1. One way of defining $\Lambda_{\rm QCD}$ (which does not depend on QCD perturbation theory) is as follows. The running coupling constant, $\overline{g}(Q)$, is the solution to the equation

$$\frac{d\overline{g}}{dt} = \beta(\overline{g})\,, \tag{1}$$

with boundary condition $\overline{g}(0) = g$, where $t \equiv \ln(Q/\mu)$, and μ is an arbitrary parameter with dimensions of mass introduced by the renormalization procedure. To solve eq. (1), introduce the indefinite integral

$$y(z) \equiv \int^z \frac{dz'}{\beta(z')}\,. \tag{2}$$

Then, the solution to eq. (1) is

$$t = y(\overline{g}) - y(g)\,.$$

Note that $y(g)$ is just the integration constant that is fixed by the boundary condition for the differential equation. We now define $\Lambda_{\rm QCD}$ through the following equation:

$$y(g) \equiv -\tfrac{1}{2}\ln\left(\frac{\Lambda^2_{\rm QCD}}{\mu^2}\right)\,. \tag{3}$$

(a) Working to lowest nontrivial order in QCD perturbation theory, show that $\Lambda_{\rm QCD}$ defined in eq. (3) coincides with the definition given in class.

In class, we defined $\Lambda \equiv \Lambda_{\rm QCD}$ by the following equation obtained by employing the one-loop β-function,

$$\Lambda^2 = \mu^2 \exp\left(\frac{-4\pi}{b_0\alpha_s(\mu)}\right)\,, \tag{4}$$

where $b_0 = 11 - \frac{2}{3}n_F$ for QCD with an SU(3) color group and n_F flavors of quarks, and $\alpha_s(\mu) \equiv g_s^2(\mu)/(4\pi)$.

To verify that this definition coincides with $\Lambda_{\rm QCD}$ defined in eq. (3) in the one-loop approximation, recall that the one-loop QCD β-function is given by

$$\beta(g_s) = -\frac{b_0 g_s^3}{16\pi^2}\,. \tag{5}$$

Plugging this result into eq. (2) yields,

$$y(g_s) = -\frac{16\pi^2}{b_0}\int^{g_s}\frac{dz}{z^3} = \frac{8\pi^2}{b_0 g_s^2}\,. \tag{6}$$

Solving eq. (3) for $\Lambda_{\rm QCD}$, we therefore obtain

$$\Lambda^2_{\rm QCD} = \mu^2 e^{-2y(g_s)} = \mu^2 \exp\left(\frac{-16\pi^2}{b_0 g_s^2(\mu)}\right) = \mu^2 \exp\left(\frac{-4\pi}{b_0\alpha_s(\mu)}\right)\,, \tag{7}$$

where $g_s \equiv g_s(\mu)$. Thus, we have confirmed eq. (4).

(b) Show that $\Lambda_{\rm QCD}$ defined in eq. (3) is independent of the arbitrary mass parameter μ.

Starting with eq. (7),

$$\Lambda_{\rm QCD} = \mu e^{-y(g_s)} \,. \tag{8}$$

Taking a derivative with respect to μ then yields,

$$\frac{d\Lambda_{\rm QCD}}{d\mu} = e^{-y(g_s)}\left[1 - \mu\frac{dy}{d\mu}\right] \,. \tag{9}$$

Using the chain rule,

$$\frac{dy}{d\mu} = \frac{dy}{dg_s}\frac{dg_s}{d\mu} = \frac{dg_s}{d\mu}\frac{d}{dg_s}\int^{g_s}\frac{dz}{\beta(z)} = \frac{1}{\beta(g_s)}\frac{dg_s}{d\mu} \,. \tag{10}$$

Recalling the definition of the β-function,

$$\beta(g_s) = \mu\frac{dg_s}{d\mu} \,, \tag{11}$$

and inserting this result into eq. (10), it follows that

$$\frac{dy}{d\mu} = \frac{1}{\mu} \,. \tag{12}$$

Plugging this result back into eq. (9) yields,

$$\frac{d\Lambda_{\rm QCD}}{d\mu} = 0 \,, \tag{13}$$

as was to be shown.

Note that this proof does not rely on perturbation theory, and thus is completely general. This means that eqs. (2) and (3) provide a non-perturbative definition of $\Lambda_{\rm QCD}$ which is independent of the arbitrary mass scale μ that is introduced by the renormalization procedure.

2. Consider an extension of QCD (called supersymmetric QCD), where we add to QCD a color octet neutral Majorana fermion called the gluino ($\tilde{g}$), and color triplet scalar particles, called squarks ($\tilde{q}$), which possess the same electroweak quantum numbers as the corresponding quarks. Take all particles of this model to be massless. The squarks and gluinos possess the following interactions and corresponding Feynman rules:

$g\tilde{q}\tilde{q}$	$-ig_s(p_1+p_2)_\mu \boldsymbol{T^a}$
$gg\tilde{q}\tilde{q}$	$ig_s^2\, g_{\mu\nu}(\boldsymbol{T^a T^b} + \boldsymbol{T^b T^a})$
$\tilde{g}\tilde{q}q$	$-ig_s\sqrt{\tfrac{1}{2}}(1\pm\gamma_5)\boldsymbol{T^a}$
$g\tilde{g}\tilde{g}$	$-g_s f^{abc}\gamma_\mu$

where in the $g\tilde{q}\tilde{q}$ vertex, a $\tilde{q}$ enters the vertex with momentum p_1 and leaves with momentum p_2. In the rule for the $g\tilde{g}\tilde{g}$ vertex, a is the adjoint color index of the gluon and b (c) is the adjoint color index of the gluino that leaves (enters) the vertex. In the rule for the $\tilde{g}\tilde{q}q$ vertex, use the positive (negative) sign if the outgoing $\tilde{q}$ is the partner of a right (left) handed quark, and vice versa for an incoming $\tilde{q}$. In particular, for every quark flavor, there are two corresponding squark partners (called $\tilde{q}_R$ and $\tilde{q}_L$). The $g\tilde{q}\tilde{q}$ Feynman rule applies to both $g\tilde{q}_L\tilde{q}_L$ and to $g\tilde{q}_R\tilde{q}_R$. However, there is no $g\tilde{q}_L\tilde{q}_R$ interaction since the gluon couples diagonally to pairs of scalars or fermions. In your calculation, take the gauge group to be SU(N) with structure constants f^{abc} and denote the generators in the defining (fundamental) representation of SU(N) by $\boldsymbol{T^a}$. (Of course, for QCD, one should take $N = 3$.)

(a) Using dimensional regularization and the $\overline{\text{MS}}$ renormalization scheme, compute the lowest order contribution to the QCD β-function in a non-abelian gauge theory based on SU(N) color coupled to n_f quark flavors, $2n_f$ squark partners and a gluino. This requires a number of steps:

(i) Start with the result for $Z_g = Z_{1F}Z_2^{-1}Z_3^{-1/2}$ derived in class for ordinary QCD. Draw Feynman diagrams corresponding to the new supersymmetric contributions to Z_{1F}, Z_2 and Z_3.

(ii) Argue that the one-loop supersymmetric contributions to $Z_{1F}Z_2^{-1}$ cancel exactly. (Recall that in QED, $Z_{1F}Z_2^{-1} = 1$.) As a result, one need only consider the supersymmetric contributions to Z_3.

(iii) Using the result for Z_3 in ordinary QCD obtained in class, the gluino contribution to Z_3 can be obtained by inspection. Keep in mind that the gluino transforms under the adjoint representation of SU(N) color. Moreover, the gluino is a Majorana fermion which possesses half the number of degrees of freedom of a Dirac fermion. This yields an extra factor of 1/2.

(iv) Thus, the only new computation required is the squark loop contribution to Z_3. Compute this contribution, and then combining this with the result of (iii), obtain the supersymmetric QCD one-loop β-function.

In class, the one-loop QCD β-function, which is defined via eq. (5) was found to be $b_0 = \frac{11}{3}N - \frac{2}{3}n_F$, in an SU($N$) gauge theory with n_F flavors of quarks. This was obtained by identifying the residue of the ϵ^{-1} pole of the renormalization constant Z_g in the MS renormalization scheme,

$$Z_g = Z_{1F}Z_2^{-1}Z_3^{-1/2} = 1 - \frac{\alpha_s}{8\pi\epsilon}\left[\tfrac{11}{3}C_A - \tfrac{4}{3}T_F n_F\right]\,, \tag{14}$$

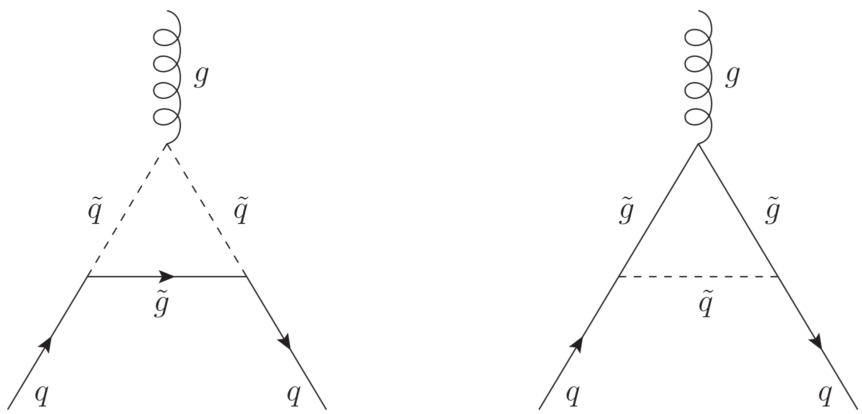
where the group theory factors for SU(N) are $C_A = N$ and $T_F = \frac{1}{2}$. In general,

$$\boldsymbol{T^a_R T^a_R} \equiv C_R \mathbb{1}_{d_R}\,, \qquad \text{Tr}(\boldsymbol{T^a_R T^b_R}) = \tfrac{1}{2}\delta^{ab}\,, \tag{15}$$

in representation R, where $\mathbb{1}_{d_R}$ is the $d_R \times d_R$ identity operator and d_R is the dimension of the representation. C_A is the quadratic Casimir operator in the adjoint representation [$C_A = N$ for SU(N)], and $T_F = \frac{1}{2}$ in the fundamental representation.

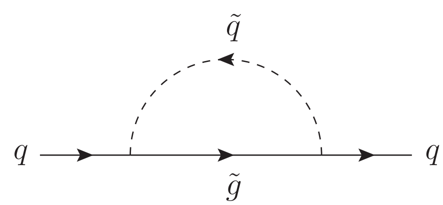
In order to compute the one-loop β-function of supersymmetric QCD, we must consider the new contributions to Z_{1F}, Z_2 and Z_3, respectively. We shall depict the squarks ($\tilde{q}_L$ and $\tilde{q}_R$, collectively denoted by $\tilde{q}$) with dashed lines, the gluinos by solid lines with no arrows (since the gluino is a neutral Majorana color octet fermion), and as usual, we will denote the quarks by solid lines with arrows denoted the direction of flow of the fermion number, and the gluons by curly lines.

First, the new diagrams contributing to the one-loop computation of Z_{1F} are shown below,



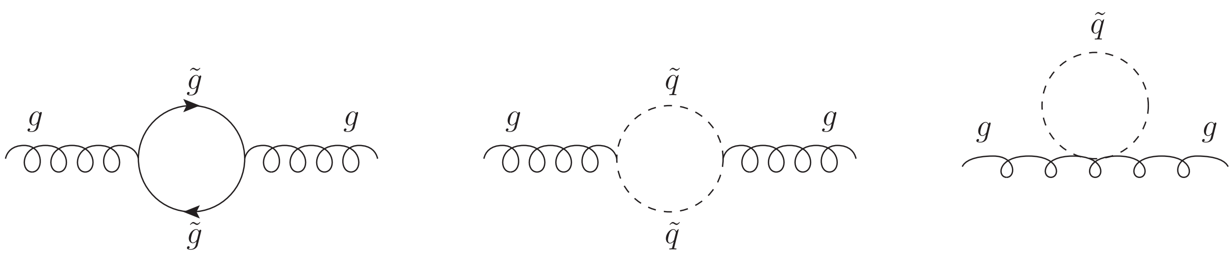
where in the first diagram above one must sum over $\tilde{q}_L\tilde{q}_L$ and $\tilde{q}_R\tilde{q}_R$ intermediate states, while in the second diagram above, one must sum the contributions from the $\tilde{q}_L$ and $\tilde{q}_R$ loops.

One new diagram, shown below, contributes to the one-loop computation of Z_2,



where again one must sum the contributions from the $\tilde{q}_L$ and $\tilde{q}_R$ loops.

Finally, the new diagrams contributing to the one-loop computation of Z_3 are shown below,



where in the second diagram above one must sum over the $\tilde{q}_L\tilde{q}_L$ and $\tilde{q}_R\tilde{q}_R$ intermediate states, while in the third diagram above one must sum the contributions from the $\tilde{q}_L$ and $\tilde{q}_R$ loops.

Remarkably, the supersymmetric contributions to $Z_{1F}Z_2^{-1}$ cancel exactly. Here is a slick argument to explain why this occurs. Suppose one adds to the theory a new color triplet quark Q but does not add the corresponding scalar superpartners. Gauge invariance requires that the $\overline{Q}Qg$ interaction is exactly the same as the $\bar{q}qg$ interaction, with the same Feynman rule, $-ig_s\gamma^\mu$. This implies that

$$Z_g = Z'_{1F}Z_2'^{-1}Z_3^{-1/2}\,, \tag{16}$$

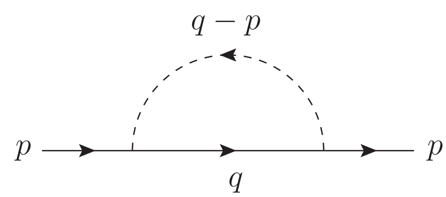
where Z'_{1F} is the $\bar{Q}Qg$ vertex counterterms and Z'_2 is the wave function renormalization constant for the field Q. Indeed, the statement that

$$Z_{1F}Z_2^{-1}Z_3^{-1/2} = Z'_{1F}Z_2'^{-1}Z_3^{-1/2}\,, \tag{17}$$

is a Slavnov-Taylor identity that is satisfied due to the gauge symmetry (in the same way that $Z_g = Z_1Z_3^{-3/2}$ where Z_1 is the ggg vertex counterterm and Z_3 is the wave function renormalization constant for the gluon, as a consequence of the gauge symmetry).

The observation above implies that one can calculate Z_g using eq. (16) instead of eq. (14). If we now reconsider the change in Z_g as a consequence of adding supersymmetric particles, we see that there are no supersymmetric contributions to Z'_{1F} or Z'_2, since by assumption no scalar superpartner exists for Q. On the other hand, the supersymmetric contributions to Z_3 exhibited in the diagrams above are all still present. That is, the supersymmetric contributions to Z_3 are unchanged due to the presence of the field Q. Since the calculation of Z_g cannot depend on whether eq. (14) or eq. (16) is employed, it follows that there are no supersymmetric contributions to the product $Z_g Z_3^{1/2}$. Given that there are supersymmetric contributions *separately* to Z_{1F} and Z_2, the only possible conclusion is that these contributions exactly cancel in the product $Z_{1F}Z_2^{-1}$, as asserted above.

