2 Z_m = 1 - \frac{\alpha_R}{4\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\,.$$

Inserting these results back into eq. (121) yields,

$$\Sigma(p) = -\not{p}\frac{\alpha_R}{4\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)(A-1) + \frac{m_R\alpha_R}{\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)(B-1)\,, \tag{122}$$

where A and B are the following loop integrals,

$$A \equiv 2(1-\epsilon)\int_0^1 dx\,(1-x)x^{-\epsilon}\left[\frac{m_R^2 - p^2(1-x)}{\mu^2}\right]^{-\epsilon}$$

$$B \equiv (1-\tfrac{1}{2}\epsilon)\int_0^1 dx\,x^{-\epsilon}\left[\frac{m_R^2 - p^2(1-x)}{\mu^2}\right]^{-\epsilon}\,.$$

Expanding about $\epsilon = 0$,

$$A = 1 - \epsilon\left\{1 + 2\int_0^1 (1-x)\ln x\,dx + 2\int_0^1 (1-x)\ln\left[\frac{m_R^2 - p^2(1-x)}{\mu^2}\right]\,dx\right\} + \mathcal{O}(\epsilon^2)\,,$$

$$B = 1 - \epsilon\left\{\frac{1}{2} + \int_0^1 \ln x\,dx + \int_0^1 \ln\left[\frac{m_R^2 - p^2(1-x)}{\mu^2}\right]\,dx\right\} + \mathcal{O}(\epsilon^2)\,.$$

We record below the relevant integrals:

$$\int_0^1 (1-x)\ln x = -\frac{3}{4}\,,$$

$$\int_0^1 \ln x\,dx = -1\,,$$

$$\int_0^1 \ln\left[\frac{m_R^2 - p^2(1-x)}{\mu^2}\right]\,dx = \frac{m_R^2}{p^2}\ln\left(\frac{m_R^2}{\mu^2}\right) + \left(1 - \frac{m_R^2}{p^2}\right)\ln\left(\frac{m_R^2 - p^2}{\mu^2}\right) - 1\,.$$

$$\int_0^1 (1-x)\ln\left[\frac{m_R^2 - p^2(1-x)}{\mu^2}\right]\,dx = \frac{m_R^2}{2p^4}\ln\left(\frac{m_R^2}{\mu^2}\right) + \frac{1}{2}\left(1 - \frac{m_R^4}{p^4}\right)\ln\left(\frac{m_R^2 - p^2}{\mu^2}\right) - \frac{1}{4} - \frac{m_R^2}{2p^2}\,.$$

Inserting these results into the expressions for A and B and performing some simplification yields,

$$A = 1 + \epsilon\left[1 + \frac{m_R^2}{p^2} - \ln\left(\frac{m_R^2 - p^2}{\mu^2}\right) + \frac{m_R^4}{p^4}\ln\left(1 - \frac{p^2}{m_R^2}\right)\right] + \mathcal{O}(\epsilon^2)\,,$$

$$B = 1 + \epsilon\left[\frac{3}{2} - \ln\left(\frac{m_R^2 - p^2}{\mu^2}\right) + \frac{m_R^2}{p^2}\ln\left(1 - \frac{p^2}{m_R^2}\right)\right] + \mathcal{O}(\epsilon^2)\,.$$

Using these explicit expressions for A and B in eq. (122),

$$\Sigma(p) = -\not{p}\frac{\alpha_R}{4\pi}\left[1 + \frac{m_R^2}{p^2} - \ln\left(\frac{m_R^2 - p^2}{\mu^2}\right) + \frac{m_R^4}{p^4}\ln\left(1 - \frac{p^2}{m_R^2}\right)\right]$$
$$+\frac{m_R\alpha_R}{\pi}\left[\frac{3}{2} - \ln\left(\frac{m_R^2 - p^2}{\mu^2}\right) + \frac{m_R^2}{p^2}\ln\left(1 - \frac{p^2}{m_R^2}\right)\right]\,,$$

after taking the $\epsilon \to 0$ limit.

The one-loop correction to the inverse propagator is

$$\begin{aligned}\Gamma^{(2)}(p) &= \not{p} - m_R - \Sigma(p) \\ &= \not{p}\left\{1+\frac{\alpha_R}{4\pi}\left[1+\frac{m_R^2}{p^2}-\ln\left(\frac{m_R^2-p^2}{\mu^2}\right)+\frac{m_R^4}{p^4}\ln\left(1-\frac{p^2}{m_R^2}\right)\right]\right\} \\ &\qquad -m_R\left\{1+\frac{\alpha_R}{\pi}\left[\frac{3}{2}-\ln\left(\frac{m_R^2-p^2}{\mu^2}\right)+\frac{m_R^2}{p^2}\ln\left(1-\frac{p^2}{m_R^2}\right)\right]\right\}\,. \end{aligned} \tag{123}$$

In this expression $m_R \equiv m_R(\mu)$ is the renormalized mass, which differs from the physical pole mass. The definition of the $\overline{\text{MS}}$ mass is obtained by setting $\mu = m_R$. That is, the $\overline{\text{MS}}$ mass is defined as $m_R(m_R)$. Thus, we set $\mu = m_R$ in eq. (123) and obtain,

$$\begin{aligned}\Gamma^{(2)}(p) &= \not{p}\left\{1+\frac{\alpha_R}{4\pi}\left[1+\frac{m_R^2}{p^2}-\left(1-\frac{m_R^4}{p^4}\right)\ln\left(1-\frac{p^2}{m_R^2}\right)\right]\right\} \\ &\qquad -m_R\left\{1+\frac{\alpha_R}{\pi}\left[\frac{3}{2}-\left(1-\frac{m_R^2}{p^2}\right)\ln\left(1-\frac{p^2}{m_R^2}\right)\right]\right\}\,, \end{aligned} \tag{124}$$

where $m_R \equiv m_R(m_R)$.

