Applications of Young tableaux

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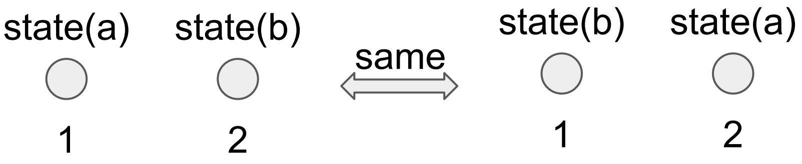
Outline

- Sn permutation group
    - 2 identical particles with 2 different states
    - 3 identical particles with 3 different states
- Standard Young tableaux rules
- Decomposition of Irrep of
    - SU(2)
        - Composition of 2 Electrons
        - Composition of 3 Electrons
        - Selection Rules and CG coefficients
    - SU(3)
        - Composition of 3 quarks
        - Composition of 1 quark and antiquark
    - SU(N)

Example: 2 identical particles in 2 different states (a,b)

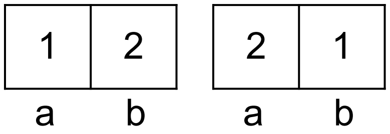


Box: state
Number: particle

Identical particles $\phi_1(a)\phi_2(b)$ $\phi_1(b)\phi_2(a)$

Totally symmetric state:

$$\psi_s = \phi_1(a)\phi_2(b) + \phi_2(a)\phi_1(b)$$
$$= \phi_1(a)\phi_2(b)\left[\mathbf{I} + \hat{\mathbf{P}}_{12}\right]$$



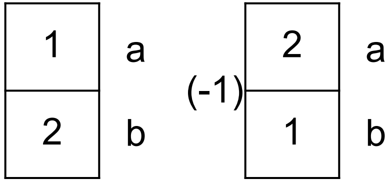
$\hat{\mathbf{P}}_{ij}:\ swap\, particle\, 1\, with\, particle\, 2$

Idempotent

$$s = \sum_{g \in S_n} g$$

Totally antisymmetric state:

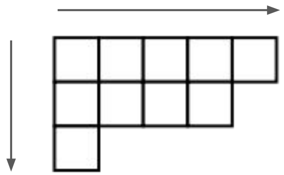
$$\psi_A = \phi_1(a)\phi_2(b) - \phi_2(a)\phi_1(b)$$
$$= \phi_1(a)\phi_2(b)\left[\mathbf{I} - \hat{\mathbf{P}}_{12}\right]$$



$$a = \sum_{g \in S_n} (-1)^g g$$

Standard Young Tableaux Rules revisited

- The number in the row doesn’t decrease when goes from left to right
- The number in the column has to increase downwards



- If there are N particles, it’s impossible to have the number of the row that is greater than N

Example: 3 identical particles in 3 different states (a,b,c)

Totally symmetric state:

$$\psi_s = \phi_1(a)\phi_2(b)\phi_3(c) + \phi_2(a)\phi_1(b)\phi_3(c) + \phi_1(a)\phi_3(b)\phi_2(c) + \phi_3(a)\phi_2(b)\phi_1(c) + \phi_3(a)\phi_1(b)\phi_2(c) + \phi_2(a)\phi_3(b)\phi_1(c)$$

$$= \phi_1(a)\phi_2(b)\phi_3(c)\left[\mathbf{I} + \hat{\mathbf{P}}_{12} + \hat{\mathbf{P}}_{13} + \hat{\mathbf{P}}_{23} + \hat{\mathbf{P}}_{13}\hat{\mathbf{P}}_{12} + \hat{\mathbf{P}}_{12}\hat{\mathbf{P}}_{13}\right] \qquad s = \sum_{g \in S_n} g$$

1	2	3
a	b	c

Totally antisymmetric state:

$$\psi_A = \phi_1(a)\phi_2(b)\phi_3(c) - \phi_2(a)\phi_1(b)\phi_3(c) - \phi_3(a)\phi_2(b)\phi_1(c) - \phi_1(a)\phi_3(b)\phi_2(c) + \phi_3(a)\phi_1(b)\phi_2(c) + \phi_2(a)\phi_3(b)\phi_1(c)$$

$$= \phi_1(a)\phi_2(b)\phi_3(c)\left[\mathbf{I} - \hat{\mathbf{P}}_{12} - \hat{\mathbf{P}}_{13} - \hat{\mathbf{P}}_{23} + \hat{\mathbf{P}}_{13}\hat{\mathbf{P}}_{12} + \hat{\mathbf{P}}_{12}\hat{\mathbf{P}}_{13}\right] \qquad a = \sum_{g \in S_n} (-1)^g g$$