It is instructive to verify this conclusion by an explicit computation. To compute the supersymmetric contributions to Z_2 (denoted by $(\delta Z_2)_{\rm SUSY}$ below), we analyze the graph,



Applying the Feynman rules yields

$$\tfrac{1}{2}(-ig_s\mu^\epsilon \boldsymbol{T^a})(-ig_s\mu^\epsilon \boldsymbol{T^a})\Bigg\{\int\frac{d^nq}{(2\pi)^n}\left(\frac{(1-\gamma_5)i\not{q}(1+\gamma_5)}{q^2}\right)\left(\frac{i}{(q-p)^2}\right) + \int\frac{d^nq}{(2\pi)^n}\left(\frac{(1+\gamma_5)i\not{q}(1-\gamma_5)}{q^2}\right)\left(\frac{i}{(q-p)^2}\right)\Bigg\}, \tag{18}$$

after including contributions from both $\tilde{q}_L$ and $\tilde{q}_R$ (under the assumption that the squarks and gluinos are massless). Although one might worry about the presence of γ_5 in dimensional regularization, this is an example where we can assume that γ_5 anticommutes with γ_μ without fear of inconsistencies. When the two integrals are added the γ_5 disappears, and we are left with,

$$\begin{aligned}8\pi\alpha_s C_F\mu^{2\epsilon}\gamma^\mu\int\frac{d^nq}{(2\pi)^n}\frac{q_\mu}{q^2(q-p)^2} &= \frac{i\alpha_s C_F\mu^{2\epsilon}}{2\pi}\gamma^\mu B_\mu(-p;0,0) = -\frac{i\alpha_s C_F\mu^{2\epsilon}}{2\pi}\not{p}B_1(p^2;0,0)\\ &= \frac{i\alpha_s C_F}{4\pi\epsilon}\not{p} + \text{finite terms}\,,\end{aligned} \tag{19}$$

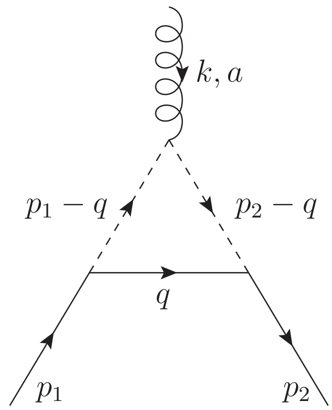
after employing $\boldsymbol{T^aT^a} = C_F\mathbb{1}_N$ [where $C_F = (N^2-1)/(2N)$ in the case of an SU(N) gauge theory] and putting $g_s^2 = 4\pi\alpha_s$. Adding the counterterm,

$$i\not{p}(Z_2 - 1)\,, \tag{20}$$

and noting that the factor of -1 cancels the tree-level result, it follows that

$$(\delta Z_2)_{\rm SUSY} = -\frac{i\alpha_s C_F}{4\pi\epsilon}\,. \tag{21}$$

Next we examine the Feynman graphs contributing to the vertex correction. The first graph is exhibited below.



Applying the Feynman rules yields,

$$\tfrac{1}{2}(-ig_s\mu^\epsilon \boldsymbol{T^b})(-ig_s\mu^\epsilon \boldsymbol{T^a})(-ig_s\mu^\epsilon \boldsymbol{T^b})$$

$$\times\Bigg\{\int\frac{d^nq}{(2\pi)^n}\left(\frac{(1-\gamma_5)i\not{q}(1+\gamma_5)}{q^2}\right)\left(\frac{i}{(q-p_1)^2}\right)\left(\frac{i}{(q-p_2)^2}\right)(p_1+p_2-2q_\mu)$$

$$+\int\frac{d^nq}{(2\pi)^n}\left(\frac{(1+\gamma_5)i\not{q}(1-\gamma_5)}{q^2}\right)\left(\frac{i}{(q-p_1)^2}\right)\left(\frac{i}{(q-p_2)^2}\right)(p_1+p_2-2q_\mu)\Bigg\}\,,$$

after including contributions from both $\tilde{q}_L$ and $\tilde{q}_R$. When the two integrals are added the γ_5 disappears, and we are left with,

$$-16\pi\alpha_s(\boldsymbol{T^bT^aT^b})g_s\gamma^\nu\int\frac{d^nq}{(2\pi)^n}\frac{q_\mu q_\nu}{q^2(q-p_1)^2(q-p_2)^2}\;+\;\text{finite terms}$$

$$=-32\pi\alpha_s(\boldsymbol{T^bT^aT^b})g_s\gamma^\nu\int_0^1 x\,dx\int_0^1 dy\int\frac{d^nq}{(2\pi)^n}$$

$$\times\frac{q_\mu q_\nu}{\left[q^2-2q\cdot[p_1x(1-y)+p_2(1-x)]+p_1^2x(1-y)+p_2^2(1-x)\right]^3}\;+\;\text{finite terms}$$

$$=-\frac{i\alpha_s}{2\pi}(\boldsymbol{T^bT^aT^b})g_s\gamma_\nu\int_0^1 x\,dx\int_0^1 dy\;+\;\text{finite terms}$$

$$=-\frac{i\alpha_s}{4\pi\epsilon}(C_F-\tfrac{1}{2}C_A)g_s\boldsymbol{T^a}\gamma_\mu\;+\;\text{finite terms}\,. \tag{22}$$

In deriving eq. (22), we made use of the following result that is easily obtained from the class handout entitled *Useful formulae for computing one-loop integrals*,

$$\int\frac{d^nq}{(2\pi)^n}\frac{q_\mu q_\nu}{(q^2+2q\cdot p-m^2+i\varepsilon)^3}=\frac{i}{64\pi^2\epsilon}g_{\mu\nu}\;+\;\text{finite terms}\,. \tag{23}$$

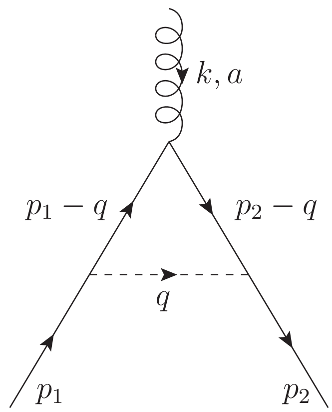
In addition, the group theoretical factor $\boldsymbol{T^bT^aT^b}$ was simplified as follows,

$$\boldsymbol{T^bT^aT^b}=\boldsymbol{T^b}\big\{\boldsymbol{T^bT^a}+if^{abc}\boldsymbol{T^c}\big\}=C_F\boldsymbol{T^a}+\tfrac{1}{2}if^{abc}(\boldsymbol{T^bT^c}-\boldsymbol{T^cT^b})$$

$$=C_F\boldsymbol{T^a}-\tfrac{1}{2}f^{abc}f^{bcd}\boldsymbol{T^d}=(C_F-\tfrac{1}{2}C_A)\boldsymbol{T^a}\,, \tag{24}$$

where $\boldsymbol{T^bT^b}=C_F\mathbb{1}_N$ and $f^{abc}f^{dbc}=C_A\delta^{ad}$.

The second graph that contributes to the vertex correction is exhibited below.



Applying the Feynman rules yields,

$$\tfrac{1}{2}(-ig_s\mu^\epsilon \boldsymbol{T^b})(-ig_s\mu^\epsilon \boldsymbol{T^c})(-g_s\mu^\epsilon f^{abc})$$
$$\times\Bigg\{\int\frac{d^nq}{(2\pi)^n}\left(\frac{(1-\gamma_5)i(\not p_2-\not q)\gamma_\mu\, i(\not p_1-\not q)(1+\gamma_5)}{(q-p_1)^2(q-p_2)^2}\right)\left(\frac{i}{q^2}\right)$$
$$+\int\frac{d^nq}{(2\pi)^n}\left(\frac{(1+\gamma_5)i(\not p_2-\not q)\gamma_\mu\, i(\not p_1-\not q)(1-\gamma_5)}{(q-p_1)^2(q-p_2)^2}\right)\left(\frac{i}{q^2}\right)\Bigg\},$$

after including contributions from both $\tilde q_L\tilde q_L$ and $\tilde q_R\tilde q_R$ intermediate states. The divergent piece of the integrals can be identified by keeping only the $\not q$ terms in the numerators above. When the two integrals are added the γ_5 disappears, and we are left with,

$$-8\pi i\alpha_s g_s f^{abc}T^bT^c\int\frac{d^nq}{(2\pi)^n}\frac{\not q\gamma_\mu\not q}{q^2(q-p_1)^2(q-p_2)^2}+\text{finite terms}$$
$$=4\pi\alpha_s C_A g_s\boldsymbol{T^a}\gamma^\nu\int\frac{d^nq}{(2\pi)^n}\frac{2q_\mu q_\nu-g_{\mu\nu}q^2}{q^2(q-p_1)^2(q-p_2)^2}+\text{finite terms}$$
$$=8\pi\alpha_s C_A g_s\boldsymbol{T^a}\gamma^\nu\int_0^1 x\,dx\int_0^1 dy\int\frac{d^nq}{(2\pi)^n}$$
$$\times\frac{2q_\mu q_\nu-g_{\mu\nu}q^2}{\left[q^2-2q\cdot[p_1x(1-y)+p_2(1-x)]+p_1^2x(1-y)+p_2^2(1-x)\right]^3}+\text{finite terms}$$
$$-\frac{i\alpha_s C_A}{4\pi\epsilon}g_s\boldsymbol{T^a}\gamma_\mu\int_0^1 x\,dx\int_0^1 dy$$
$$-\frac{i\alpha_s C_A}{8\pi\epsilon}g_s\boldsymbol{T^a}\gamma_\mu+\text{finite terms}. \tag{25}$$

after employing eq. (23). In addition, the group theoretical factor $if^{abc}\boldsymbol{T^b}\boldsymbol{T^c}$ was simplified as follows

$$if^{abc}\boldsymbol{T^b}\boldsymbol{T^c}=\tfrac{1}{2}if^{abc}(\boldsymbol{T^b}\boldsymbol{T^c}-\boldsymbol{T^c}\boldsymbol{T^b})=-\tfrac{1}{2}f^{abc}f^{bcd}\boldsymbol{T^d}=-\tfrac{1}{2}C_A\boldsymbol{T^a}\,. \tag{26}$$

Adding eqs. (22) and (25), we see that the term proportional to C_A exactly cancels. Hence, the result for the supersymmetric contribution to the one-loop vertex is,

$$-\frac{i\alpha_s C_F}{4\pi\epsilon} g_s \boldsymbol{T^a} \gamma_\mu + \text{finite terms.} \tag{27}$$

Finally, we add the counterterm,

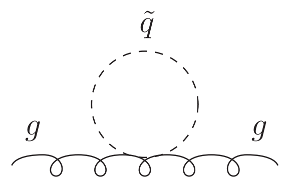
$$-i(Z_{1F} - 1) g_s \boldsymbol{T^a} \gamma_\mu \,, \tag{28}$$

and notice that the factor of -1 cancels the tree-level result. Hence, we can conclude that

$$(\delta Z_{1F})_{\rm SUSY} = -\frac{i\alpha_s C_F}{4\pi\epsilon} \,. \tag{29}$$

In light of eqs. (21) and (29), it follows that $(\delta Z_{1F})_{\rm SUSY} = (\delta Z_2)_{\rm SUSY}$, as advertised.

We now turn our attention to the supersymmetric corrections to the gluon self-energy. Note that the graph shown below,

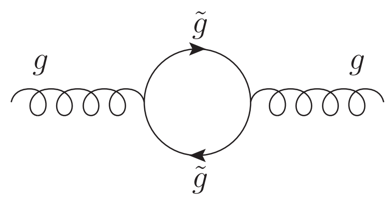


yields a loop integral that is zero by the rules of dimensional regularization,[1]

$$\int \frac{d^q}{(2\pi)^n} \frac{1}{q^2} = 0 \,, \tag{30}$$

Thus, we only need consider the gluino loop diagram and the squark loop diagram.

Consider first, the gluino loop diagram exhibited below.



The gluino is a Majorana fermion. Hence the symmetry factor of this graph is $\frac{1}{2}$. Otherwise, we may use the result obtained in class for QCD without supersymmetric particles,

$$Z_3 = 1 + \frac{\alpha_s}{8\pi\epsilon}\left[\left(\frac{13}{3} - a\right) C_A - \frac{8}{3} T_F n_F\right] \,, \tag{31}$$

to conclude that a fermion in the fundamental representation of SU(N) contributes

$$-\frac{\alpha_s T_F}{3\pi\epsilon} \,, \tag{32}$$

to the gluon wave function renormalization constant Z_3.

[1]For more details on the origin of this rule, see Section III of the class handout entitled, *Electron wave function and mass renormalization in QED*. In particular, note eq. (19) of this handout.

We can use eq. (32) to determine the contribution of the gluino to Z_3. The gluino is in the adjoint representation of SU(N). Hence, its contribution to Z_3 must be given by

$$-\frac{\alpha_s T_A}{6\pi\epsilon}\,, \tag{33}$$

after including the symmetry factor of $\frac{1}{2}$ mentioned above. The group theoretical factor T_A is defined by

$$\mathrm{Tr}(\boldsymbol{T_A^a}\boldsymbol{T_A^b}) = T_A\delta^{ab}\,. \tag{34}$$

The matrix elements of the generator in the adjoint representation are given by,

$$(\boldsymbol{T_A^a})_{bc} = -if^{abc}\,. \tag{35}$$

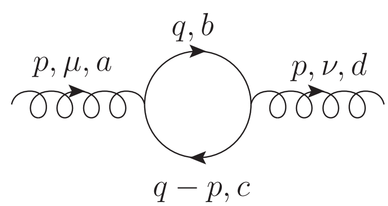
Hence, eq. (34) yields,

$$\mathrm{Tr}(\boldsymbol{T_A^a}\boldsymbol{T_A^b}) = (\boldsymbol{T_A^a})_{cd}(\boldsymbol{T_A^b})_{dc} = f^{acd}f^{bcd} = C_A\delta_{ab}\,, \tag{36}$$

where we used the antisymmetry of the structure constants. Comparing with eq. (34), we conclude that $T_A = C_A$. Hence, the gluino loop contribution to Z_3 is

$$(\delta Z_3)_{\mathrm{SUSY},\tilde{g}} = -\frac{\alpha_s C_A}{6\pi\epsilon}\,. \tag{37}$$