The physical pole mass, denoted by m_e, corresponds to a zero of the inverse propagator. That is, m_e is defined by the condition

$$\Gamma^{(2)}(p)\Big|_{\not{p}=m_e} = 0\,. \tag{125}$$

One can expand the $\overline{\text{MS}}$ mass perturbatively in terms of the physical mass m_e,

$$m_R(m_R) = m_e\left[1+\frac{\alpha_R}{\pi}\kappa + \mathcal{O}(\alpha_R^2)\right]\,. \tag{126}$$

Inserting this into eq. (124) and imposing the condition specified in eq. (125), we can solve for κ. At one-loop accuracy,

$$\begin{aligned}\Gamma^{(2)}(p) &= \not{p}\left\{1+\frac{\alpha_R}{4\pi}\left[1+\frac{m_e^2}{p^2}-\left(1-\frac{m_e^4}{p^4}\right)\ln\left(1-\frac{p^2}{m_e^2}\right)\right]\right\} \\ &\qquad -m_e\left\{1+\frac{\alpha_R}{\pi}\left[\kappa+\frac{3}{2}-\left(1-\frac{m_e^2}{p^2}\right)\ln\left(1-\frac{p^2}{m_e^2}\right)\right]\right\}\,, \end{aligned} \tag{127}$$

Setting $\not{p} = m$ and $p^2 = \not{p}\not{p} = m^2$, we end up with

$$\frac{\alpha_R}{2\pi} - \frac{\alpha_R}{\pi}\left(\kappa+\tfrac{3}{2}\right) = 0\,.$$

It follows that $\kappa = -1$. Inserting this result back into eq. (126), we conclude that the relation between the $\overline{\text{MS}}$ mass and the physical mass m_e is given to one-loop accuracy by

$$m_R(m_R) = m_e\left(1-\frac{\alpha_R}{\pi}\right)\,.$$

The inverse relation can also be obtained to one-loop accuracy,

$$m_e = m_R(m_R)\left(1+\frac{\alpha_R}{\pi}\right)\,.$$

3. In this problem, you will investigate the behavior of the renormalization group functions in QED under a change of renormalization scheme. You should assume throughout the problem that you are working in a class of renormalization schemes that are mass-independent. In particular, if e_1 and e_2 are coupling constants defined in two different schemes, then I can expand one in the other, e.g.,

$$e_1 = e_2 + Ae_2^3 + \ldots \tag{128}$$

for some appropriate mass-independent coefficient A.

(a) Show that there is a one-to-one correspondence between the fixed points [i.e., the zeros of $\beta(e)$] of both schemes, and the value of the first derivative of $\beta(e)$ at the corresponding fixed point is independent of scheme.

In this problem, we consider a class of mass-independent renormalization schemes. Then, if e_1 and e_2 are coupling constants in two different renormalization schemes, then it must be possible to relate the two couplings as indicated in eq. (128). Using the definition of the β-function, it follows that the corresponding β-functions obtained in the two renormalization schemes are,

$$\beta_1(e_1) = \mu\frac{de_1}{d\mu}\,, \tag{129}$$

$$\beta_2(e_2) = \mu\frac{de_2}{d\mu} = \mu\frac{de_1}{d\mu}\frac{de_2}{de_1} = \beta_1(e_1)\frac{de_2}{de_1}\,. \tag{130}$$

Hence, if there exists a zero of $\beta_1(e_1)$, then there is a corresponding zero in $\beta_2(e_2)$, since de_2/de_1 does not blow up in light of eq. (128). In particular, if $\beta_1(e_1^*) = 0$, then $\beta_2(e_2^*) = 0$, where e_1^* and e_2^* are perturbatively related by eq. (128).

If we take the derivative of the β-function with respect to the coupling and evaluate it at the fixed point (where the β function vanishes), then

$$\begin{aligned}\left.\frac{d\beta_2}{de_2}\right|_{e_2=e_2^*} = \left.\frac{d}{de_2}\left[\beta_1(e_1)\frac{de_2}{de_1}\right]\right|_{e_2=e_2^*} &= \left.\frac{d\beta_1(e_1)}{de_2}\frac{de_2}{de_1}\right|_{e_2=e_2^*} + \beta_1(e_1)\frac{d}{de_2}\left.\left(\frac{de_2}{de_1}\right)\right|_{e_2=e_2^*} \\ &= \left.\frac{d\beta_1(e_1)}{de_1}\frac{de_1}{de_2}\frac{de_2}{de_1}\right|_{e_2=e_2^*} + \beta_1(e_1)\frac{d}{de_2}\left.\left(\frac{de_2}{de_1}\right)\right|_{e_2=e_2^*} \\ &= \left.\frac{d\beta_1(e_1)}{de_1}\right|_{e_1=e_1^*} + \beta_1(e_1^*)\frac{d}{de_2}\left.\left(\frac{de_2}{de_1}\right)\right|_{e_2=e_2^*}.\end{aligned} \tag{131}$$

where we have used that result obtained above that $e_1 = e_1^*$ when $e_2 = e_2^*$. Since $\beta_1(e_1^*) = 0$, eq. (131) yields

$$\left.\frac{d\beta_2}{de_2}\right|_{e_2=e_2^*} = \left.\frac{d\beta_1(e_1)}{de_1}\right|_{e_1=e_1^*}, \tag{132}$$

as was to be shown.

(b) Show that the values of γ_m and γ_i ($i = 2, 3$) at the corresponding fixed points [as defined in part (a)] are independent of scheme.

Changing renormalization schemes amounts to a finite renormalization. That is, the ratio of renormalization constants defined in two different schemes must be finite and perturbatively related. Schematically, one can write,

$$\frac{Z_{m1}(e_1)}{Z_{m2}(e_2)} = 1 + A_m e_2^2 + \cdots \,, \tag{133}$$

$$\frac{Z_{i1}(e_1)}{Z_{i2}(e_2)} = 1 + A_i e_2^2 + \cdots \,, \tag{134}$$

where Z_{m1} and Z_{m2} are the multiplicative mass renormalization constants in the two schemes and Z_{i1} and Z_{i2} are the wave function renormalization constants in the two schemes (for $i = 2, 3$ corresponding to the electron and photon wave function renormalization constants, respectively).

By definition,

$$m_R \gamma_m = \mu \frac{dm_R}{d\mu} \,, \qquad \gamma_i = \tfrac{1}{2}\mu \frac{d}{d\mu} \ln Z_i \,, \quad \text{for } i = 2, 3. \tag{135}$$

Hence,

$$\begin{aligned} \gamma_{i1}(e_1) = \tfrac{1}{2}\mu \frac{d}{d\mu} \ln Z_{i1}(e_1) &= \tfrac{1}{2}\mu \frac{d}{d\mu} \left[\ln Z_{i2}(e_2) + \ln(1 + A_i e_2^2 + \cdots) \right] \\ &= \gamma_{i2}(e_2) + \tfrac{1}{2} A_i \, \mu \frac{d}{d\mu}(e_2^2 + \cdots) = \gamma_{i2}(e_2) + \beta_2(e_2)\left[A_i e_2 + \cdots\right] . \end{aligned} \tag{136}$$

At the fixed point, $\beta_2(e_2^*) = 0$, and it follows that

$$\gamma_{i1}(e_1^*) = \gamma_{i2}(e_2^*) \,, \tag{137}$$

since $e_1 = e_1^*$ when $e_2 = e_2^*$ as noted in part (a).