1	a
2	b
3	c

Mixed symmetry states:

$$\psi_{\lambda_1} = \phi_1(a)\phi_2(b)\phi_3(c) + \phi_2(a)\phi_1(b)\phi_3(c) - \phi_3(a)\phi_2(b)\phi_1(c) - \phi_2(a)\phi_3(b)\phi_1(c)$$

$$= \phi_1(a)\phi_2(b)\phi_3(c)\left[\mathbf{I} + \hat{\mathbf{P}}_{12} - \hat{\mathbf{P}}_{13} - \hat{\mathbf{P}}_{12}\hat{\mathbf{P}}_{13}\right] \qquad e_{\lambda_1} = s_{\lambda_1} a_{\lambda_1} = \left(\mathbf{I} + \hat{\mathbf{P}}_{12}\right)\left(\mathbf{I} - \hat{\mathbf{P}}_{13}\right)$$

$\lambda_1 =$

1	2
3	

$$s_{\lambda_1} = \mathbf{I} + \hat{\mathbf{P}}_{12}$$
$$a_{\lambda_1} = \mathbf{I} - \hat{\mathbf{P}}_{13}$$

$$\psi_{\lambda_2} = \phi_1(a)\phi_2(b)\phi_3(c) + \phi_3(a)\phi_2(b)\phi_1(c) - \phi_2(a)\phi_1(b)\phi_3(c) - \phi_3(a)\phi_1(b)\phi_2(c)$$

$$= \phi_1(a)\phi_2(b)\phi_3(c)\left[\mathbf{I} + \hat{\mathbf{P}}_{13} - \hat{\mathbf{P}}_{12} - \hat{\mathbf{P}}_{13}\hat{\mathbf{P}}_{12}\right] \qquad e_{\lambda_2} = s_{\lambda_2} a_{\lambda_2} = \left(\mathbf{I} + \hat{\mathbf{P}}_{13}\right)\left(\mathbf{I} - \hat{\mathbf{P}}_{12}\right)$$

$\lambda_2 =$

1	3
2	

$$s_{\lambda_2} = \mathbf{I} + \hat{\mathbf{P}}_{13}$$
$$a_{\lambda_2} = \mathbf{I} - \hat{\mathbf{P}}_{12}$$

# of Standard Young Tableaux = dim of irreps $\quad n_\lambda = \frac{n!}{\Pi g_i} = \frac{3!}{3} = 2$

3	1
1	

What about those occupy the same states?

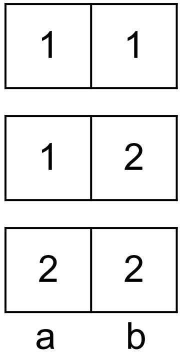
Box: particle
Number: spin state

Example: 2 electrons with ½ spin states

Totally symmetric state: $\phi_a(\uparrow)\phi_b(\uparrow)$ $|J=1,\, m=1\rangle$

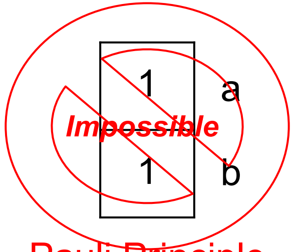
$$\psi_s = \phi_a(\uparrow)\phi_b(\downarrow) + \phi_a(\downarrow)\phi_b(\uparrow) \quad |J=1,\, m=0\rangle$$

$$\phi_a(\downarrow)\phi_b(\downarrow) \quad |J=1,\, m=1\rangle$$



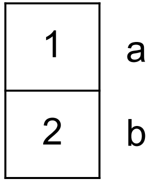
Triplet

Totally antisymmetric state:



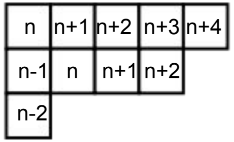
Pauli Principle

$$\psi_A = \phi_a(\uparrow)\phi_b(\downarrow) - \phi_a(\downarrow)\phi_b(\uparrow) \quad |J=0,\, m=0\rangle$$