Although it is unnecessary, I shall provide an explicit computation of eq. (37). Consider the gluino contribution to the gluon self-energy,



where the adjoint color indices (a, b, c, d) have been specified along with the four-momenta and the Lorentz indices of the external gluons. Applying the Feynman rules yields,

$$-\tfrac{1}{2}f^{abc}f^{dcb}(-g_s\mu^\epsilon)^2\int\frac{d^nq}{(2\pi)^n}\,\mathrm{Tr}\left\{\left(\frac{i\not{q}}{q^2}\right)\gamma_\mu\left(\frac{i(\not{q}-\not{p})}{(q-p)^2}\right)\gamma_\nu\right\}\,, \tag{38}$$

after including the minus sign for the fermionic gluino loop and the symmetry factor of $\frac{1}{2}$ (since the gluino is a neutral Majorana fermion). Evaluating the trace and employing Feynman's trick yields,

$$\begin{aligned}
2f^{abc}f^{dcb}g_s^2\mu^{2\epsilon}&\int_0^1 dx\int\frac{d^nq}{(2\pi)^n}\,\frac{q_\mu(q-p)_\nu + q_\nu(q-p)_\mu - g_{\mu\nu}q\!\cdot\!(q-p)}{\big[(1-x)q^2 + x(q-p)^2\big]^2}\\
&= 2f^{abc}f^{dcb}g_s^2\mu^{2\epsilon}\int_0^1 dx\int\frac{d^nq}{(2\pi)^n}\,\frac{q_\mu(q-p)_\nu + q_\nu(q-p)_\mu - g_{\mu\nu}q\!\cdot\!(q-p)}{(q^2-2q\!\cdot\!px+xp^2)^2}\\
&= 2f^{abc}f^{dcb}g_s^2\mu^{2\epsilon}\int_0^1 dx\int\frac{d^nq}{(2\pi)^n}\,\frac{2q_\mu q_\nu - q^2g_{\mu\nu} - q_\mu p_\nu - q_\nu p_\mu + g_{\mu\nu}q\!\cdot\!p}{(q^2-2q\!\cdot\!px+xp^2)^2}\,.
\end{aligned} \tag{39}$$

We now make use of the following results that are easily derived from the formulae given the class handout entitled *Useful formulae for computing one-loop integrals*,

$$\int \frac{d^n q}{(2\pi)^n} \frac{q_\mu}{(q^2 + 2q\cdot p - m^2 + i\varepsilon)^2} = -\frac{i}{16\pi^2\epsilon}\, p_\mu \; + \text{finite terms}\,, \tag{40}$$

$$\int \frac{d^n q}{(2\pi)^n} \frac{q_\mu q_\nu}{(q^2 + 2q\cdot p - m^2 + i\varepsilon)^2} = \frac{i}{16\pi^2\epsilon}\left[p_\mu p_\nu + \tfrac{1}{2} g_{\mu\nu}(p^2 + m^2)\right] \; + \text{finite terms}\,. \tag{41}$$

In addition, the group theory factor, $f^{abc}f^{dcb}$, simplifies to,

$$f^{abc}f^{dcb} = -f^{abc}f^{dbc} = -C_A\delta^{ad}\,. \tag{42}$$

Hence, eq. (39) yields,

$$-\frac{i\alpha_s C_A}{\pi\epsilon}(p^2 g_{\mu\nu} - p_\mu p_\nu)\delta_{ad}\int_0^1 x(1-x)dx = -\frac{i\alpha_s C_A}{6\pi\epsilon}(p^2 g_{\mu\nu} - p_\mu p_\nu)\delta_{ad}\,. \tag{43}$$

As expected from gauge invariance, the end result is transverse.

We now add the counterterm (by generalizing the result for QED given in class),

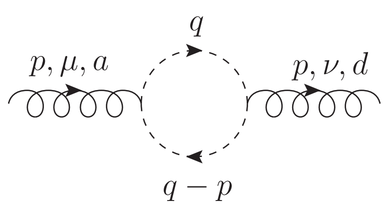
$$-i(Z_3 - 1)(p^2 g_{\mu\nu} - p_\mu p_\nu)\delta^{ad}\,. \tag{44}$$

It follows that the gluino loop contribution to Z_3 is

$$(\delta Z_3)_{\text{SUSY},\tilde{g}} = -\frac{\alpha_s C_A}{6\pi\epsilon}\,. \tag{45}$$

thereby confirming the result of eq. (37).

Finally, we consider the squark-loop contribution to the gluon self-energy,



Applying the Feynman rules yields,

$$\text{Tr}(\boldsymbol{T^a T^d})(-ig_s\mu^\epsilon)^2 \int \frac{d^n q}{(2\pi)^n}\left(\frac{i}{q^2}\right)\left(\frac{i}{(q-p)^2}\right)(2q-p)_\mu(2q-p)_\nu$$

$$= g_s^2\mu^{2\epsilon}T_F\delta^{ab}\int_0^1 dx \int \frac{d^n q}{(2\pi)^n}\frac{4q_\mu q_\nu + p_\mu p_\nu - 2q_\mu p_\nu - 2q_\nu p_\mu}{\left[(1-x)q^2 + x(q-p)^2\right]^2}$$

$$= g_s^2\mu^{2\epsilon}T_F\delta^{ad}\int_0^1 dx \int \frac{d^n q}{(2\pi)^n}\frac{4q_\mu q_\nu + p_\mu p_\nu - 2q_\mu p_\nu - 2q_\nu p_\mu}{(q^2 - 2q\cdot px + xp^2)^2}$$

$$= 4ig_s^2\mu^{2\epsilon}(4\pi)^{\epsilon-2}T_F\delta^{ad}\Gamma(\epsilon)\int_0^1 dx\left[-p^2 x(1-x)\right]^{-\epsilon}\left\{\left(x^2 - x + \tfrac{1}{4}\right)p_\mu p_\nu + \frac{1}{2(\epsilon-1)}g_{\mu\nu}p^2 x(1-x)\right\}$$

$$= \frac{ig_s^2 T_F\delta^{ad}}{4\pi^2\epsilon}\left\{p_\mu p_\nu\int_0^1\left(x-\tfrac{1}{2}\right)^2 dx - \tfrac{1}{2}g_{\mu\nu}p^2\int_0^1 x(1-x)dx\right\} + \text{finite terms}$$

$$= -\frac{i\alpha_s T_F\delta^{ad}}{12\pi^2\epsilon}(p^2 g_{\mu\nu} - p_\mu p_\nu) \; + \text{finite terms}\,. \tag{46}$$

Once again, the end result is transverse as expected from gauge invariance. Summing over n_F flavors of both $\tilde{q}_L$ and $\tilde{q}_R$ yields

$$-\frac{i\alpha_s T_F n_F \delta^{ad}}{6\pi^2\epsilon}(p^2 g_{\mu\nu} - p_\mu p_\nu) + \text{finite terms}\,. \tag{47}$$

In light of the counterterm given in eq. (44), it follows that the squark loop contribution to Z_3 is

$$(\delta Z_3)_{\text{SUSY},\tilde{q}} = -\frac{\alpha_s T_F n_F}{6\pi\epsilon}\,. \tag{48}$$

Adding the results of eqs. (37) and (48) yields,

$$(\delta Z_3)_{\text{SUSY}} = -\frac{\alpha_s}{6\pi\epsilon}\big(C_A + T_F n_F\big)\,. \tag{49}$$

We are now ready to compute the β-function of supersymmetric QCD. Recall that,

$$Z_g = Z_{1F} Z_2^{-1} Z_3^{-1/2}\,. \tag{50}$$

Moreover, there are no supersymmetric contributions to $Z_{1F}Z_2^{-1}$. Hence,

$$(\delta Z_g)_{\text{SUSY}} = -\tfrac{1}{2}(\delta Z_3)_{\text{SUSY}} = \frac{\alpha_s}{12\pi\epsilon}\big(C_A + T_F n_F\big)\,. \tag{51}$$

Adding this contribution to eq. (14) yields,

$$Z_g = Z_{1F} Z_2^{-1} Z_3^{-1/2} = 1 - \frac{\alpha_s}{8\pi\epsilon}\big(3C_A - 2T_F n_F\big)\,, \tag{52}$$

In class, we obtained

$$\beta(g_s) = g_s^2 \frac{dZ_g^{(1)}}{dg_s}\,, \tag{53}$$

where

$$Z_g = 1 + \frac{Z_g^{(1)}}{\epsilon}\,, \tag{54}$$

in the one-loop approximation. Thus, we identify,

$$Z_g^{(1)} = -\frac{g_s^2}{32\pi^2}\big(3C_A - 2T_F n_F\big)\,. \tag{55}$$

Hence, we obtain

$$\boxed{\beta(g_s) = -\frac{g_s^3}{16\pi^2}\big(3C_A - 2T_F n_F\big)\,.} \tag{56}$$

That is, if we write

$$\beta(g_s) = -\frac{b_0 g_s^3}{16\pi^2}\,, \tag{57}$$

then it follows that,

$$b_0 = \begin{cases} \frac{11}{3}C_A - \frac{4}{3}T_F n_F\,, & \text{for QCD}\,, \\ 3C_A - 2T_F n_F\,, & \text{for supersymmetric QCD}\,. \end{cases} \tag{58}$$

For an SU(N) gauge group, $C_A = N$ and $T_F = \frac{1}{2}$.

(b) Does the QCD running coupling constant run faster or slower at large momentum scales in a supersymmetric theory as compared to the non-supersymmetric one?

If we put $N = 3$ and $n_F = 6$ flavors in eq. (58), we obtain,

$$b_0 = \begin{cases} 7\,, & \text{for QCD}\,, \\ 3\,, & \text{for supersymmetric QCD}\,. \end{cases} \tag{59}$$

The one-loop running coupling is given by,

$$\alpha_s(Q^2) = \frac{4\pi}{b_0 \ln(Q^2/\Lambda^2)}\,, \tag{60}$$

where $\Lambda \equiv \Lambda_{QCD}$ is the subject of problem 1 in this Problem Set. The Particle Data Group provides the following world average,

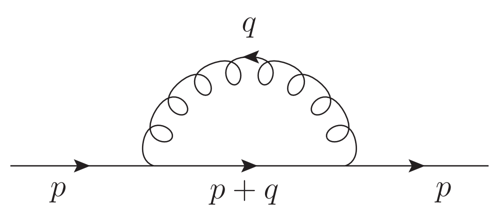
$$\alpha_s(m_Z^2) = 0.1179 \pm 0.001\,. \tag{61}$$

In QCD, we should really choose $n_F = 5$ near the mass of the Z boson since the top quark is significantly heavier than the Z and thus should be decoupled from the effective theory at m_Z. The numerical determination of Λ obtained in the one-loop approximation is not especially reliable. Suffice it to say that roughly, $\Lambda \sim \mathcal{O}(100~\text{MeV})$.

In any case, at mass scales above the top quark, we can assume $n_F = 6$ and compare the running of $\alpha_s(Q^2)$ with or without supersymmetry. Indeed, since b_0 is smaller in supersymmetric QCD as compared to non-supersymmetric QCD, then it follows that the value of $\alpha_s(Q^2)$, for $Q^2 \gg m_Z$, would be *larger* in supersymmetric QCD, given the known value of $\alpha_s(m_Z^2)$. That is, the running coupling constant decreases faster with increasing Q^2 in non-supersymmetric QCD as compared to supersymmetric QCD.

(c) Compute the one-loop $\mathcal{O}(\alpha_s)$ relation between the $\overline{\text{MS}}$ running top-quark mass, $m_t(m_t)$, and the "pole mass" (denoted by M_t) in ordinary QCD. Ignore all electroweak contributions.

In order to determine the relation between the $\overline{\text{MS}}$ mass and the pole mass of the quark in QCD, we must compute the one-loop quark self energy. That is, we evaluate the following diagram,



We evaluate this graph in the Feynman gauge ($a = 1$). Applying the Feynman rules yields,

$$\begin{aligned} -i\Sigma(p) &= \boldsymbol{T^a T^b}(-ig_s\mu^\epsilon)^2 \int \frac{d^n q}{(2\pi)^n} \gamma_\nu \left(\frac{i(\not{p} + \not{q} + m)}{(q+p)^2 - m^2}\right) \gamma_\mu \left(\frac{-ig^{\mu\nu}\delta^{ab}}{q^2}\right) \\ &= -C_F g_s^2 \mu^{2\epsilon} \int \frac{d^n q}{(2\pi)^n} \frac{\gamma_\mu(\not{p} + \not{q} + m)\gamma^\mu}{q^2\big[(q+p)^2 - m^2\big]}\,, \end{aligned} \tag{62}$$

after employing $\delta^{ab}\boldsymbol{T^a T^b} = C_F \mathbb{1}$.

Using n-dimensional Dirac matrix algebra,

$$\gamma_\mu(\not{p}+\not{q}+m)\gamma^\mu = (2-n)(\not{p}+\not{q})+nm\,. \tag{63}$$

Putting $n=4-2\epsilon$ and employing the Passarino-Veltman functions defined in problem 1 of Problem Set 3,

$$\begin{aligned} -i\Sigma(p) &= g_s^2 C_F \mu^{2\epsilon}\int\frac{d^nq}{(2\pi)^n}\frac{2(1-\epsilon)(\not{p}+\not{q})-(4-2\epsilon)m}{q^2\big[(q+p)^2-m^2\big]} \\ &= \frac{ig_s^2C_F}{16\pi^2}\big\{\big[2(1-\epsilon)\not{p}-(4-2\epsilon)m\big]B_0(p^2;0,m^2)+2(1-\epsilon)\not{p}B_1(p^2;0,m^2)\big\}\,. \end{aligned} \tag{64}$$

We now make use of eqs. (10) and (84) of the Solutions to Problem Set 4,

$$B_0(p^2;0,m^2) = \Delta - \int_0^1 \ln\left(\frac{xm^2-p^2x(1-x)}{\mu^2}\right)dx + \mathcal{O}(\epsilon)\,, \tag{65}$$

$$B_1(p^2;0,m^2) = -\tfrac{1}{2}\Delta + \int_0^1 x\ln\left(\frac{xm^2-p^2x(1-x)}{\mu^2}\right)dx + \mathcal{O}(\epsilon)\,, \tag{66}$$

where

$$\Delta \equiv \frac{1}{\epsilon}-\gamma+\ln(4\pi)\,. \tag{67}$$

It follows that

$$\begin{aligned} -i\Sigma(p) = {} & \frac{ig_s^2C_F}{16\pi^2}(\not{p}-4m)\Delta \\ & -\frac{ig_s^2C_F}{16\pi^2}\left\{\not{p}-2m+2\int_0^1\big[\not{p}(1-x)-2m\big]\ln\left(\frac{xm^2-p^2x(1-x)}{\mu^2}\right)dx\right\}+\mathcal{O}(\epsilon)\,. \end{aligned} \tag{68}$$

In the $\overline{\rm MS}$ renormalization scheme, the term proportional to Δ is exactly canceled by the counterterms. Thus, the one-loop correction to the inverse propagator is given by $i\Gamma^{(2)}(p)$ [cf. eq. (123) of the solutions to Problem Set 3], where

$$\begin{aligned} \Gamma^{(2)}(p) &= \not{p}-m-\Sigma(p) \\ &= \not{p}-m-\frac{g_s^2C_F}{16\pi^2}\left\{\not{p}-2m+2\int_0^1[\big[\not{p}(1-x)-2m\big]\ln\left(\frac{xm^2-p^2x(1-x)}{\mu^2}\right)dx\right\}, \end{aligned} \tag{69}$$

where $m\equiv m(\mu)$. It is common practice to define the $\overline{\rm MS}$ mass parameter by $m(\mu=m)$. That is we simply set $\mu=m$ in eq. (69) to obtain,

$$\Gamma^{(2)}(p) = \not{p}-m-\frac{g_s^2C_F}{16\pi^2}\left\{\not{p}-2m+2\int_0^1\big[\not{p}(1-x)-2m\big]\ln\left(\frac{xm^2-p^2x(1-x)}{m^2}\right)dx\right\}, \tag{70}$$

where $m\equiv m(m)$ is the $\overline{\rm MS}$ mass parameter.