A similar analysis applies in the case of γ_m. Using $m = Z_m m_R$, it follows that

$$0 = \mu \frac{dm}{d\mu} = \mu \frac{d}{d\mu}(Z_m m_R) = m_R \mu \frac{dZ_m}{d\mu} + Z_m \mu \frac{dm_R}{d\mu} = m_R \mu \frac{dZ_m}{d\mu} + m_R \gamma_m Z_m \,. \tag{138}$$

Hence,

$$\gamma_m = -\frac{\mu}{Z_m} \frac{dZ_m}{d\mu} = -\mu \frac{d}{d\mu} \ln Z_m \,, \tag{139}$$

which has a similar form to the definition of γ_i given in eq. (135). Thus, we can simply repeat the steps given in eqs. (136) and (137) and conclude that

$$\gamma_{m1}(e_1^*) = \gamma_{m2}(e_2^*) \,, \tag{140}$$

(c) One can compute $\beta(e)$ as a power series in e in perturbation theory. Show that the coefficients of the first two terms are independent of scheme, but the coefficient of all succeeding terms are scheme-dependent.

The perturbative expansion of the QED β function has the form,

$$\beta_1(e_1) = b_0 e_1^3 + b_1 e_1^5 + \mathcal{O}(e_1^7) \,. \tag{141}$$

Then, as noted in part (a),

$$\beta_2(e_2) = \beta_1(e_1)\frac{de_2}{de_1}\,. \tag{142}$$

Using eq. (128),

$$\frac{de_2}{de_1} = \frac{1}{de_1/de_2} = \frac{1}{1+3Ae_2^2+\cdots} = 1-3Ae_2^2+\cdots\,. \tag{143}$$

Plugging this result back into eq. (142),

$$\begin{aligned}\beta_2(e_2) &= (b_0e_1^3+b_1e_1^5+\cdots)(1-3Ae_2^2+\cdots)\\ &= \bigl[b_0(e_2+Ae_2^3+\cdots)^3+b_1(e_2+Ae_2^3+\cdots)^5+\cdots\bigr](1-3Ae_2^2+\cdots)\\ &= \bigl[b_0e_2^3(1+3Ae_2^2)+b_1e_2^5\bigr](1-3Ae_2^2)+\mathcal{O}(e_2^7)\\ &= b_0e_2^3+b_1e_2^5+\mathcal{O}(e_2^7)\,. \end{aligned} \tag{144}$$

That is, the first two coefficients of the β-function are scheme-independent.

(d) Likewise, if one computes γ_m and γ_i ($i=2,3$) in perturbation theory, show that only the leading terms are scheme-independent, whereas all higher order terms are scheme-dependent.

In eq. (136), we obtained

$$\gamma_{i1}(e_1) = \gamma_{i2}(e_2)+\beta_2(e_2)\bigl[A_ie_2+\mathcal{O}(e_2^3)\bigr]\,. \tag{145}$$

The perturbative expansion of the QED γ_i functions have the form,

$$\gamma_{i2}(e_2) = \gamma_0e_2^2+\mathcal{O}(e_2^4)\,. \tag{146}$$

Eqs. (145) and (146) then yield,

$$\gamma_{i1}(e_1) = \gamma_0e_2^2+\mathcal{O}(e_2^4)+\bigl[b_0e_2^3+\mathcal{O}(e_2^5)\bigr]\bigl[A_ie_2+\mathcal{O}(e_2^3)\bigr]\,, \tag{147}$$

after employing eq. (144). If we now use eq. (128) to express the right hand side of eq. (147) in terms of e_1, it follows that

$$\gamma_{i1}(e_1) = \gamma_0e_1^2+\mathcal{O}(e_1^4)\,, \tag{148}$$

where the $\mathcal{O}(e_1^4)$ terms clearly differ from the $\mathcal{O}(e_2^4)$ terms in eq. (146). A similar derivation leads to the same conclusion regarding γ_m. Thus, only the lowest order term in the perturbative expansion of γ_m and γ_i ($i=2,3$) are scheme-independent.

4. Consider QED coupled to a neutral scalar field:

$$\mathscr{L} = \mathscr{L}_{QED}+\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{2}m^2\phi^2-\frac{\lambda}{4!}\phi^4-g\overline{\psi}\psi\phi\,. \tag{149}$$

Define a separate β-function of each coupling constant: β_e, β_g and β_λ.

The model described by eq. (149) possesses three couplings: e, λ and g. The corresponding renormalization constants can be defined as follows,

$$\phi = Z_\phi^{1/2}\phi_R\,, \tag{150}$$

$$\lambda = \mu^{2\epsilon} Z_\lambda \lambda_R\,, \tag{151}$$

$$\psi = Z_2^{1/2}\psi_R\,, \tag{152}$$

$$A^\mu = Z_3^{1/2} A_R^\mu\,, \tag{153}$$

$$e = \mu^\epsilon Z_1 Z_2^{-1} Z_3^{-1/2} e_R\,, \tag{154}$$

where renormalized quantities are designated with a subscript R, whereas bare quantities have no corresponding subscript. In addition, it is convenient to introduce the vertex renormalization constant Z_4, which is defined such that

$$g\overline{\psi}\psi\phi = \mu^\epsilon Z_4 g_R \overline{\psi}_R \psi_R \phi_R\,. \tag{155}$$

However the left hand side of eq. (155) can also be rewritten as,

$$g\overline{\psi}\psi\phi = g Z_2 Z_\phi^{1/2} \overline{\psi}_R \psi_R \phi_R\,. \tag{156}$$

Hence, it follows that

$$g = \mu^\epsilon Z_4 Z_2^{-1} Z_\phi^{-1/2} g_R\,. \tag{157}$$

(a) Is the QED Ward identity, $Z_1 = Z_2$, modified in this theory? At one-loop, will β_e be the same or different from what you obtained in problem 2?

The Ward identity, $Z_1 = Z_2$ is not modified. There are many ways to see this. For example, consider the derivation of the Ward identity presented in the class handout entitled *Current Conservation and the QED Ward Identity*. The addition of the new terms added to the QED Lagrangian given in eq. (149) does not modify the gauge symmetry since the field ϕ is neutral. In particular, the conserved Noether current, $j_\mu(x) = -e\bar{\psi}\gamma_\mu\psi$ is unchanged, Hence, the derivation of $Z_1 = Z_2$ given in this class handout is not modified. Note that the QED computations of Z_1 and Z_2 receive additional contributions from the exchange of the scalar field. However, as shown in the Appendix to this problem, the scalar contributions to Z_1 and Z_2 are equal.

Employing $Z_1 = Z_2$ in eq. (154) yields,

$$e = \mu^\epsilon Z_3^{-1/2} e_R\,. \tag{158}$$

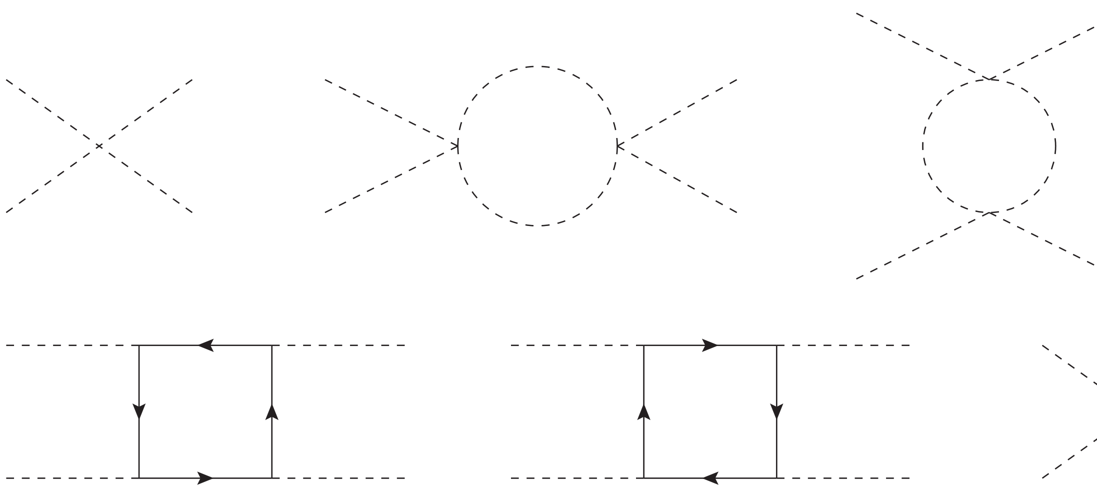
Hence, the derivation of eq. (106) is unchanged, since there are no contributions from the neutral scalar to the self energy of the photon in the one-loop approximation that would modify the computation of Z_3.

(b) Compute β_g and β_λ, assuming that λ is of order g^2. Work consistently to lowest nontrivial order in perturbation theory.

We now turn to the computation of β_g and β_λ, assuming that λ is of order g^2. To compute these quantities, we will need to compute Z_ϕ, Z_λ, Z_2, Z_4. In fact, we have already computed Z_ϕ in problem 1 of Problem Set 2. Thus, in the MS renormalization scheme, eq. (12) of the Solutions to Problem Set 2 yields,

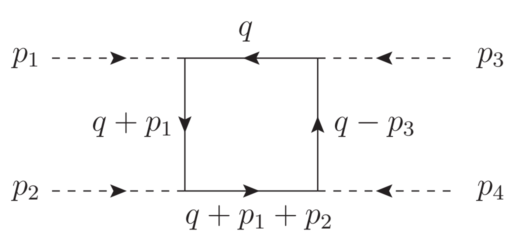
$$Z_\phi = 1 - \frac{g_R^2}{8\pi^2\epsilon}\,. \tag{159}$$

Consider the computation of Z_λ, which requires us to examine the Green function with four external scalar fields. The relevant Feynman diagrams in the one-loop approximation are:



Dashed lines represent scalars and the directed arrows represent fermions. In class, we have already evaluated the four diagrams exhibited in the first line above (consisting of the tree-level contribution plus the one-loop s, t and u-channel contributions, respectively, where "crossed" indicates the u-channel diagram (not shown) in which two external scalar lines of the t-channel graph are crossed. The diagram with the $+$ at the vertex is the four-point counterterm. In order to compute Z_λ in the MS renormalization scheme, all we must do is isolate the divergences of the one-loop diagrams.

Let us focus on the first box diagram above,



where the four-momenta are labeled and flow in the direction of the arrows as indicated. Employing the Feynman rules, the diagram above is given by

$$-(-ig_R\mu^\epsilon)^4\int\frac{d^nq}{(2\pi)^n}\,\mathrm{Tr}\left\{\left(\frac{i(\not{q}+M)}{q^2-M^2}\right)\left(\frac{i(\not{q}-\not{p}_3+M)}{(q-p_3)^2-M^2}\right)\left(\frac{i(\not{q}+\not{p}_1+\not{p}_2+M)}{(q+p_1+p_2)^2-M^2}\right)\left(\frac{i(\not{q}+\not{p}_1+M)}{(q+p_1)^2-M^2}\right)\right\}, \tag{160}$$

where we have included an overall minus sign for the closed fermion loop. In eq. (160), M denotes the fermion mass.[4] In order to identify the divergent term of this loop integral, it is sufficient to retain only the $\not{q}$ terms in the numerator. Noting that

$$\mathrm{Tr}\,\not{q}\not{q}\not{q}\not{q} = 4q^4, \tag{161}$$

it suffices to examine,

$$-4g_R^4\mu^{4\epsilon}\int\frac{d^nq}{(2\pi)^n}\frac{q^4}{(q^2-M^2)\big[(q-p_3)^2-M^2\big]\big[(q+p_1+p_2)^2-M^2\big]\big[(q+p_1)^2-M^2\big]}\,. \tag{162}$$

Writing $q^4 = q^2(q^2-M^2)+M^2q^2$, we can drop the M^2q^2 piece, which does not contribute to the divergence, and likewise we can set $\mu^\epsilon = 1$. We are then left with

$$-4g_R^4\int\frac{d^nq}{(2\pi)^n}\frac{q^2}{\big[(q-p_3)^2-M^2\big]\big[(q+p_1+p_2)^2-M^2\big]\big[(q+p_1)^2-M^2\big]}\,. \tag{163}$$

Writing $q^2 = (q+p_1+p_2)^2 - M^2 + M^2 - q\cdot(p_1+p_2) - (p_1+p_2)^2$, we can drop the last three terms, which do not contribute to the divergence. What remains is

$$-4g_R^4\int\frac{d^nq}{(2\pi)^n}\frac{1}{\big[(q-p_3)^2-M^2\big]\big[(q+p_1)^2-M^2\big]}\,. \tag{164}$$

Finally, changing the integration variable, $q\to q-p_3$ yields,

$$-4g_R^4\int\frac{d^nq}{(2\pi)^n}\frac{1}{(q^2-M^2)\big[(q+p_1+p_3)^2-M^2\big]} = -\frac{ig_R^4}{4\pi^2}B_0\big((p_1+p_3)^2;M^2,M^2\big)\,, \tag{165}$$

where the Passarino-Veltman function, B_0 is defined in eq. (2). In light of eqs. (4) and (55), it follows that the divergent part of the loop integral given in eq. (160) is

$$-\frac{ig_R^4}{4\pi^2\epsilon}\,. \tag{166}$$

It is straightforward to show that the second box diagram (in which the arrows of the fermion lines are reversed) yields the same divergent result given in eq. (166).

The divergence contributed by the pure scalar graphs was computed in class, so we just record that result here,

$$-i\lambda_R\left(1-\frac{3\lambda_R}{32\pi^2\epsilon}\right)\,, \tag{167}$$

where the term proportional to 1 corresponds to the tree diagram and the divergence originates from the sum of the one-loop s, t and u channel diagrams, where each of the three diagrams contributes the same divergence. Finally, we add the diagram that contains the counterterm, whose Feynman rule was given in class by,

$$-i\lambda_R\mu^{2\epsilon}(Z_\lambda Z_\phi^2-1)\,. \tag{168}$$

[4]For notational simplicity, we omit the usual $i\varepsilon$ factors that appear in the denominators of the propagator factors.