The pole mass, denoted by M, is equal to the pole of the one-loop propagator, or equivalently the zero of the inverse propagator. Hence, we can identify the pole mass by the equation,

$$\Gamma^{(2)}(p)\big|_{\not{p}=M} = 0\,. \tag{71}$$

Hence eq. (70) yields,

$$M - m = \frac{g_s^2 C_F}{16\pi^2}\left\{M - 2m + 2\int_0^1 \big[M(1-x) - 2m\big]\ln\left(\frac{xm^2 - M^2 x(1-x)}{m^2}\right)dx\right\}. \tag{72}$$

Since $M = m[1+\mathcal{O}(g_s^2)]$, it is consistent within the one-loop approximation to set $M = m$ on the right hand side of eq. (72). Hence,

$$M - m = -\frac{g_s^2 C_F m}{16\pi^2}\left\{1 + 4\int_0^1 (1+x)\ln x\,dx\right\} = \frac{g_s^2 C_F m}{4\pi^2}\,. \tag{73}$$

Setting $g_s^2 = 4\pi\alpha_s$, we end up with,

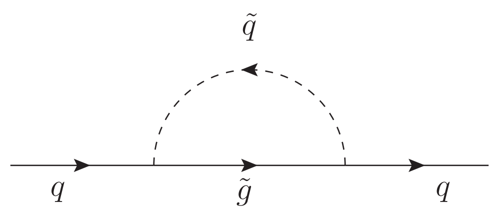
$$M = m\left(1 + \frac{C_F\alpha_s}{\pi}\right)\,. \tag{74}$$

Noting that $C_F = (N^2-1)/(2N)$ for an SU(N) gauge theory, we set $N=3$ to obtain the one-loop pole mass in terms of the $\overline{\rm MS}$ mass parameter,

$$M = m\left(1 + \frac{4\alpha_s}{3\pi}\right)\,. \tag{75}$$

(d) Repeat part (c) for supersymmetric QCD. Which new Feynman graphs contribute? How is the one-loop relation of part (c) modified? For simplicity, you may take the gluino to be massless and the top-squarks to be degenerate in mass with the top-quark.

In supersymmetric QCD, two new graphs contribute,



where we must sum over contributions from $\tilde{q} = \tilde{q}_L$ and $\tilde{q}_R$, respectively. In this calculation, we shall take the gluino to be massless and the quark and squarks to be mass-degenerate. Defining the four-momenta as in part (c) and applying the Feynman rules yields,

$$-i\Sigma(p)_{\rm SUSY} = \tfrac{1}{2}\boldsymbol{T^a T^b}(-ig_s\mu^\epsilon)^2\int\frac{d^n q}{(2\pi)^n}\left\{(1-\gamma_5)\left(\frac{i(\not{p}+\not{q})\delta^{ab}}{(q+p)^2}\right)(1+\gamma_5)\right.$$
$$\left.+(1+\gamma_5)\left(\frac{i(\not{p}+\not{q})\delta^{ab}}{(q+p)^2}\right)(1-\gamma_5)\right\}\left(\frac{i}{q^2-m^2}\right)\,, \tag{76}$$

after adding the two contributions from $\tilde{q} = \tilde{q}_L$ and $\tilde{q}_R$. Once again we employ $\delta^{ab}\boldsymbol{T^a T^b} = C_F\mathbb{1}$.

It then follows that,

$$-i\Sigma(p)_{\rm SUSY} = 2g_s^2 C_F \mu^{2\epsilon} \int \frac{d^n q}{(2\pi)^n} \frac{\not{p} + \not{q}}{(q^2 - m^2)(q+p)^2}\,, \tag{77}$$

after employing $\delta^{ab}\boldsymbol{T^a T^b} = C_F \mathbb{1}$. It is convenient to change integration variables, $q \to p + q$ and rewrite eq. (77) as,

$$\begin{aligned}
-i\Sigma(p)_{\rm SUSY} &= 2g_s^2 C_F \mu^{2\epsilon} \int \frac{d^n q}{(2\pi)^n} \frac{\not{q}}{q^2\big[(q-p)^2 - m^2\big]} = -\frac{ig_s^2 C_F}{8\pi^2}\not{p} B_1(p^2; 0, m^2) \\
&= \frac{ig_s^2 C_F}{16\pi^2}\not{p}\Delta - \frac{ig_s^2 C_F}{8\pi^2}\not{p} \int_0^1 x \ln\left(\frac{xm^2 - p^2 x(1-x)}{\mu^2}\right) dx\,.
\end{aligned} \tag{78}$$

As in part (c), we set $\mu = m$, in which case we can identify $m = m(m)$ with the $\overline{\rm MS}$ mass parameter. The term proportional to Δ in eq. (78) is absorbed by the counterterms, and we are left after renormalization with,

$$-i\Sigma(p)_{\rm SUSY} = -\frac{ig_s^2 C_F}{8\pi^2}\not{p} \int_0^1 x \ln\left(\frac{m^2 x - p^2 x(1-x)}{m^2}\right) dx. \tag{79}$$

Adding this result to the one obtained in part (c) yields,

$$\begin{aligned}
\Gamma^{(2)}(p) = \not{p} - m - \frac{g_s^2 C_F}{16\pi^2}\Bigg\{ \not{p} - 2m + 2\int_0^1 \big[\not{p}(1-x) - 2m\big] \ln\left(\frac{xm^2 - p^2 x(1-x)}{m^2}\right) dx \\
+ 2\not{p} \int_0^1 x \ln\left(\frac{m^2 x - p^2 x(1-x)}{m^2}\right) dx \Bigg\}\,.
\end{aligned} \tag{80}$$

Following the same steps employed in part (c), we can identify the pole mass M,

$$M - m = -\frac{g_s^2 C_F m}{16\pi^2}\left\{1 + 4\int_0^1 (1+x)\ln x\, dx - 4\int_0^1 x \ln x\, dx\right\} = \frac{3g_s^2 C_F m}{16\pi^2}\,. \tag{81}$$

Setting $g_s^2 = 4\pi\alpha_s$, we end up with,

$$M = m\left(1 + \frac{3C_F \alpha_s}{4\pi}\right)\,. \tag{82}$$

Finally, we set $N = 3$ to obtain the one-loop pole mass in terms of the $\overline{\rm MS}$ mass parameter in supersymmetric QCD,

$$M = m\left(1 + \frac{\alpha_s}{\pi}\right)\,. \tag{83}$$

<u>REMARKS</u>:

You will not find either eq. (82) or eq. (83) in the literature. The reason for this is that theorists do not like to employ the $\overline{\rm MS}$ renormalization scheme in supersymmetric models, because in $n \neq 4$ spacetime dimensions, the equality of the number of bosonic and fermionic degrees of freedom in supersymmetric theories is spoiled. Thus, the $\overline{\rm MS}$ subtraction scheme does not respect supersymmetry. However, there is a related scheme called dimensional reduction (DR) that preserves supersymmetry, and this is the preferred choice for supersymmetric theories. The analogue of $\overline{\text{MS}}$ renormalization is called $\overline{\text{DR}}$ renormalization. Operationally, the only change to the computations presented in this problem occur in part (c).

One way to implement the $\overline{\text{DR}}$ renormalization procedure in part (c) is to treat the factor of $g_{\mu\nu}$ that appears in the gluon propagator as a four-dimensional object.[2] Consequently, one must replace eq. (63) by

$$\gamma_\mu(\not p + \not q + m)\gamma^\mu = -2(\not p + \not q) + 4m\,, \tag{84}$$

in which case eq. (64) is replaced by

$$\begin{aligned}
-i\Sigma(p) &= 2g_s^2 C_F \mu^{2\epsilon}\int \frac{d^n q}{(2\pi)^n}\frac{\not p + \not q - 2m}{q^2\big[(q+p)^2 - m^2\big]} \\
&= \frac{ig_s^2 C_F}{8\pi^2}\big\{(\not p - 2m)B_0(p^2;0,m^2) + \not p B_1(p^2;0,m^2)\big\} \\
&= \frac{ig_s^2 C_F}{16\pi^2}(\not p - 4m)\Delta - \frac{ig_s^2 C_F}{8\pi^2}\int_0^1 \big[\not p(1-x) - 2m\big]\ln\left(\frac{xm^2 - p^2 x(1-x)}{\mu^2}\right)dx + \mathcal{O}(\epsilon)\,.
\end{aligned} \tag{85}$$

Hence, eq. (72) is modified to

$$M - m = \frac{g_s^2 C_F}{8\pi^2}\int_0^1 \big[M(1-x) - 2m\big]\ln\left(\frac{xm^2 - M^2 x(1-x)}{m^2}\right)dx\,. \tag{86}$$

Setting $M = m$ on the right hand side of eq. (86) yields,

$$M - m = -\frac{g_s^2 C_F m}{4\pi^2}\int_0^1 (1+x)\ln x\,dx = \frac{5g_s^2 C_F m}{16\pi^2}\,. \tag{87}$$

Setting $g_s^2 = 4\pi\alpha_s$, we end up with,

$$M = m\left(1 + \frac{5C_F\alpha_s}{4\pi}\right)\,, \qquad \text{gluon-quark loop contribution in } \overline{\text{DR}} \text{ renormalization.} \tag{88}$$

The contribution of the squark–gluino loop [which is given by the last term within the braces in eq. (81)],

$$M = m\left(1 - \frac{C_F\alpha_s}{4\pi}\right)\,, \qquad \text{gluino-squark loop contribution in } \overline{\text{DR}} \text{ renormalization,} \tag{89}$$

is the same in both $\overline{\text{MS}}$ and $\overline{\text{DR}}$ renormalization. Adding the two contributions above yields the one-loop pole mass in terms of the $\overline{\text{DR}}$ mass parameter in supersymmetric QCD,

$$M = m\left(1 + \frac{C_F\alpha_s}{\pi}\right)\,, \tag{90}$$

which is the result that can be found in the literature [instead of the $\overline{\text{MS}}$ result obtained in eq. (82)].

[2]The actual dimensional reduction procedure is more complicated in general, but in the present application the modification suggested here suffices.

3. Consider a theory of a single massless scalar real field:

$$\mathscr{L} = \partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda}{4!}\phi^4\,.$$

In class, we computed the effective potential ($V_{\rm eff}$) in the one-loop approximation. The renormalized $V_{\rm eff}$ depends on the parameter μ (which is either the mass scale of dimensional regularization or the off-shell subtraction point used in the definition of λ). The unrenormalized $V_{\rm eff}$ is independent of μ.

(a) Deduce the renormalization group equation (RGE) satisfied by the renormalized $V_{\rm eff}$. Your equation should involve the beta-function $\beta(\lambda_R)$ and the anomalous dimension $\gamma_d(\lambda_R)$, where λ_R is the renormalized coupling.

In class, we derived the following expression for the effective potential of a quantum field theory of a real massive scalar field prior to renormalization, where the divergences have been regulated using dimensional regularization,

$$V_{\rm eff}(\phi) = \tfrac{1}{2}m^2\phi^2\left[1 - \frac{\lambda}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi) + \frac{3}{2}\right)\right] - \frac{\lambda}{4!}\phi^4\left[1 - \frac{3\lambda}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi) + \frac{3}{2}\right)\right]$$
$$+\frac{1}{64\pi^2}\left[(m^2 + \tfrac{1}{2}\lambda\phi^2)^2\ln(m^2 + \tfrac{1}{2}\lambda\phi^2) - m^4\ln m^2\right]\,, \qquad (91)$$

where λ and m are bare parameters and ϕ is the unrenormalized (bare) scalar field. We then showed that the divergences could be completely absorbed by replacing λ and m with the corresponding renormalized parameters, λ_R and m_R, respectively, and by replacing the bare field ϕ with the corresponding renormalized field ϕ_R.[3] We carry out the renormalization procedure by substituting,

$$\phi = Z_\phi^{1/2}\phi_R\,, \qquad m^2 = Z_m m_R^2\,, \qquad \lambda = \mu^{2\epsilon}Z_\lambda\lambda_R\,. \qquad (92)$$

Consequently, the effective potential apparently depends on μ. However, this is an illusion since one observes from eq. (91) that $V_{\rm eff}(\phi)$ is explicitly independent of μ. That is,

$$\frac{d}{d\mu}V_{\rm eff}(\phi) = 0\,. \qquad (93)$$

When $V_{\rm eff}(\phi)$ is re-expressed in terms of the normalized parameters and fields, then we can apply eq. (93) by invoking the chain rule,

$$\mu\frac{d}{d\mu} = \mu\frac{\partial}{\partial\mu} + \mu\frac{d\lambda_R}{d\mu}\frac{\partial}{\partial\lambda_R} + \mu\frac{dm_R}{d\mu}\frac{\partial}{\partial m_R} + \mu\frac{d\phi_R}{d\mu}\frac{\partial}{\partial\phi_R}\,. \qquad (94)$$

We define as usual,

$$\beta(\lambda_R) \equiv \mu\frac{d\lambda_R}{d\mu}\,, \qquad m_R\gamma_m(\lambda_R) \equiv \mu\frac{dm_R}{d\mu}\,. \qquad (95)$$

[3]Since $Z_\phi = 1 + \mathcal{O}(\lambda^2)$, it follows that $\phi_R = \phi$ in a one-loop analysis.

Since the bare field ϕ knows nothing about μ, if follows that

$$0 = \mu\frac{d\phi}{d\mu} = \mu\frac{d}{d\mu}\big(Z_\phi^{1/2}\phi_R\big) = \phi_R\mu\frac{dZ_\phi^{1/2}}{d\mu} + Z_\phi^{1/2}\mu\frac{d\phi_R}{d\mu}\,. \tag{96}$$

Consequently,

$$\mu\frac{d\phi_R}{d\mu} = -\phi_R Z_\phi^{-1/2}\mu\frac{dZ_\phi^{1/2}}{d\mu} = -\tfrac{1}{2}\phi_R\mu\frac{d}{d\mu}\ln Z_\phi = -\phi_R\gamma_d(\lambda_R)\,, \tag{97}$$

where γ_d is the anomalous dimension. That is, we can rewrite the chain rule given in eq. (94) as

$$\mu\frac{d}{d\mu} = \mu\frac{\partial}{\partial\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} + m_R\gamma_m(\lambda_R)\frac{\partial}{\partial m_R} - \gamma_d(\lambda_R)\phi_R\frac{\partial}{\partial\phi_R}\,. \tag{98}$$

That is, eq. (93) yields the renormalization group equation (RGE),

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} + m_R\gamma_m(\lambda_R)\frac{\partial}{\partial m_R} - \gamma_d(\lambda_R)\phi_R\frac{\partial}{\partial\phi_R}\right)V_{\rm eff}(\phi_R) = 0\,. \tag{99}$$

In the case of the massless theory ($m_R = 0$), the RGE satisfied by the effective potential simplifies to,

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} - \gamma_d(\lambda_R)\phi_R\frac{\partial}{\partial\phi_R}\right)V_{\rm eff}(\phi_R) = 0\,. \tag{100}$$

(b) By dimensional analysis, the renormalized $V_{\rm eff}$ can be written as:

$$V_{\rm eff}(\phi_R) = \frac{Y(\lambda_R, t)\phi_R^4}{4!}\,, \tag{101}$$

where $t = \log(\phi_R/\mu)$ and ϕ_R is the renormalized scalar field. Assume that $V_{\rm eff}$ is defined in the physical scheme where,

$$\left.\frac{d^2V_{\rm eff}}{d\phi_R^2}\right|_{\phi_R=0} = 0, \qquad\qquad \left.\frac{d^4V_{\rm eff}}{d\phi_R^4}\right|_{\phi_R=\mu} = \lambda_R\,. \tag{102}$$

Rewrite the RGE of part (a) as an equation for $Y(\lambda_R, t)$. Solve the resulting equation for Y as a function of a suitably defined running coupling constant $\overline{\lambda}(t)$.