Collecting all of the results obtained above, the -1 in eq. (168) cancels the tree-level result, and we are left with,

$$i\Gamma^{(4)} = -i\lambda_R \left(Z_\lambda Z_\phi^2 - \frac{3\lambda_R}{32\pi^2\epsilon} \right) - \frac{ig_R^4}{2\pi^2\epsilon} + \text{finite terms}\,, \tag{169}$$

where Z_ϕ is given in eq. (159). Moreover, at tree-level, $i\Gamma_0^{(4)} = -i\lambda_R$. Hence, inserting the result for Z_ϕ and working consistently within the one-loop approximation, it follows that,

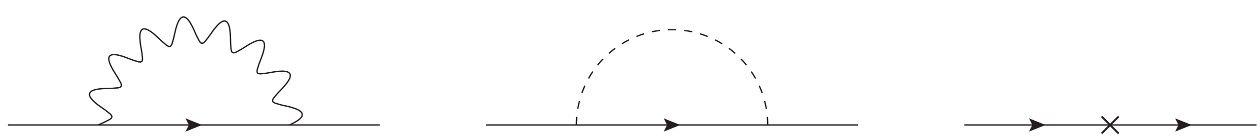
$$i\Gamma^{(4)} = -i\lambda_R \left(Z_\lambda - \frac{g_R^2}{4\pi^2\epsilon} - \frac{3\lambda_R}{32\pi^2\epsilon} \right) - \frac{ig_R^4}{2\pi^2\epsilon} + \text{finite terms}\,. \tag{170}$$

Note that in obtaining this result, we used $g_R^2 Z_\lambda = g_R^2$, since the terms omitted are higher order in the perturbation theory. Since by assumption of the problem, λ_R is of order g_R^2, we see that all terms in eq. (170) are of the same order in the coupling.

The MS renormalization procedure instructs us to choose Z_λ such that the divergences cancel in $i\Gamma^{(4)}$. Thus, we conclude that

$$Z_\lambda = 1 + \frac{1}{\epsilon}\left(\frac{3\lambda_R^2}{32\pi^2} + \frac{g_R^2}{4\pi^2} - \frac{g_R^4}{2\pi^2\lambda_R} \right)\,. \tag{171}$$

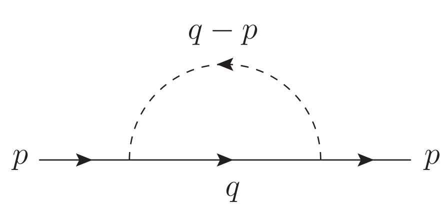
Next, consider the computation of Z_2, which can be obtained by examining the fermion self-energy graphs. The relevant Feynman diagrams in the one-loop approximation are:



The first diagram was computed in class. In the MS renormalization scheme, it yielded

$$i\not{p}\frac{e_R^2}{16\pi^2\epsilon} - iM\frac{e_R^2}{4\pi^2\epsilon}\,, \tag{172}$$

where M is the fermion mass. Thus, we focus on the scalar exchange diagram,



which yields,

$$(-ig_R\mu^\epsilon)^2 \int \frac{d^nq}{(2\pi)^n} \left(\frac{i(\not{q} + M)}{q^2 - M^2} \right) \left(\frac{i}{\left[(q-p)^2 - m^2\right]} \right)\,. \tag{173}$$

Employing the Passarino-Veltman functions defined in eqs. (2) and (3), it follows that

$$g_R^2\mu^{2\epsilon}\int\frac{d^nq}{(2\pi)^n}\frac{\not{q}+M}{(q^2-M^2)\bigl[(q-p)^2-m^2\bigr]}=\frac{ig_R^2\mu^{2\epsilon}}{16\pi^2}\bigl[MB_0(p^2;M,m)-\not{p}B_1(p^2;M,m)\bigr]$$
$$=\frac{ig_R^2}{16\pi^2\epsilon}\bigl(M+\tfrac{1}{2}\not{p}\bigr)+\text{finite terms}\,, \qquad (174)$$

in light of eqs. (4), (55) and (85).

Finally, we add the diagram that contains the counterterm, whose Feynman rule was given in class by,

$$i\not{p}(Z_2-1)-iM(Z_mZ_2-1)\,. \qquad (175)$$

Collecting all of the results obtained above, the factors of -1 in eq. (175) cancel the tree-level result, and we are left with,

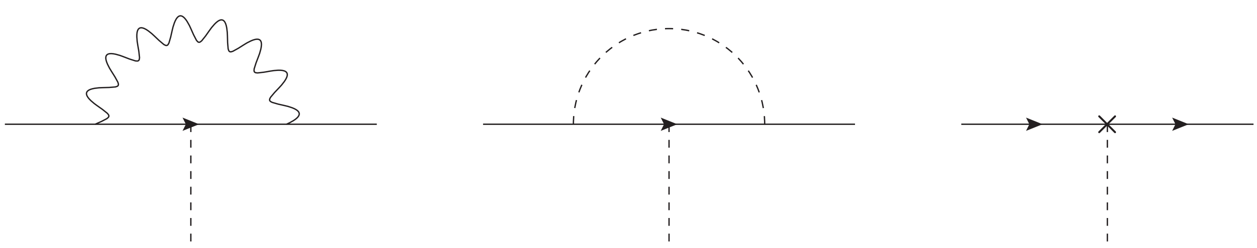
$$i\Gamma^{(2)}(p)=i\not{p}\left[Z_2+\frac{e_R^2}{16\pi^2\epsilon}+\frac{g_R^2}{32\pi^2\epsilon}\right]-iM\left[Z_mZ_2+\frac{e_R^2}{4\pi^2\epsilon}-\frac{g_R^2}{16\pi^2\epsilon}\right]+\text{finite terms}\,. \qquad (176)$$

The MS renormalization procedure instructs us to choose Z_2 and Z_m such that the divergences cancel in $i\Gamma^{(2)}$. Moreover, at tree-level, $i\Gamma^{(2)}(p)$ is equal to the negative of the inverse tree-level fermion propagator, $i(\not{p}-M)$. Thus, we conclude that

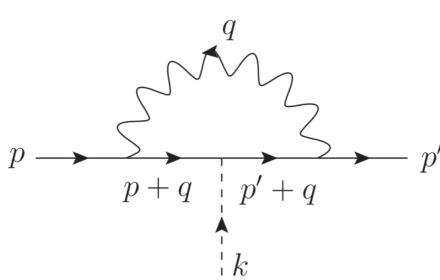
$$Z_2=1-\frac{e_R^2}{16\pi^2\epsilon}-\frac{g_R^2}{32\pi^2\epsilon}\,. \qquad (177)$$

We will not need to make use of the expression for Z_m, so we do not record its result here.