Plugging eq. (101) into the RGE given in eq. (100) yields,

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} - \gamma_d(\lambda_R)\phi_R\frac{\partial}{\partial\phi_R}\right)Y(\lambda_R, t)\phi_R^4 = 0\,. \tag{103}$$

The above equation can be further simplified by noting that for a function of $t \equiv \ln(\phi/\mu)$, one can always write,

$$\left(\mu\frac{\partial}{\partial\mu} - \gamma_d\phi_R\frac{\partial}{\partial\phi_R}\right) f\big(\ln(\phi_R/\mu)\big) = \left(\mu\frac{\partial t}{\partial\mu}\frac{\partial}{\partial t} - \gamma_d\phi_R\frac{\partial t}{\partial\phi_R}\frac{\partial}{\partial t}\right) f(t) = -(1+\gamma_d)\frac{\partial f}{\partial t}\,. \tag{104}$$

This result allows us to rewrite eq. (103) as

$$\left[\beta\frac{\partial}{\partial\lambda_R} - (1+\gamma_d)\frac{\partial}{\partial t} - 4\gamma_d\right]Y(\lambda_R,t) = 0\,. \tag{105}$$

It is therefore convenient to define a rescaled β and γ_d via,

$$\overline{\beta} \equiv \frac{\beta}{1+\gamma_d}\,, \qquad\qquad \overline{\gamma}_d \equiv \frac{\gamma_d}{1+\gamma_d}\,. \tag{106}$$

Then, eq. (105) simplifies even further,

$$\left(-\frac{\partial}{\partial t} + \overline{\beta}\frac{\partial}{\partial\lambda_R} - 4\overline{\gamma}_d\right)Y(\lambda_R,t) = 0\,. \tag{107}$$

To solve eq. (107), it is convenient to introduce a running coupling that is the solution to the following differential equation,

$$\frac{d\overline{\lambda}(t)}{dt} = \overline{\beta}\big(\overline{\lambda}(t)\big)\,, \qquad \text{subject to } \overline{\lambda}(t=0) = \lambda_R. \tag{108}$$

Note that $\overline{\lambda}(t)$ is not quite the same as the running coupling defined in class since the right hand side of eq. (108) involves $\overline{\beta}$ rather than β. Nevertheless, the form of the differential equation given in eq. (107) is of the same form as the one we solved in class. The solution to eq. (107) can be obtained directly using the following technique. Introduce the function

$$y(\lambda_R) = \int_{\lambda_0}^{\lambda_R}\frac{dz}{\overline{\beta}(z)}\,, \tag{109}$$

and note that

$$\overline{\beta}\frac{\partial}{\partial\lambda_R} = \overline{\beta}\frac{\partial y}{\partial\lambda_R}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\,. \tag{110}$$

Hence, eq. (107) can be rewritten in the following form,

$$\left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial y} - 4\overline{\gamma}_d\right)Y(\lambda_R,t) = 0\,. \tag{111}$$

In light if the fact that

$$\left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right)f(t+y) = 0\,, \tag{112}$$

for an arbitrary function f, it follows that the solution to eq. (111) is

$$Y(\lambda_R,t) = \exp\left\{4\int_{\lambda_0}^{\lambda_R}\frac{\overline{\gamma}_d(z)}{\overline{\beta}(z)}\,dz\right\}f(t+y)\,. \tag{113}$$

To verify this assertion, note that the exponential function is independent of t and depends on y implicitly through λ_R. Hence,

$$\begin{aligned}\left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right)\exp\left\{4\int_{\lambda_0}^{\lambda_R}\frac{\overline{\gamma}_d(z)}{\overline{\beta}(z)}\,dz\right\} &= 4\exp\left\{4\int_{\lambda_0}^{\lambda_R}\frac{\overline{\gamma}_d(z)}{\overline{\beta}(z)}\,dz\right\}\frac{\partial}{\partial y}\int_{\lambda_0}^{\lambda_R(y)}\frac{\overline{\gamma}_d(z)}{\overline{\beta}(z)}\,dz \\ &= 4\exp\left\{4\int_{\lambda_0}^{\lambda_R}\frac{\overline{\gamma}_d(z)}{\overline{\beta}(z)}\,dz\right\}\overline{\beta}\frac{\partial}{\partial\lambda_R}\int_{\lambda_0}^{\lambda_R}\frac{\overline{\gamma}_d(z)}{\overline{\beta}(z)}\,dz \\ &= 4\,\overline{\gamma}_d(\lambda_R)\exp\left\{4\int_{\lambda_0}^{\lambda_R}\frac{\overline{\gamma}_d(z)}{\overline{\beta}(z)}\,dz\right\}\,. \end{aligned} \tag{114}$$

Thus, we have shown that the solution to eq. (107) is[4]

$$Y(\lambda_R, t) = \exp\left\{4\int_{\lambda_0}^{\lambda_R} \frac{\overline{\gamma}_d(z)}{\overline{\beta}(z)}\,dz\right\} f\left(t + \int_{\lambda_0}^{\lambda_R} \frac{dz}{\overline{\beta}(z)}\right)\,, \tag{115}$$

where f is an arbitrary function. The form of this solution simplifies considerably by introducing the running coupling defined in eq. (108), which is equivalent to,

$$t = \int_{\lambda_R}^{\overline{\lambda}(t)} \frac{dz}{\overline{\beta}(z)}\,. \tag{116}$$

It follows that,

$$t + \int_{\lambda_0}^{\lambda_R} \frac{dz}{\overline{\beta}(z)} = \int_{\lambda_0}^{\overline{\lambda}(t)} \frac{dz}{\overline{\beta}(z)}\,. \tag{117}$$

Hence,

$$Y(\lambda_R, t) = \exp\left\{4\int_{\lambda_0}^{\lambda_R} \frac{\overline{\gamma}_d(z)}{\overline{\beta}(z)}\,dz\right\} f\left(\int_{\lambda_0}^{\overline{\lambda}(t)} \frac{dz}{\overline{\beta}(z)}\right)\,. \tag{118}$$

On the other hand, if we put $\lambda_R \to \overline{\lambda}(t)$ and $t \to 0$ in eq. (115), we obtain,

$$Y(\overline{\lambda}(t), 0) = \exp\left\{4\int_{\lambda_0}^{\overline{\lambda}(t)} \frac{\overline{\gamma}_d(z)}{\overline{\beta}(z)}\,dz\right\} f\left(\int_{\lambda_0}^{\overline{\lambda}(t)} \frac{dz}{\overline{\beta}(z)}\right)\,. \tag{119}$$

Dividing eqs. (118) and (119) eliminates λ_0, and we end up with

$$Y(\lambda_R, t) = \exp\left\{-4\int_{\lambda_R}^{\overline{\lambda}(t)} \frac{\overline{\gamma}_d(z)}{\overline{\beta}(z)}\,dz\right\} Y(\overline{\lambda}(t), 0)\,. \tag{120}$$

Introducing a new variable, $z = \overline{\lambda}(t')$, it follows that $dt' = dz/\overline{\beta}(z)$. Noting that $t' = 0$ when $z = \lambda_R$ and $t' = t$ when $z = \overline{\lambda}(t)$,

$$\int_{\lambda_R}^{\overline{\lambda}(t)} \frac{\overline{\gamma}_d(z)}{\overline{\beta}(z)}\,dz = \int_0^t \overline{\gamma}_d\big(\overline{\lambda}(t')\big)\,dt'\,. \tag{121}$$

Hence,

$$Y(\lambda_R, t) = \exp\left\{-4\int_0^t \overline{\gamma}_d\big(\overline{\lambda}(t')\big)\,dt'\right\} Y(\overline{\lambda}(t), 0)\,. \tag{122}$$

In order to fix the function $Y(\overline{\lambda}(t), 0)$, we impose the renormalization condition,

$$\left.\frac{d^4 V_{\rm eff}}{d\phi_R^4}\right|_{\phi_R=\mu} = \lambda_R\,. \tag{123}$$

Plugging in eq. (101) with $t = \ln(\phi_R/\mu)$ yields

$$Y(\lambda_R, 0) = \lambda_R\,. \tag{124}$$

[4] In the mathematics literature, the technique that yields eq. (115) is often called the method of characteristics. See, e.g., Gustavo López, *Partial Differential Equations of First Order and Their Applications to Physics* (World Scientific, Singapore,1999).

We can now replace $\lambda_R \to \overline{\lambda}(t)$ in eq. (124), which implies that $Y(\overline{\lambda}(t), 0) = \overline{\lambda}(t)$. Hence, eq. (122) yields,

$$\boxed{Y(\lambda_R, t) = \overline{\lambda}(t) \exp\left\{-4\int_0^t \overline{\gamma}_d\big(\overline{\lambda}(t')\big)\right\}\,.} \tag{125}$$

(c) Assuming that β is constant (independent of λ_R) and $\gamma_d = 0$, use the result of part (b) to obtain a formula for the renormalized $V_{\rm eff}$. Compare this result to the one-loop effective potential computed in class.

If $\gamma_d = 0$, then $\overline{\beta} = \beta$, which is assumed to be a constant. In this approximation, eq. (116) yields,

$$t = \frac{1}{\beta}\int_{\lambda_R}^{\overline{\lambda}(t)} dz = \frac{\overline{\lambda}(t) - \lambda_R}{\beta}\,. \tag{126}$$

Hence, $\overline{\lambda}(t) = \lambda_R + \beta t$, and eq. (125) yields,

$$Y(\lambda_R, t) = \lambda_R + \beta \ln\left(\frac{\phi_R}{\mu}\right)\,. \tag{127}$$

That is,

$$V_{\rm eff}(\phi_R) = \frac{\lambda_R}{4!}\phi_R^4 + \frac{\beta\lambda_R\phi_R^4}{4!}\ln\left(\frac{\phi_R}{\mu}\right)\,. \tag{128}$$

Let us compare this result with the Coleman-Weinberg potential obtained in class,

$$V_{\rm eff}(\phi_R) = \frac{\lambda_R}{4!}\phi_R^4 + \frac{\lambda_R^2\phi_R^4}{256\pi^2}\left[\ln\left(\frac{\phi_R^2}{\mu^2}\right) - \frac{25}{6}\right]\,. \tag{129}$$

The factor of 25/6 can always be reabsorbed into a redefinition of the renormalized coupling. Thus, a comparison of eqs. (128) and (129) would yield,

$$\beta(\lambda_R) = \frac{3\lambda_R^2}{16\pi^2}\,, \tag{130}$$

which we recognize as the one-loop β-function of the scalar field theory!

(d) Repeat part (c), but now use the one-loop approximations for β and γ_d. (*HINT: γ_d is still zero in this approximation. Why?*) The resulting $V_{\rm eff}$ is now the renormalization group improved effective potential. Recall that $V_{\rm eff}$ in the one-loop approximation had a local maximum at $\phi_R = 0$ and a local minimum for a nonzero value of ϕ_R. Is the extremum of the renormalization group improved $V_{\rm eff}$ at $\phi_R = 0$ a minimum or a maximum? Is the discrete $\phi_R \to -\phi_R$ symmetry spontaneously broken?

We shall now evaluate eq. (125) in the one-loop approximation with no additional assumptions. In class, we showed that $Z_\phi = 1 + \mathcal{O}(\lambda_R^2)$. That is, in the one-loop approximation, $Z_\phi = 1$ which implies that $\gamma_d = 0$. Hence, eq. (106) implies that $\overline{\beta} = \beta$. In class we derived eq. (130), which then yields the one-loop running coupling,

$$\overline{\lambda}(t) = \lambda_R \left[1 - \frac{3\lambda_R t}{16\pi^2}\right]^{-1} . \tag{131}$$

Writing $t = \frac{1}{2}\ln(\phi^2/\mu^2)$ and using eq. (125) yields the one-loop renormalization group improved effective potential,

$$V_{\rm eff}(\phi_R) = \frac{\frac{1}{4!}\lambda_R\phi_R^4}{1 - \frac{3\lambda_R}{32\pi^2}\ln\left(\frac{\phi_R^2}{\mu^2}\right)} . \tag{132}$$

Note that if we expand out the denominator in a perturbation series, we recover the one-loop Coleman-Weinberg potential (up to the non-logarithmic 25/6, which is not picked up by the renormalization group improvement). Indeed, in this way, we recover the leading logs to all orders in perturbation theory!

Let us now check for the extrema of the renormalization group improved effective potential. We compute the first derivative of $V_{\rm eff}(\phi_R)$ and obtain,

$$\frac{dV_{\rm eff}}{d\phi_R} = \frac{\frac{1}{6}\lambda_R\phi_R^3}{1 - \frac{3\lambda_R}{32\pi^2}\ln\left(\frac{\phi_R^2}{\mu^2}\right)} + \frac{\frac{1}{128\pi^2}\lambda_R^2\phi_R^3}{\left[1 - \frac{3\lambda_R}{32\pi^2}\ln\left(\frac{\phi_R^2}{\mu^2}\right)\right]^2} . \tag{133}$$

One extrema of the effective potential is at $\phi_R = 0$. But in contrast to the one-loop Coleman-Weinberg potential, the $\phi_R = 0$ extremum is a local minimum! To verify this assertion, one can check that for very small but non-zero values of ϕ_R^2, the coefficient of ϕ_R^4 in eq. (132) is positive (whereas it is negative for the one-loop Coleman-Weinberg potential).

We can also look for extrema of the effective potential for $\phi_R \neq 0$. Setting $dV_{\rm eff}/d\phi_R = 0$ then yields,

$$1 - \frac{3\lambda_R}{32\pi^2}\ln\left(\frac{\phi_R^2}{\mu^2}\right) + \frac{3\lambda_R}{64\pi^2} = 0 . \tag{134}$$

which implies that the extremum occurs at a field value of

$$\phi_R^2 = \mu^2 \exp\left[\frac{32\pi^2}{3\lambda_R^2} - \frac{1}{2}\right] . \tag{135}$$

One can check that this is a minimum. However, it is not a reliable extremum, since it occurs at a very large value of ϕ_R in the perturbative limit of $\lambda_R \ll 1$. That is, this extremum lies outside of the range of validity of the renormalization group improved computation.

In the case of the original one-loop Coleman-Weinberg effective potential, the maximum at $\phi_R = 0$ and the minimum at $\phi_R \neq 0$ were both unreliable as these field values were outside the validity of the one-loop computation. In contrast, the renormalization group improved effective potential has a minimum at $\phi_R = 0$ which is reliable (although the minimum at $\phi_R \neq 0$ is not). Hence, we conclude that a local minimum of the effective potential exists at $\phi_R = 0$ and the discrete symmetry, $\phi_R \to -\phi_R$ is *not* spontaneously broken.

REFERENCE:

A very nice reference to the material presented in this problem can be found in M. Sher, "Electroweak Higgs Potentials and Vacuum Stability," Phys. Rept. **179**, 273–418 (1989).

4. Consider scalar electrodynamics where the bare tree-level scalar mass parameter is zero,

$$\mathscr{L} = (D_\mu\phi)(D^\mu\phi)^* - \lambda(\phi\phi^*)^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\,, \tag{136}$$

where $D_\mu \equiv \partial_\mu + ieA_\mu$.

(a) Compute the one-loop effective potential in the Landau gauge ($\xi = 0$) in two different schemes: the $\overline{\rm MS}$ scheme and the physical scheme analogous to the one defined in eq. (102). Assume that the renormalized couplings have the property that λ_R is of $\mathcal{O}(e_R^4)$, and keep only terms of the same order in $V_{\rm eff}$.

It is convenient to write,

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}\,, \tag{137}$$

where ϕ_1 and ϕ_2 are real fields. Then, eq. (136) becomes,

$$\begin{aligned}\mathscr{L} = {} & \tfrac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2) + eA^\mu(\phi_1\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_1) + \tfrac{1}{2}e^2A_\mu A^\mu(\phi_1^2+\phi_2^2) \\ & -\tfrac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\,.\end{aligned} \tag{138}$$

In class, we derived the following formula for the effective potential in the Landau gauge (corresponding to $\xi = 0$),[5]

$$V^{(1)}(\Phi) = -\tfrac{1}{2}i\,{\rm Str}\int\frac{d^nq}{(2\pi)^n}\ln\left(\frac{q^2 - M_i^2(\Phi) + i\varepsilon}{q^2 - M_i^2(0) + i\varepsilon}\right)\,, \tag{139}$$

where the supertrace instructs us to perform a sum over all fields in the theory that couple to the scalar weighted by the number of degrees of freedom of the field. In addition, bosonic contributions appear in the sum with a plus sign whereas fermionic contributions appear in the sum with a minus sign. That is,

$${\rm Str}\,\{\cdots\} = \sum_i (-1)^{2J_i}(2J_i+1)C_i\{\cdots\}\,, \tag{140}$$

where J_i is the spin of the field i and C_i counts any additional internal degrees of freedom (e.g., electric charge and color).

However, eq. (139) is not quite right for the vector boson contribution, since as noted in class,the computation actually yields $n - 1 = 3 - 2\epsilon$ degrees of freedom for the vector boson contribution in the Landau gauge. Thus, we must remember to include this correction when applying eq. (140).

Applying eq. (139) for the Lagrangian given in eq. (138), we can use the results obtained in class to identifying the relevant field-dependent squared masses. In particular, in order to compute the field-dependent squared masses, it is sufficient to shift only one of the scalar fields [due to the O(2) symmetry of the scalar sector], $\phi_1 \to \phi_1 + \Phi$. After the shift, the terms of the Lagrangian that are quadratic in the scalar fields yield two scalar squared masses,

$$-\tfrac{1}{4}\lambda\big[(\phi_1+\Phi)^2 + \phi_2^2\big]^2 = -\tfrac{1}{4}\lambda(\phi_1^2+\phi_2^2+2\phi_1\Phi+\Phi^2)^2 = -\tfrac{1}{2}\lambda\Phi^2(3\phi_1^2+\phi_2^2)+\cdots \tag{141}$$

[5] I have reinserted the $i\varepsilon$ factors that were omitted in the class lecture.

The coefficients of $-\frac{1}{2}\phi_1^2$ and $-\frac{1}{2}\phi_2^2$ are the field-dependent scalar squared masses, i.e. $3\lambda\Phi^2$ and $\lambda\Phi^2$. Likewise, the term of the Lagrangian that is quadratic in the vector boson fields yield the vector boson squared mass,

$$\tfrac{1}{2}e^2 A_\mu A^\mu\big[(\phi_1+\Phi)^2+\phi_2^2\big]=\tfrac{1}{2}e^2\Phi^2 A_\mu A^\mu+\cdots \tag{142}$$

The coefficient of $\frac{1}{2}A_\mu A^\mu$ is the field-dependent vector boson squared mass, i.e. $e^2\Phi^2$.

Hence, eq. (139) yields,

$$V^{(1)}(\Phi)=-\tfrac{1}{2}i\int\frac{d^nq}{(2\pi)^n}\left\{\ln\left(\frac{q^2-\lambda\Phi^2+i\varepsilon}{q^2+i\varepsilon}\right)+\ln\left(\frac{q^2-3\lambda\Phi^2+i\varepsilon}{q^2+i\varepsilon}\right)\right.$$
$$\left.+(3-2\epsilon)\ln\left(\frac{q^2-e^2\Phi^2+i\varepsilon}{q^2+i\varepsilon}\right)\right\}. \tag{143}$$

We can now reinterpret this equation as the effective potential of a single complex field ϕ by setting $\Phi^2=\phi_1^2+\phi_2^2=2\phi^*\phi$ in eq. (143),[6]

$$V^{(1)}(\phi)=-\tfrac{1}{2}i\int\frac{d^nq}{(2\pi)^n}\left\{\ln\left(\frac{q^2-2\lambda\phi^*\phi+i\varepsilon}{q^2+i\varepsilon}\right)+\ln\left(\frac{q^2-6\lambda\phi^*\phi+i\varepsilon}{q^2+i\varepsilon}\right)\right.$$
$$\left.+(3-2\epsilon)\ln\left(\frac{q^2-2e^2\phi^*\phi+i\varepsilon}{q^2+i\varepsilon}\right)\right\}. \tag{144}$$

In class, we derived the following formula,

$$\int\frac{d^nq}{(2\pi)^n}\ln\left(\frac{q^2-M^2+i\varepsilon}{q^2-m^2+i\varepsilon}\right)=-i(4\pi)^{\epsilon-2}\Gamma(\epsilon-2)\big[(M^2)^{2-\epsilon}-(m^2)^{2-\epsilon}\big]. \tag{145}$$

It follows that the effective potential including the tree-level and one-loop contributions is given by,

$$V_{\rm eff}(\phi)=\lambda(\phi^*\phi)^2-\frac{5\lambda^2}{8\pi^2}(\phi^*\phi)^2\left[\frac{1}{\epsilon}-\gamma+\ln(4\pi)+\frac{3}{2}\right]-\frac{3e^4}{16\pi^2}(\phi^*\phi)^2\left[\frac{1}{\epsilon}-\gamma+\ln(4\pi)+\frac{5}{6}\right]$$
$$+\frac{\lambda^2}{16\pi^2}\left\{(\phi^*\phi)^2\ln(2\lambda\phi^*\phi)+9(\phi^*\phi)^2\ln(6\lambda\phi^*\phi)\right\}+\frac{3e^4}{16\pi^2}(\phi^*\phi)^2\ln(2e^2\phi^*\phi). \tag{146}$$

Note that the 5/6 appearing in eq. (146) arises since in the $\epsilon\to 0$ limit,

$$\lim_{\epsilon\to 0}\,(-\tfrac{1}{2}i)\;(-2\epsilon)\int\frac{d^nq}{(2\pi)^n}\ln\left(\frac{q^2-2e^2\phi^*\phi+i\varepsilon}{q^2+i\varepsilon}\right)=\frac{e^4}{(8\pi)^2}(\phi^*\phi)^2, \tag{147}$$

after using eq. (145). Thus,

$$-\frac{3e^4}{16\pi^2}(\phi^*\phi)^2\left(\frac{3}{2}-\frac{2}{3}\right)=-\frac{3e^4}{16\pi^2}(\phi^*\phi)^2\left(\frac{5}{6}\right). \tag{148}$$

[6] In this calculation, we performed the shift $\phi_1\to\phi_1+\Phi$ by making use of the O(2) symmetry of the scalar sector. But, the same O(2) symmetry implies that had we shifted $\phi_1\to\phi_1+\Phi_1$ and $\phi_2\to\phi_2+\Phi_2$, the end result would have been eq. (143) with $\Phi^2=\Phi_1^2+\Phi_2^2$.

The problem indicates that $\lambda \sim \mathcal{O}(e^4)$. Thus, for the purposes of a consistent treatment of perturbative effects, we henceforth drop all terms of $\mathcal{O}(\lambda^2)$. Thus, our final expression is,

$$V_{\rm eff}(\phi) = \lambda(\phi^*\phi)^2 - \frac{3e^4}{16\pi^2}(\phi^*\phi)^2\left[\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right] + \frac{3e^4}{16\pi^2}(\phi^*\phi)^2\left[\ln(2e^2\phi^*\phi) - \tfrac{5}{6}\right]. \quad (149)$$

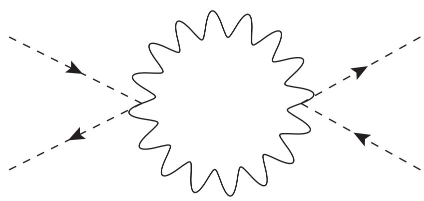
Eq. (149) is an expression in terms of the bare field and parameters. We now introduce the renormalized scalar field and renormalized couplings,

$$\phi = Z_\phi^{1/2}\phi_R\,, \quad (150)$$

$$\lambda = Z_\lambda^{(1)}\mu^{2\epsilon}\lambda_R + Z_\lambda^{(2)}\mu^{2\epsilon}e_R^4\,, \quad (151)$$

$$e = Z_e\mu^\epsilon e_R\,. \quad (152)$$

The renormalization of λ includes an $\mathcal{O}(e_R^4)$ correction due to the one-loop diagram,



which must be included in eq. (151) since by assumption λ_R and e_R^4 are taken to be parameters of the same order. In the approximation where we keep only terms of $\mathcal{O}(\lambda_R)$ and $\mathcal{O}(e_R^4)$, we may put $Z_\phi = Z_\lambda^{(1)} = Z_e = 1$. Hence,

$$\begin{aligned} V_{\rm eff}(\phi) &= \lambda_R\mu^{2\epsilon}(\phi_R^*\phi_R)^2 + e_R^4\mu^{2\epsilon}(\phi_R^*\phi_R)^2\left[Z_\lambda^{(2)} - \frac{3}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right)\right] \\ &\qquad + \frac{3e_R^4\mu^{4\epsilon}}{16\pi^2}(\phi_R^*\phi_R)^2\left[\ln(2e^2\phi_R^*\phi_R) - \tfrac{5}{6}\right]. \\ &= \lambda_R(\phi_R^*\phi_R)^2 + e_R^4(\phi_R^*\phi_R)^2\left[Z_\lambda^{(2)} - \frac{3}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right)\right] \\ &\qquad + \frac{3e_R^4}{16\pi^2}(\phi_R^*\phi_R)^2\left[\ln\left(\frac{2e^2\phi_R^*\phi_R}{\mu^2}\right) - \frac{5}{6}\right], \end{aligned} \quad (153)$$

after dropping terms of $\mathcal{O}(\epsilon)$. If we choose the $\overline{\rm MS}$ renormalization scheme, then we should identify,

$$Z_\lambda^{(2)} = \frac{3}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right). \quad (154)$$

Hence, the renormalized effective potential is in the $\overline{\rm MS}$ scheme is given by

$$\boxed{V_{\rm eff}(\phi)_{\overline{\rm MS}} = \lambda_R(\phi_R^*\phi_R)^2 + \frac{3e_R^4}{16\pi^2}(\phi_R^*\phi_R)^2\left[\ln\left(\frac{2e^2\phi_R^*\phi_R}{\mu^2}\right) - \frac{5}{6}\right].} \quad (155)$$

To evaluate the effective potential in the physical scheme, it is more convenient to employ the real fields ϕ_1 and ϕ_2 introduced in eq. (137). Since $\phi^*\phi = \frac{1}{2}(\phi_1^2 + \phi_2^2)$, we shall introduce the notation $\varphi^2 \equiv \phi_1^2 + \phi_2^2 = 2\phi^*\phi$, where φ is a real field. In this notation, eq. (153) reads,

$$V_{\rm eff}(\varphi) = \tfrac{1}{4}\lambda_R\varphi_R^4 + \tfrac{1}{4}e_R^4\varphi_R^4\left[Z_\lambda^{(2)} - \frac{3}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right)\right] + \frac{3e_R^4}{64\pi^2}\varphi_R^4\left[\ln\left(\frac{e^2\varphi_R^2}{\mu^2}\right) - \frac{5}{6}\right]. \tag{156}$$

The appropriate subtraction conditions analogous to those of eq. (102) are,[7]

$$\left.\frac{d^2V_{\rm eff}}{d\varphi_R^2}\right|_{\varphi_R=0} = 0, \qquad\qquad \left.\frac{d^4V_{\rm eff}}{d\varphi_R^4}\right|_{\varphi_R=\mu} = 6\lambda_R\,. \tag{157}$$

Note that the condition $(d^2V_{\rm eff}/d\varphi^2)_{\varphi=0} = 0$ is automatically satisfied by eq. (156). Imposing the second condition above yields,

$$6e_R^4\left[Z_\lambda^{(2)} - \frac{3}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right)\right] + \frac{9e_R^4}{8\pi^2}\left[\ln(e_R^2) + \tfrac{10}{3}\right] = 0\,. \tag{158}$$

Solving for $Z_\lambda^{(2)}$ yields,

$$Z_\lambda^{(2)} = \frac{3}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi) - \ln(e_R^2) - \tfrac{10}{3}\right). \tag{159}$$

Plugging this result back into eq. (156) yields,

$$V_{\rm eff}(\varphi) = \tfrac{1}{4}\lambda_R\varphi_R^4 + \frac{3e_R^4}{64\pi^2}\varphi_R^4\left[\ln\left(\frac{\varphi_R^2}{\mu^2}\right) - \frac{25}{6}\right]. \tag{160}$$

Rewriting this in term if the complex field by putting $\varphi^2 = 2\phi^*\phi$, we obtain the effective potential in the physical scheme,

$$\boxed{V_{\rm eff}(\phi)_{\rm physical} = \lambda_R(\phi_R^*\phi_R)^2 + \frac{3e_R^4}{16\pi^2}(\phi_R^*\phi_R)^2\left[\ln\left(\frac{2\phi_R^*\phi_R}{\mu^2}\right) - \frac{25}{6}\right].} \tag{161}$$

(b) Show that the $U(1)$ gauge symmetry is spontaneously broken and compute the mass of the resulting Higgs boson (m_H) in terms of the mass of the vector boson (m_V). Show that in the one-loop approximation considered here, the Higgs boson mass is scheme independent by showing that you get the same result in both schemes of part (a).