The final renormalization constant we must evaluate is Z_4. The relevant Feynman diagrams in the one-loop approximation are:



We first focus on the photon exchange graph,



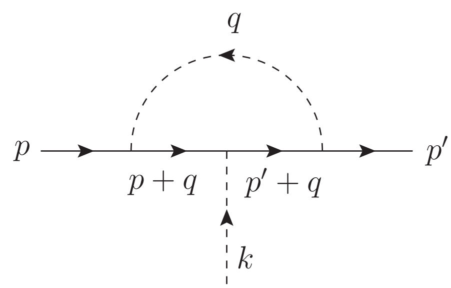
which, in the Feynman gauge, is given by

$$(-ig_R\mu^\epsilon)(-ie_R\mu^\epsilon)^2\int\frac{d^nq}{(2\pi)^n}\gamma_\beta\left(\frac{i(\not{p}'+\not{q})+M}{(p'+q)^2-M^2}\right)\left(\frac{i(\not{p}+\not{q})+M}{(p+q)^2-M^2}\right)\gamma_\alpha\left(\frac{-ig^{\alpha\beta}}{q^2}\right)\,. \qquad (178)$$

The divergent contribution arises due to the term $(\gamma_\beta \not{q}\not{q}\gamma_\alpha)g^{\alpha\beta} = nq^2$ which appears in the numerator of the integrand. Moreover, we do not change the divergent contribution by setting $n = 4$, $\mu^\epsilon = 1$ and replacing this factor of q^2 with $(p' + q)^2 - M^2$, since the extra terms only contribute to the finite part of the integral. This allows us to cancel one of the denominators. We end up with

$$-4g_R e_R^2 \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2\big[(q+p)^2 - M^2\big]} = -\frac{ig_R e_R^2}{4\pi^2} B_0(p^2, 0, M^2) = -\frac{ig_R e_R^2}{4\pi^2\epsilon} + \text{finite terms.} \quad (179)$$

Next, we focus on the scalar exchange graph,



which is given by

$$(-ig_R\mu^\epsilon)^3 \int \frac{d^n q}{(2\pi)^n} \left(\frac{i(\not{p}' + \not{q}) + M}{(p'+q)^2 - M^2}\right) \left(\frac{i(\not{p} + \not{q}) + M}{(p+q)^2 - M^2}\right) \left(\frac{i}{q^2 - m^2}\right). \quad (180)$$

The divergent contribution arises due to $\not{q}\not{q} = q^2$ which appears in the numerator of the integrand. Moreover, we do not change the divergent contribution by setting $\mu^\epsilon = 1$ and replacing this factor of q^2 with $(p' + q)^2 - M^2$, since the extra terms only contribute to the finite part of the integral. This allows us to cancel one of the denominators. We end up with

$$g_R^3 \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - m^2)\big[(q+p)^2 - M^2\big]} = \frac{ig_R^3}{16\pi^2} B_0(p^2, m^2, M^2) = \frac{ig_R^3}{16\pi^2\epsilon} + \text{finite terms.} \quad (181)$$

Finally, we add the diagram that contains the counterterm. The Feynman rule for the fermion-fermion-scalar vertex counterterm is easily obtained (using the same method employed in class),

$$-i\mu^\epsilon g_R(Z_4 - 1)\,. \quad (182)$$

Collecting all of the results obtained above, the -1 in eq. (182) cancels the tree-level result, and we are left with,

$$i\Gamma^{(3)}(p, -p', k) = -ig_R Z_4 - \frac{ig_R e_R^2}{4\pi\epsilon} + \frac{ig_R^3}{16\pi^2\epsilon} + \text{finite terms}\,. \quad (183)$$

The MS renormalization procedure instructs us to choose Z_4 such that the divergences cancel in $i\Gamma^{(3)}$. Moreover, at tree-level, $i\Gamma_0^{(3)}(p) = -ig_R$. Hence, it follows that

$$Z_4 = 1 + \frac{g_R^2}{16\pi^2\epsilon} - \frac{e_R^2}{4\pi^2\epsilon}\,. \quad (184)$$

It is convenient to introduce one additional renormalization constant,

$$g = \mu^\epsilon Z_g g_R\,, \tag{185}$$

In light of eq. (157), it follows that

$$Z_g = Z_4 Z_2^{-1} Z_\phi^{-1/2} = 1 + \frac{5g_R^2}{32\pi^2\epsilon} - \frac{3e_R^2}{16\pi^\epsilon}\,, \tag{186}$$

after making use of eqs. (159), (177) and (184).

We are now ready to compute the β functions, β_e, β_g and β_λ in the one loop approximation. Before proceeding with the calculation, it is convenient to summarize the results obtained above that provide the ingredients necessary for computing all the β-functions. The relations between the bare and the renormalized couplings are

$$e == \mu^\epsilon Z_3^{-1/2} e_R\,, \tag{187}$$

$$g = \mu^\epsilon Z_4 Z_2^{-1} Z_\phi^{-1/2} g_R = \mu^\epsilon Z_g g_R\,, \tag{188}$$

$$\lambda = \mu^{2\epsilon} Z_\lambda \lambda_R\,, \tag{189}$$

where

$$Z_3 = 1 - \frac{e_R^2}{12\pi^2\epsilon}\,, \tag{190}$$

$$Z_4 = 1 + \frac{g_R^2}{16\pi^2\epsilon} - \frac{e_R^2}{4\pi^2\epsilon}\,, \tag{191}$$

$$Z_2 = 1 - \frac{e_R^2}{16\pi^2\epsilon} - \frac{g_R^2}{32\pi^2\epsilon}\,, \tag{192}$$

$$Z_\phi = 1 - \frac{g_R^2}{8\pi^2\epsilon}\,, \tag{193}$$

$$Z_g = Z_4 Z_2^{-1} Z_\phi^{-1/2} = 1 + \frac{5g_R^2}{32\pi^2\epsilon} - \frac{3e_R^2}{16\pi^\epsilon}\,, \tag{194}$$

$$Z_\lambda = 1 + \frac{1}{\epsilon}\left(\frac{3\lambda_R^2}{32\pi^2} + \frac{g_R^2}{4\pi^2} - \frac{g_R^4}{2\pi^2\lambda_R}\right)\,. \tag{195}$$

First, in light of part (a), the calculation of β_e is unchanged from the calculation performed in problem 2. Hence, we simply quote the result obtained in eq. (106),

$$\boxed{\beta(e_R) = \frac{e_R^3}{12\pi^2}\,.} \tag{196}$$

Second, in light of eqs. (188) and (194), we see that β_g is a function of g_R and e_R. Thus, we define,

$$\beta_g(g_R, e_R, \epsilon) \equiv \mu\frac{dg_R}{d\mu}\,. \tag{197}$$

The bare coupling is independent of μ. Thus, eq. (188) yields,

$$0 = \mu\frac{dg}{d\mu} = \mu\frac{d}{d\mu}\big(\mu^\epsilon Z_g g_R\big) = \epsilon g_R Z_g + \left(\mu\frac{dg_R}{d\mu}\frac{\partial}{\partial g_R} + \mu\frac{de_R}{d\mu}\frac{\partial}{\partial e_R}\right)(Z_g g_R)\,, \tag{198}$$

after employing the chain rule.