In the $\overline{\rm MS}$ scheme, we shall employ eq. (155). It is again convenient to write this potential in terms of the real fields, ϕ_1 and ϕ_2. For notational convenience, we henceforth drop the subscript R on the renormalized fields and couplings. Then,

$$V_{\rm eff}(\phi_1,\phi_2) = \tfrac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2 + \frac{3e^4}{64\pi^2}(\phi_1^2+\phi_2^2)^2\left[\ln\left(\frac{e^2(\phi_1^2+\phi_2^2)}{\mu^2}\right) - \frac{5}{6}\right]. \tag{162}$$

[7] The factor of 6 in eq. (157) is motivated by the observation that if $V_{\rm eff}$ is replaced by the tree-level scalar potential (denoted by V_0) then $(d^4V_0/d\varphi_R^4)\|_{\varphi=0} = 6\lambda$. However, in eq. (157) one must choose the subtraction scheme by evaluating $d^4V_{\rm eff}/d\varphi_R^4$ at $\varphi = \mu \neq 0$, in order to avoid infrared divergences (which arise due to the fact that the bare scalar mass has been set to zero).

The first derivative of the effective potential with respect to ϕ_i $(i = 1, 2)$ is

$$\frac{\partial V_{\rm eff}}{\partial \phi_i} = \lambda(\phi_1^2 + \phi_2^2)\phi_i + \frac{3e^4(\phi_1^2 + \phi_2^2)\phi_i}{16\pi^2}\left[\ln\left(\frac{e^2(\phi_1^2 + \phi_2^2)}{\mu^2}\right) - \frac{1}{3}\right] . \tag{163}$$

The extrema of the potential are obtained by setting $\partial V_{\rm eff}/\partial\phi_i = 0$ for $i = 1, 2$. Clearly, $\phi_1 = \phi_2 = 0$ corresponds to one of the extrema. However, it is easy to check that this corresponds to a local maximum of the effective potential. Assuming $\phi_1^2 + \phi_2^2 \neq 0$, one finds additional extrema correspond to the solution to the equation,

$$\lambda + \frac{3e^4}{16\pi^2}\left[\ln\left(\frac{e^2(\phi_1^2 + \phi_2^2)}{\mu^2}\right) - \frac{1}{3}\right] = 0 . \tag{164}$$

That is,

$$\phi_1^2 + \phi_2^2 = \frac{\mu^2}{e^2}\exp\left(\frac{1}{3} - \frac{16\pi^2\lambda}{3e^4}\right) . \tag{165}$$

Since $\lambda/e^4 \sim \mathcal{O}(1)$, this is a consistent solution in the context of perturbation theory. Moreover, one can check that eq. (165) corresponds to a local minimum of the effective potential. Thus, the U(1) gauge symmetry is spontaneously broken. We can use the U(1) symmetry, which is isomorphic to the SO(2) symmetry,

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \longrightarrow \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} , \tag{166}$$

to rotate the vacuum expectation value into, say, ϕ_2. Denoting the scalar field vacuum expectation value by v, it follows that

$$\langle\Omega\,|\,\phi_1^2 + \phi_2^2\,|\,\Omega\rangle = v^2 . \tag{167}$$

Thus,[8]

$$e^2v^2 = \mu^2\exp\left(\frac{1}{3} - \frac{16\pi^2\lambda}{3e^4}\right) . \tag{168}$$

Let us now compute the masses of the vector boson and the Higgs boson in this model. It is evident from eq. (142) that the photon acquires a mass,

$$m_\gamma = ev . \tag{169}$$

To determine the Higgs mass, we first define shifted scalar fields,

$$\tilde{\phi}_1 = \phi_1 - v , \qquad \tilde{\phi}_2 = \phi_2 . \tag{170}$$

[8] Although eq. (167) appears to depend on the arbitrary parameter μ, the results of the previous problem teach us that $V_{\rm eff}$ is in fact independent of μ. A change in μ would result in a compensating change in the coupling constants. Since v is related to a physical parameter (namely the mass of the gauge boson), it follows that we are free to choose μ at our convenience. Indeed, if we were to choose $\mu^2 = e^2v^2$, then eq. (168) would yield $\lambda = e^4/(16\pi^2)$. In this case, we have traded the original parameters λ and e for v and e. This is yet another example of dimensional transmutation, in which a dimensionful parameter has been generated starting from a theory with no dimensionful parameters.

Then, in terms of the shifted fields,

$$V_{\rm eff}(\tilde{\phi}_1,\tilde{\phi}_2)=\tfrac{1}{4}\lambda(\tilde{\phi}_1^2+\tilde{\phi}_2^2+2v\tilde{\phi}_1+v^2)^2 \\ +\frac{3e^4}{64\pi^2}(\tilde{\phi}_1^2+\tilde{\phi}_2^2+2v\tilde{\phi}_1+v^2)^2\left[\ln\left(\frac{e^2(\tilde{\phi}_1^2+\tilde{\phi}_2^2+2v\tilde{\phi}_1+v^2)}{\mu^2}\right)-\frac{5}{6}\right].\quad(171)$$

Next, we compute the derivatives,

$$\frac{\partial V_{\rm eff}}{\partial\phi_1}=\lambda(\tilde{\phi}_1^2+\tilde{\phi}_1^2+2v\tilde{\phi}_1+v^2)\tilde{\phi}_1 \\ +\frac{3e^4(\tilde{\phi}_1^2+\tilde{\phi}_2^2+2v\tilde{\phi}_1+v^2)\tilde{\phi}_1}{16\pi^2}\left[\ln\left(\frac{e^2(\tilde{\phi}_1^2+\tilde{\phi}_2^2+2v\tilde{\phi}_1+v^2)}{\mu^2}\right)-\frac{1}{3}\right],$$

$$\frac{\partial V_{\rm eff}}{\partial\phi_2}=\lambda(\tilde{\phi}_1^2+\tilde{\phi}_2^2+2v\tilde{\phi}_1+v^2)(\tilde{\phi}_2+v) \\ +\frac{3e^4(\tilde{\phi}_1^2+\tilde{\phi}_2^2+2v\tilde{\phi}_1+v^2)(\tilde{\phi}_2+v)}{16\pi^2}\left[\ln\left(\frac{e^2(\tilde{\phi}_1^2+\tilde{\phi}_2^2+2v\tilde{\phi}_1+v^2)}{\mu^2}\right)-\frac{1}{3}\right],\quad(172)$$

Of course, setting these derivatives to zero yields the minimum corresponding to $\tilde{\phi}_1=\tilde{\phi}_2=0$ in light of eq. (164).

We now compute the second derivatives and evaluate them at $\tilde{\phi}_1=\tilde{\phi}_2=0$.

$$\left.\frac{\partial^2 V_{\rm eff}}{\partial\phi_1^2}\right|_{\tilde{\phi}_1=\tilde{\phi}_2=0}=3v^2\left\{\lambda+\frac{3e^4}{16\pi^2}\left[\ln\left(\frac{e^2v^2}{\mu^2}\right)-\frac{1}{3}\right]+\frac{e^4}{8\pi^2}\right\},$$

$$\left.\frac{\partial^2 V_{\rm eff}}{\partial\phi_1\partial\phi_2}\right|_{\tilde{\phi}_1=\tilde{\phi}_2=0}=0\,,$$

$$\left.\frac{\partial^2 V_{\rm eff}}{\partial\phi_2^2}\right|_{\tilde{\phi}_1=\tilde{\phi}_2=0}=v^2\left\{\lambda+\frac{3e^4}{16\pi^2}\left[\ln\left(\frac{e^2v^2}{\mu^2}\right)-\frac{1}{3}\right]\right\}.\quad(173)$$

However, the potential minimum condition [cf. eq. (164)] is,

$$\lambda+\frac{3e^4}{16\pi^2}\left[\ln\left(\frac{e^2v^2}{\mu^2}\right)-\frac{1}{3}\right]=0\,.\quad(174)$$

Hence, the scalar squared-mass matrix is given by

$$\mathcal{M}^2=\begin{pmatrix} m_H^2 & 0\\ 0 & 0\end{pmatrix},\quad(175)$$

where

$$m_H^2=\frac{3e^2v^2}{8\pi^2}\,.\quad(176)$$

That is, $\widetilde{\phi}_2$ is identified with the massless Goldstone boson, which is absorbed by the gauge boson via the Higgs mechanism to produce a massive photon, whereas $\widetilde{\phi}_1$ is the Higgs boson with mass m_H. Combining eqs. (169) and (176), we conclude that

$$\boxed{\frac{m_H^2}{m_\gamma^2} = \frac{3e^2}{8\pi^2}\,.} \tag{177}$$

Indeed, the mass of the Higgs boson is independent of the arbitrary mass parameter μ.

We would also like to prove that the mass of the Higgs boson is independent of the scheme used to obtain the effective potential. Note that we can rewrite the $\overline{\rm MS}$ effective potential given in eq. (162) as follows,

$$\begin{aligned} V_{\rm eff}(\phi_1,\phi_2) &= \tfrac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2 + \frac{3e^4}{64\pi^2}(\phi_1^2+\phi_2^2)^2\left[\ln\left(\frac{e^2(\phi_1^2+\phi_2^2)}{\mu^2}\right)-\frac{5}{6}\right] \\ &= \tfrac{1}{4}\lambda'(\phi_1^2+\phi_2^2)^2 + \frac{3e^4}{64\pi^2}(\phi_1^2+\phi_2^2)^2\left[\ln\left(\frac{\phi_1^2+\phi_2^2}{\mu^2}\right)-\frac{25}{6}\right]\,, \end{aligned} \tag{178}$$

where

$$\lambda' = \lambda + \frac{3e^4}{16\pi^2}\left[\ln(e^2)+\frac{10}{3}\right]\,. \tag{179}$$

In light of eq. (160), the $\overline{\rm MS}$ scheme and the physical scheme are simply related by a redefinition of the coupling λ. But the Higgs boson mass obtained in eq. (176) is independent of λ. Thus, if one repeats the derivation of the Higgs mass starting from eq. (160), one must also obtain eq. (176). Indeed, the mass of the Higgs boson is scheme independent.

Note that the same conclusion can be achieved by rewriting eq. (178) as

$$V_{\rm eff}(\phi_1,\phi_2) = \tfrac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2 + \frac{3e^4}{64\pi^2}(\phi_1^2+\phi_2^2)^2\left[\ln\left(\frac{\phi_1^2+\phi_2^2}{\mu'^2}\right)-\frac{25}{6}\right]\,, \tag{180}$$

where

$$\ln\left(\frac{\mu^2}{\mu'^2}\right) = \ln(e^2)+\frac{10}{3}\,. \tag{181}$$

That is, the physical scheme actually is equivalent to the $\overline{\rm MS}$ scheme with a different choice of the parameter μ.

Thus, by eliminating μ from the expression of $V_{\rm eff}$, one can obtain a result that is independent of the choice of renormalization schemes. For example, we can use eq. (168) to eliminate μ^2 from the $\overline{\rm MS}$ effective potential. It then follows that eq. (162) can be rewritten as

$$V_{\rm eff}(\phi_1,\phi_2) = \frac{3e^4}{64\pi^2}(\phi_1^2+\phi_2^2)^2\left[\ln\left(\frac{\phi_1^2+\phi_2^2}{v^2}\right)-\frac{1}{2}\right]\,. \tag{182}$$

Note that λ does not appear at all in eq. (182). This is consistent with the remarks of footnote 8, which indicates that it is possible to express the effective potential in terms of two physical parameters, e and v, or equivalently in terms of m_γ and m_H.

Likewise, one can derive eq. (182) starting from eq. (161) and using the effective potential in the physical scheme to solve for v. Thus, eq. (182) is independent of the choice of scheme. Of course, one can now compute the Higgs mass directly from eq. (182) and recover the result of eq. (177).

(c) Consider the dependence of the one-loop effective potential on the gauge parameter ξ. If one employs the gauge fixing term exhibited in eq. (136), the calculation of the effective potential using the tadpole method is unwieldy due to the mixing of the photon field and the derivative of the scalar field in the shifted Lagrangian. This problem is ameliorated by employing the alternative gauge fixing term,

$$\mathscr{L}_{\rm GF} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi e\phi_1\phi_2)^2 \,. \tag{183}$$

Employing this new gauge fixing term, repeat the computations of part (a). Show that in the one-loop approximation considered here, the Higgs boson mass is independent of ξ.

Although the result obtained in eq. (143) of part (a) is correct, we glossed over a subtlety in the computation. Recall that in applying eq. (139) to the Lagrangian given in eq. (138), we were instructed to shift the field, $\phi_1 \to \phi_1 + \Phi$ (for a constant Φ), and then to identify the Φ-dependent masses by examining the resulting terms that were quadratic in the fields. However, we neglected to note a quadratic term that mixes A_μ and $\partial_\mu\phi_2$, due to the term

$$eA^\mu\big[(\phi_1+\Phi)\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_1\big] \tag{184}$$

after noting that $\partial_\mu\Phi = 0$, since Φ is constant. Why then were we able to ignore this issue in part (a)?

One solution to this conundrum is to introduce an alternative gauge fixing term, for example the one proposed in eq. (183), called the $\overline{R}_\xi$ gauge fixing term by Boris Kastening.[9] Expanding out the squared expression yields,

$$\mathscr{L}_{\rm GF} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + e\phi_1\phi_2\partial_\mu A^\mu - \tfrac{1}{2}\xi e^2\phi_1^2\phi_2^2 \,. \tag{185}$$

When we perform the shift, $\phi_1 \to \phi_1 + \Phi$, on the gauge fixing term and add the result to eq. (184), we end up with,

$$-\frac{1}{2\xi}(\partial_\mu A^\mu)^2 - 2eA^\mu\phi_2\partial_\mu\phi_1 - \tfrac{1}{2}\xi e^2(\phi_1+\Phi)^2\phi_2^2 + e\partial_\mu\big[A^\mu(\phi_1+\Phi)\phi_2\big] \,. \tag{186}$$

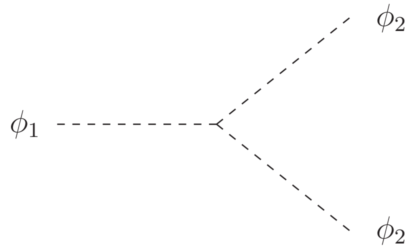
The last term above is a total divergence, which does not contribute to the action and thus can be dropped. Hence, we have succeeded in eliminating the mixing of A^μ with $\partial_\mu\phi_2$ in the shifted Lagrangian. Note that we have generated two new terms that are quadratic in the fields ϕ_1 and ϕ_2, respectively. However, the coefficient of these terms are proportional to ξ. Thus, in the Landau gauge where $\xi = 0$ no new quadratic terms in the fields appear in the shifted Lagrangian and thus the computations presented in part (a) are indeed valid.