As in the solution to problem 2, we follow eqs. (102) and (103) by writing,

$$\beta(e_R,\epsilon) = \mu\frac{de_R}{d\mu} = -\epsilon e_R + \beta(e_R)\,, \tag{199}$$

where $\beta(e_R)$ is independent of ϵ. In particular,

$$\beta(e_R) = \lim_{\epsilon\to 0}\beta(e_R,\epsilon)\,. \tag{200}$$

Hence, eq. (198) yields,

$$\bigl[\epsilon g_R + \beta_g(g_R,e_R,\epsilon)\bigr]Z_g + g_R\beta_g(g_R,e_R,\epsilon)\frac{\partial Z_g}{\partial g_R} + g_R\bigl[\beta_e(e_R) - \epsilon e_R\bigr]\frac{\partial Z_g}{\partial e_R} = 0\,. \tag{201}$$

Hence, analogous to eqs. (199) and (200),

$$\beta_g(g_R,e_R,\epsilon) = -\epsilon g_R + \beta_g(g_R,e_R)\,, \tag{202}$$

where

$$\beta_g(g_R,e_R) = \lim_{\epsilon\to 0}\beta_g(g_R,e_R,\epsilon)\,. \tag{203}$$

Hence, eqs. (201) and (202) yield,

$$\beta_g(g_R,e_R)Z_g + g_R\bigl[\beta_g(g_R,e_R) - \epsilon g_R\bigr]\frac{\partial Z_g}{\partial g_R} + g_R\bigl[\beta_e(e_R) - \epsilon e_R\bigr]\frac{\partial Z_g}{\partial e_R} = 0\,. \tag{204}$$

In the MS scheme, the renormalization constants have the following generic form,

$$Z = 1 + \sum_n \frac{a_n}{\epsilon^n}\,. \tag{205}$$

Thus, eq. (204) is a formal expansion in inverse powers of ϵ. The ϵ^0 term of eq. (204) reads,

$$\beta_g(g_R,e_R) - \epsilon g_R^2\frac{\partial Z_g}{\partial g_R} - \epsilon e_R g_R\frac{\partial Z_g}{\partial e_R} = 0\,. \tag{206}$$

Plugging in eq. (194) yields,

$$\boxed{\beta_g(g_R,e_R) = \frac{g_R}{16\pi^2}\bigl(5g_R^2 - 6e_R^2\bigr)\,.} \tag{207}$$

Finally, in light of eqs. (189) and (195), we see that β_λ is a function of g_R and λ_R. Thus, we define,

$$\beta_\lambda(g_R,\lambda_R,\epsilon) \equiv \mu\frac{d\lambda_R}{d\mu}\,. \tag{208}$$

Again, the bare coupling is independent of μ. Thus, eq. (189) yields,

$$0 = \mu\frac{d\lambda}{d\mu} = \mu\frac{d}{d\mu}\bigl(\mu^{2\epsilon}Z_\lambda\lambda_R\bigr) = 2\epsilon\lambda_R Z_\lambda + \left(\mu\frac{d\lambda}{d\mu}\frac{\partial}{\partial\lambda_R} + \mu\frac{dg_R}{d\mu}\frac{\partial}{\partial g_R}\right)\bigl(Z_\lambda\lambda_R\bigr)\,, \tag{209}$$

after employing the chain rule. It follows that

$$\big[2\epsilon\lambda_R + \beta_\lambda(g_R, \lambda_R, \epsilon)\big] Z_\lambda + \lambda_R\beta_\lambda(g_R, \lambda_R, \epsilon)\frac{\partial Z_\lambda}{\partial\lambda_R} + \lambda_R\beta_g(g_R, e_R, \epsilon)\frac{\partial Z_\lambda}{\partial g_R} = 0\,. \tag{210}$$

Hence, analogous to eqs. (202) and (203),

$$\beta_\lambda(g_R, e_R, \epsilon) = -2\epsilon\lambda_R + \beta_\lambda(\lambda_R, g_R)\,, \tag{211}$$

where

$$\beta_\lambda(\lambda_R, g_R) = \lim_{\epsilon\to 0}\beta_\lambda(g_R, \lambda_R, \epsilon)\,. \tag{212}$$

Eqs. (202), (210) and (211) yield,

$$\beta_\lambda(\lambda_R, g_R)Z_\lambda + \lambda_R\big[\beta_\lambda(\lambda_R, g_R) - 2\epsilon\lambda_R\big]\frac{\partial Z_\lambda}{\partial\lambda_R} + \lambda_R\big[\beta_g(g_R, e_R) - \epsilon g_R\big]\frac{\partial Z_\lambda}{\partial g_R} = 0\,. \tag{213}$$

Eq. (213) is a formal expansion in inverse powers of ϵ. The ϵ^0 term of eq. (204) reads,

$$\beta_\lambda(\lambda_R, g_R) - 2\epsilon\lambda_R^2\frac{\partial Z_\lambda}{\partial\lambda_R} - \epsilon\lambda_R g_R\frac{\partial Z_\lambda}{\partial g_R} = 0\,. \tag{214}$$

Plugging in eq. (195) yields,

$$\boxed{\beta_\lambda(\lambda_R, g_R) = \frac{3\lambda_R^2}{16\pi^2} + \frac{\lambda_R g_R^2}{2\pi^2} - \frac{g_R^4}{\pi^2}\,.} \tag{215}$$

(c) The equations for β_e, β_g and β_λ form a set of coupled differential equations for the three running coupling constants. Identify the fixed points of these equations, and discuss their significance.

In part (b), we obtained the following β-functions,

$$\beta(e_R) = \frac{e_R^3}{12\pi^2}\,, \tag{216}$$

$$\beta_g(g_R, e_R) = \frac{g_R}{16\pi^2}\big(5g_R^2 - 6e_R^2\big)\,, \tag{217}$$

$$\beta_\lambda(\lambda_R, g_R) = \frac{3\lambda_R^2}{16\pi^2} + \frac{\lambda_R g_R^2}{2\pi^2} - \frac{g_R^4}{\pi^2}\,. \tag{218}$$

The fixed points correspond to the values of the couplings where the β-functions vanish. For β_e, $e_R = 0$ corresponds to the well-known infrared fixed point of QED, discussed in class. Next, if we set $\beta_g = 0$, we obtain two fixed points: one at $g_R = 0$ and one at $g_R = (6/5)^{1/2}e_R$.