We would now like to repeat the analysis of part (a) in the case of a general $\overline{R}_\xi$ gauge. Two changes are immediately apparent. First, in light of eqs. (141) and (186), the squared-mass term of ϕ_2 and the cubic $\phi_1\phi_2^2$ interaction have been modified,

$$\mathscr{L} \ni -(\lambda + \xi e^2)\big[\tfrac{1}{2}\Phi^2\phi_2^2 + \Phi\phi_1\phi_2^2\big] \,. \tag{187}$$

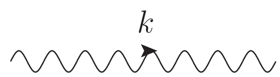
[9] The gauge fixing Lagrangian given in eq. (183) was first introduced in P.S.S. Caldas, H. Fleming and R. Lopez Garcia in Nuovo Cim. A **42**, 360 (1977) and then later rediscovered and advocated in B. Kastening, Phys. Rev. D **51**, 265 (1995). Eq. (183) is similar to the R_ξ gauge fixing term that is employed in tree-level spontaneously broken gauge theories.

In particular, the Feynman rule for the $\phi_1\phi_2^2$ vertex in the shifted theory is exhibited below,



$$-2i(\lambda + \xi e^2)\Phi$$

Second, the photon propagator has also been modified (and is the same as in the usual R_ξ gauge), as exhibited below.



$$\frac{-i}{k^2 - e^2\Phi^2 + i\varepsilon}\left[g_{\mu\nu} + \frac{(\xi - 1)k_\mu k_\nu}{k^2 - \xi e^2\Phi^2}\right]$$

Before proceeding with the analysis, there is an additional feature that we must incorporate. Namely, in the case of $\xi \neq 0$, Faddeev-Popov ghost fields exist and cannot be neglected in the computation of $V_{\rm eff}$. Thus, we need to derive the Faddeev-Popov Lagrangian that arises due to the choice of the $\overline{R}_\xi$ gauge-fixing term.

The Faddeev-Popov determinant is given by

$$\det B(x,y)\bigg|_{F=0} \equiv \det\left(\frac{\delta F(x)}{\delta\Lambda(y)}\right)\bigg|_{F=0}\,, \tag{188}$$

where Λ is the gauge transformation parameter and

$$F(x) = \partial_\mu A^\mu(x) - \xi e\phi_1(x)\phi_2(x) - f\,, \tag{189}$$

is determined by the gauge fixing term in the Lagrangian. Moreover, the functional chain rule yields,

$$B(x,y) \equiv \frac{\delta F(x)}{\delta\Lambda(y)} = \int d^4z\left(\frac{\delta F(x)}{\delta A_\mu(z)}\frac{\delta A_\mu(z)}{\delta\Lambda(y)} + \frac{\delta F(x)}{\delta\phi_1(z)}\frac{\delta\phi_1(z)}{\delta\Lambda(y)} + \frac{\delta F(x)}{\delta\phi_2(z)}\frac{\delta\phi_2(z)}{\delta\Lambda(y)}\right)\,. \tag{190}$$

To evaluate the function derivatives above, we note that under an infinitesimal U(1) gauge transformation,

$$A_\mu \longrightarrow A_\mu + \partial_\mu\Lambda\,, \qquad \phi(x) \longrightarrow \big(1 - ie\Lambda(x)\big)\phi(x)\,, \tag{191}$$

where

$$\phi(x) = \frac{\phi_1(x) + i\phi_2(x)}{\sqrt{2}}\,, \tag{192}$$

Eq. (191) implies that

$$\delta A_\mu = \partial_\mu\Lambda\,, \qquad \delta\phi_1 = e\Lambda\phi_2\,, \qquad \delta\phi_2 = -e\Lambda\phi_1\,. \tag{193}$$

Hence, eqs. (190) and (193) yields,

$$\begin{aligned} B(x,y) = \frac{\delta F(x)}{\delta\Lambda(y)} &= \int d^4z\big[\Box_x\delta^4(x-z)\delta^4(y-z) + \xi e^2\big(\phi_1(x)\phi_1(z) - \phi_2(x)\phi_2(z)\big)\delta^4(x-z)\delta^4(y-z)\big] \\ &= \big[\Box_x + \xi e^2\big(\phi_1^2(x) - \phi_2^2(x)\big)\big]\delta^4(x-y)\,, \end{aligned} \tag{194}$$

where $\Box \equiv \partial_\mu\partial^\mu$.

The Faddeev-Popov determinant has a path integral representation,

$$\det B = \mathcal{N} \int \mathcal{D}\eta^* \mathcal{D}\eta \exp\left\{-i \int d^4x\, d^4y\, \eta^*(y) B(x,y) \eta(x)\right\} \equiv \mathcal{N} \int \mathcal{D}\eta^* \mathcal{D}\eta \exp\left\{i \int d^4x\, \mathscr{L}_{\rm FP}\right\} \tag{195}$$

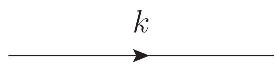
where η and η^* are the Faddeev-Popov ghost fields, and $\mathscr{L}_{\rm FP}$ is the Faddeev-Popov Lagrangian, which is then added to the Lagrangian of the spontaneously broken abelian Higgs model. Using eq. (194) and integrating by parts, we end up with

$$\mathscr{L}_{\rm FP} = \partial_\mu \eta^* \partial^\mu \eta - \xi e^2 (\phi_1^2 - \phi_2^2) \eta^* \eta \,. \tag{196}$$

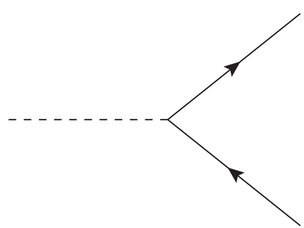
Finally, we perform the shift, $\phi_1 \to \phi_1 + \Phi$. The second term of eq. (196) yields a Φ-dependent mass term for the Faddeev-Popov ghosts and interaction terms involving the scalars,

$$-\xi e^2 \Phi^2 \eta^* \eta - \xi e^2 (2\Phi\phi_1 + \phi_1^2 - \phi_2^2) \eta^* \eta \,. \tag{197}$$

Thus, the relevant Feynman rules of the shifted theory involving the Faddeev-Popov ghost fields that we need for our computation below are:

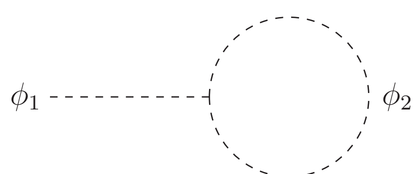


$$\frac{i}{k^2 - \xi e^2 \Phi^2 + i\varepsilon}$$



$$-2i\xi e^2 \Phi$$

We now perform the computation of the effective potential using the method of tadpoles. First, there are new contributions to the Φ-dependent mass of ϕ_2 and the $\phi_1\phi_2^2$ cubic interaction that are proportional to ξ, as noted in eq. (187). Hence, the computation performed in class of the ϕ_2 tadpole,



is modified as follows,

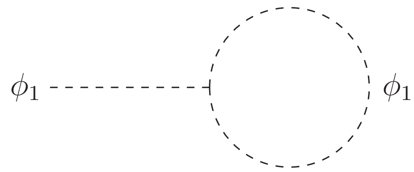
$$i\Gamma^{(1)}_\Phi(0)_{\rm G} = -i \frac{dV^{(1)}(\Phi)_{\rm G}}{d\Phi} = \frac{1}{2} \int \frac{d^n q}{(2\pi)^n} \frac{i}{q^2 - (\lambda + \xi e^2)\Phi^2 + i\varepsilon} \left[-2i(\lambda + \xi e^2)\Phi\right] \tag{198}$$

where the symmetry factor of $\frac{1}{2}$ for the ϕ_2 tadpole has been included (ϕ_2 would be the Goldstone boson if Φ were the true vacuum expectation value, hence the subscript G above). It follows that

$$V^{(1)}(\Phi)_{\rm G} = -\tfrac{1}{2} i \int \frac{d^n q}{(2\pi)^n} \ln\left(\frac{q^2 - (\lambda + \xi e^2)\Phi^2 + i\varepsilon}{q^2 + i\varepsilon}\right) , \tag{199}$$

which replaces the first term in eq. (143).

The computation performed in class of the ϕ_1 tadpole,



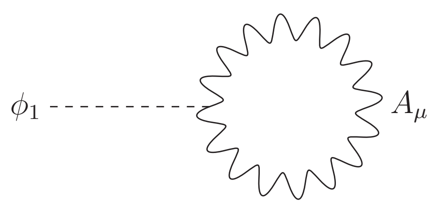
is not modified. For completeness, we summarize that computation below.

$$i\Gamma^{(1)}_{\Phi}(0)_{\rm H} = -i\frac{dV^{(1)}(\Phi)_{\rm H}}{d\Phi} = \frac{1}{2}\int\frac{d^nq}{(2\pi)^n}\,\frac{i}{q^2-3\lambda\Phi^2+i\varepsilon}\big[-6i\lambda\Phi\big] \tag{200}$$

where the symmetry factor of $\frac{1}{2}$ for the ϕ_1 tadpole has been included (ϕ_1 would be the Higgs boson if Φ were the true vacuum expectation value, hence the subscript H above). It follows that

$$V^{(1)}(\Phi)_{\rm H} = -\tfrac{1}{2}i\int\frac{d^nq}{(2\pi)^n}\,\ln\left(\frac{q^2-3\lambda\Phi^2+i\varepsilon}{q^2+i\varepsilon}\right)\,, \tag{201}$$

Next, consider the vector boson tadpole,



The Landau gauge computation of the vector boson tadpole performed in class was proportional to

$$g^{\mu\nu}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) = n-1 = 3-2\epsilon\,. \tag{202}$$

In a general R_ξ gauge, we must replace eq. (202) with,

$$g^{\mu\nu}\left(g_{\mu\nu}+\frac{(\xi-1)q_\mu q_\nu}{q^2-\xi e^2\Phi^2}\right) = n-1+\frac{\xi(q^2-e^2\Phi^2)}{q^2-\xi e^2\Phi^2}\,. \tag{203}$$

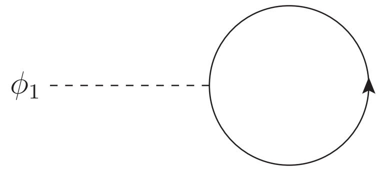
Thus, the contribution of the vector boson tadpole is given by,

$$\begin{aligned} i\Gamma^{(1)}_{\Phi}(0)_{\rm V} = -i\frac{dV^{(1)}(\Phi)_{\rm V}}{d\Phi} &= \frac{1}{2}\int\frac{d^nq}{(2\pi)^n}\frac{-i\left(g_{\mu\nu}+\frac{(\xi-1)q_\mu q_\nu}{q^2-\xi e^2\Phi^2}\right)}{q^2-e^2\Phi^2}\big[2ig_{\mu\nu}e^2\Phi\big] \\ &= (3-2\epsilon)\int\frac{d^nq}{(2\pi)^n}\frac{e^2\Phi}{q^2-e^2\Phi^2+i\varepsilon}+\int\frac{d^nq}{(2\pi)^n}\frac{\xi e^2\Phi}{q^2-\xi e^2\Phi^2+i\varepsilon}\,, \end{aligned} \tag{204}$$

after including the symmetry factor of $\frac{1}{2}$ and putting $n = 4-2\epsilon$. It follows that

$$V^{(1)}(\Phi)_{\rm V} = -\tfrac{1}{2}i\left\{(3-2\epsilon)\int\frac{d^nq}{(2\pi)^n}\ln\left(\frac{q^2-e^2\Phi^2+i\varepsilon}{q^2+i\varepsilon}\right)+\int\frac{d^nq}{(2\pi)^n}\ln\left(\frac{q^2-\xi e^2\Phi^2+i\varepsilon}{q^2+i\varepsilon}\right)\right\}\,. \tag{205}$$

Finally, we must include the contribution of the Faddeev-Popov ghosts. That is, we need to evaluate the following diagram:



Due to the complexity of the ghost fields η and η^*, the symmetry factor associated with the diagram above is 1 (rather than $\frac{1}{2}$ as in the cases of the ϕ_1, ϕ_2 and vector boson tadpoles treated above). Hence,

$$i\Gamma_\Phi^{(1)}(0)_{\rm FP} = -i\frac{dV^{(1)}(\Phi)_{\rm FP}}{d\Phi} = -\int\frac{d^nq}{(2\pi)^n}\,\frac{i}{q^2-\xi e^2\Phi^2+i\varepsilon}\left[-2i\xi e^2\Phi\right] \tag{206}$$

which includes a minus sign due to the loop of Faddeev-Popov ghosts. Thus, we obtain,

$$V^{(1)}(\Phi)_{\rm FP} = i\int\frac{d^nq}{(2\pi)^n}\,\ln\left(\frac{q^2-\xi e^2\Phi^2+i\varepsilon}{q^2+i\varepsilon}\right)\,. \tag{207}$$

Collecting our results, the one-loop effective potential in the $\overline{R}_\xi$ gauge is given by,

$$\begin{aligned}V^{(1)}(\Phi) = -\tfrac{1}{2}i\int\frac{d^nq}{(2\pi)^n}\Bigg\{&\ln\left(\frac{q^2-(\lambda+\xi e^2)\Phi^2+i\varepsilon}{q^2+i\varepsilon}\right)+\ln\left(\frac{q^2-3\lambda\Phi^2+i\varepsilon}{q^2+i\varepsilon}\right)\\ &+(3-2\epsilon)\ln\left(\frac{q^2-e^2\Phi^2+i\varepsilon}{q^2+i\varepsilon}\right)-\ln\left(\frac{q^2-\xi e^2\Phi^2+i\varepsilon}{q^2+i\varepsilon}\right)\Bigg\}\,.\end{aligned} \tag{208}$$

Indeed, eq. (208) correctly reduces to the result of the Landau gauge computation [cf. eq. (143)] after setting $\xi = 0$.

As discussed in part (a), under the assumption of $\lambda \sim \mathcal{O}(e^4)$, we must set $\lambda = 0$ in the expression for $V^{(1)}(\Phi)$ to maintain consistency of the perturbation series expansion, since the remaining terms will be of the same order as the tree-level potential. Thus, we end up with,

$$V^{(1)}(\Phi) = -\tfrac{1}{2}(3-2\epsilon)i\int\frac{d^nq}{(2\pi)^n}\,\ln\left(\frac{q^2-e^2\Phi^2+i\varepsilon}{q^2+i\varepsilon}\right)\,. \tag{209}$$

Remarkably, the ξ dependence has canceled out! Thus, at this order of perturbation theory [under the assumption that $\lambda \sim \mathcal{O}(e^4)$], the effective potential is actually gauge invariant. This means that the Higgs mass obtained in part (b) is also gauge invariant.

Unfortunately, when higher order terms that have been neglected above are taken into account, one finds that the effective potential is no longer gauge invariant and does depend on ξ. This is not surprising in light of the ξ dependence of eq. (208) when $\lambda \neq 0$. Indeed, the gauge non-invariance could have been anticipated since the effective potential was shown in class to be related to a sum of n-point Green functions evaluated at zero external momenta. In general, Green functions of gauge theories, when evaluated at external momenta that are not on-shell, are not gauge invariant as they do not represent any physical quantity.

Nevertheless, one can prove that the Higgs mass obtained from the effective potential is gauge invariant to any (consistent) order of perturbation theory. A proof of this fact can be found in R. Fukada and T. Kugo, Phys. Rev. D **13**, 3469 (1976). Similar issues are also addressed in I.J.R. Aitchison and C.M. Fraser, Annals of Physics **156**, 1 (1984). Of course, this had to be true since the Higgs mass is a physical observable.