Suppose we measure the couplings at some scale μ_0. If $g_R/e_R > (6/5)^{1/2}$ at the scale μ_0, then $g_R/e_R = (6/5)^{1/2}$ is an attractive infrared fixed point. Alternatively, if $g_R/e_R < (6/5)^{1/2}$ at the scale μ_0, then g_R will be driven toward zero in the ultraviolet regime and toward the fixed point $g_R/e_R = (6/5)^{1/2}$ in the infrared regime.

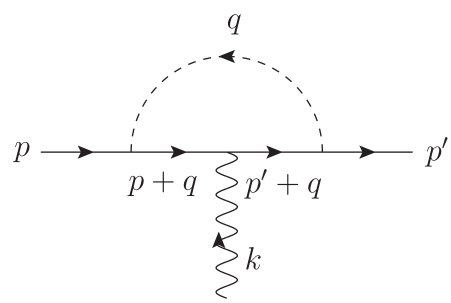
Finally, solving the equation $\beta_\lambda(\lambda_R, g_R) = 0$ yields two solutions, which we shall denote by $\lambda = \lambda_\pm$, Explicitly,

$$\lambda_+ = \tfrac{4}{3} g_R^2 \,, \qquad \lambda_- = -4g_R^2 \,. \tag{219}$$

Since the problem asks us to assume that $g_R^2 = \mathcal{O}(\lambda_R)$, we see that both fixed points are relevant. If at some scale μ_0, we have $0 < \lambda(\mu_0) < \frac{4}{3}g_R^2$, then $\beta_\lambda < 0$ and λ will be driven asymptotically to $\lambda_- = -4g_R^2$ in the ultraviolet regime. This result would be a disaster, since $\lambda_- < 0$ and a negative quartic scalar coupling implies that the scalar potential is unbounded from below (implying that no ground state exists). On the other hand, if $\lambda(\mu_0) > \frac{4}{3}g_R^2$, then $\beta_\lambda > 0$, and the coupling λ becomes large in the ultraviolet regime, eventually diverging (in analogy with the Landau pole of QED). Correspondingly, in the infrared regime, λ would be driven to its positive fixed point, $\lambda_+ = \frac{4}{3}g_R^2$.

Appendix: Proof that the scalar contributions do not modify $Z_1 = Z_2$

To compute the scalar contribution to Z_1, we must compute the divergent contribution to the following Feynman graph,



which is given by

$$(ie_R\mu^\epsilon)(-ig_R\mu^\epsilon)^2 \int \frac{d^nq}{(2\pi)^n} \left(\frac{i(\not{p}' + \not{q}) + M}{(p'+q)^2 - M^2}\right) \gamma_\mu \left(\frac{i(\not{p} + \not{q}) + M}{(p+q)^2 - M^2}\right) \left(\frac{i}{q^2 - m^2}\right) . \tag{220}$$

The divergent contribution arises due to

$$\not{q}\gamma_\mu\not{q} = (2q_\mu - \gamma_\mu\not{q})\not{q} = 2q_\mu q_\nu\gamma^\nu - \gamma_\mu q^2 \,, \tag{221}$$

which appears in the numerator of the integrand. Moreover, we do not change the divergent contribution by setting $\mu^\epsilon = 1$, since the extra terms only contribute to the finite part of the integral. Hence, we examine,

$$-e_R g_R^2 \gamma^\nu \int \frac{d^nq}{(2\pi)^n} \frac{2q_\mu q_\nu - q^2 g_{\mu\nu}}{(q^2 - m^2)\big[(p+q)^2 - M^2\big]\big[(p'+q)^2 - M^2\big]} \,. \tag{222}$$

Using eqs. (90) and (91), the above integral can be rewritten using Feynman's trick as,

$$\begin{aligned} &-2e_R g_R^2 \gamma^\nu \\ &\times \int_0^1 x\,dx \int_0^1 dy\, \frac{2q_\mu q_\nu - q^2 g_{\mu\nu}}{\big[q^2 - 2q\cdot\big[p_1 x(1-y) + p'(1-x)\big] + x(p^2 - p'^2) - xy(p^2 + m^2 - M^2) + p'^2 - M^2\big]^3} . \end{aligned} \tag{223}$$

Using the result of the handout entitled, *Useful formulae for computing one-loop integrals*,

$$\int \frac{d^n q}{(2\pi)^n} \frac{q^\mu q^\nu}{(q^2 + 2q\cdot p - m^2 + i\varepsilon)^r} = i(-1)^r (p^2+m^2)^{2-\epsilon-r}(4\pi)^{\epsilon-2}\frac{\Gamma(\epsilon+r-3)}{\Gamma(r)} \times\left[(\epsilon+r-3)p^\mu p^\nu - \tfrac{1}{2}g^{\mu\nu}(p^2+m^2)\right], \quad (224)$$

and it follows that

$$\int \frac{d^n q}{(2\pi)^n} \frac{q^2}{(q^2 + 2q\cdot p - m^2 + i\varepsilon)^r} = i(-1)^r (p^2+m^2)^{2-\epsilon-r}(4\pi)^{\epsilon-2}\frac{\Gamma(\epsilon+r-3)}{\Gamma(r)} \times\left[(2\epsilon+r-5)p^2 - (2-\epsilon)m^2\right]. \quad (225)$$

Thus,

$$\int \frac{d^n q}{(2\pi)^n} \frac{q_\mu q_\nu - q^2 g_{\mu\nu}}{(q^2 + 2q\cdot p - m^2 + i\varepsilon)^3} = -\frac{i}{32\pi^2\epsilon} g_{\mu\nu} + \text{finite terms}, \quad (226)$$

and eq. (223) reduces to

$$\frac{ie_R g_R^2}{32\pi^2\epsilon}\gamma_\mu + \text{finite terms}. \quad (227)$$

The vertex counterterm yields

$$ie_R\gamma_\mu(Z_1 - 1). \quad (228)$$

The factor of -1 above is canceled by the tree-level contribution to the vertex, $i\Gamma_0^{(3)}(p) = ie_R\gamma_\mu$.

Combining the above results with the QED contribution to the one-loop vertex obtained in class, we end up with,

$$i\Gamma_\mu^{(3)} = ie_R\gamma_\mu\left[Z_1 + \frac{e_R^2}{16\pi^2} + \frac{g_R^2}{32\pi^2\epsilon}\right]. \quad (229)$$

The MS renormalization procedure instructs us to choose Z_1 such that the divergences cancel in $i\Gamma_\mu^{(3)}$. Thus, we conclude that

$$Z_1 = 1 - \frac{e_R^2}{16\pi^2} - \frac{g_R^2}{32\pi^2\epsilon}. \quad (230)$$

Comparing with eq. (177), we have confirmed that $Z_1 = Z_2$ is not modified by the scalar contributions to Z_1 and Z_2.