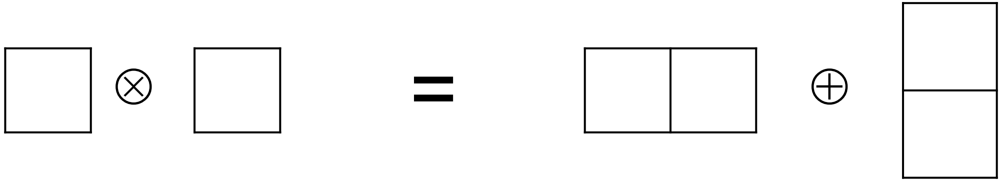
Singlet

# of Standard Young Tableaux



$$n_\lambda = \frac{N}{G} = \frac{\Pi n_i}{\Pi g_i}$$ hook

Decomposition of reducible representation of SU(2)



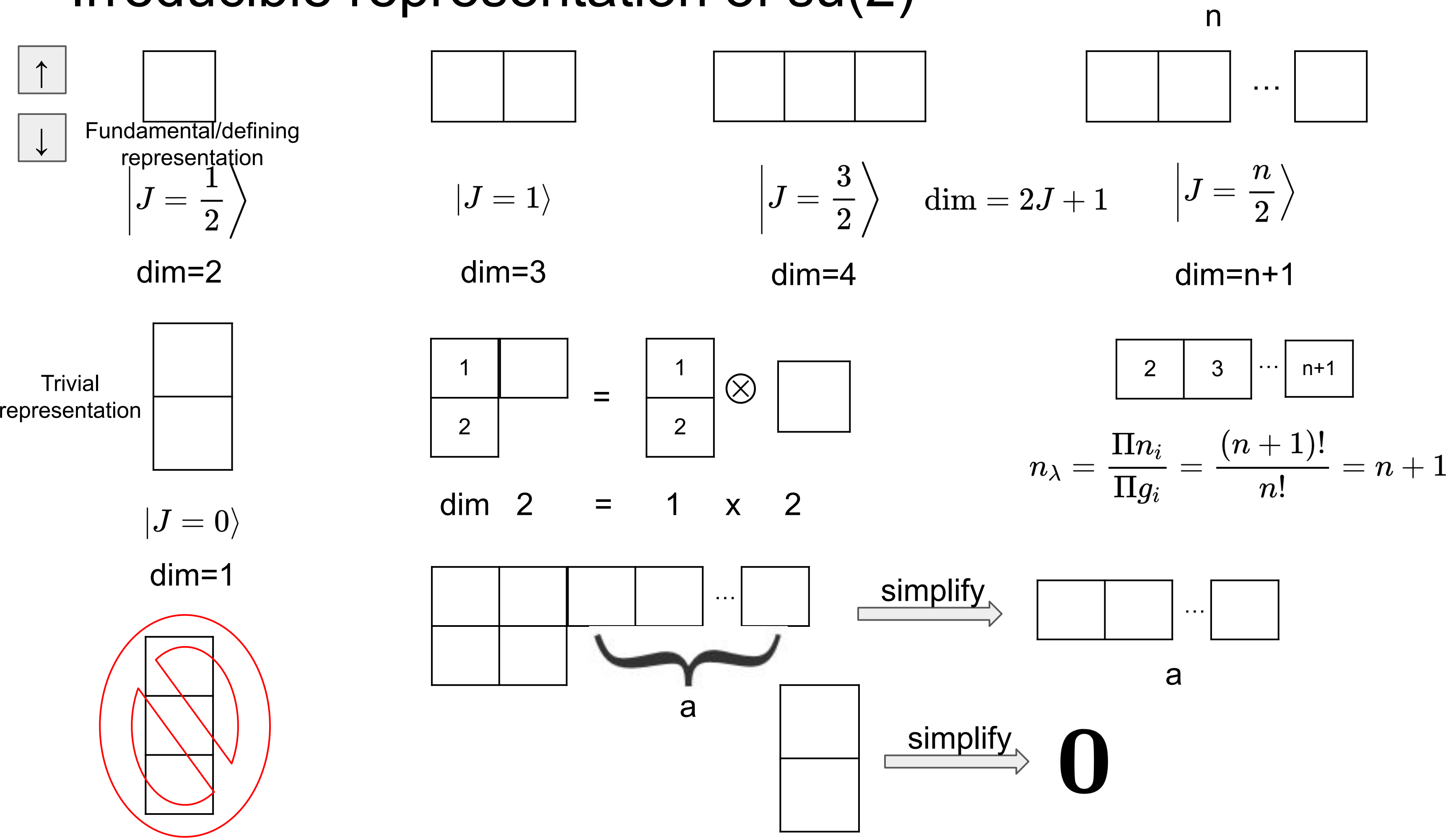
$$\left|j_1 = \frac{1}{2}\right\rangle \otimes \left|j_2 = \frac{1}{2}\right\rangle = |J=1\rangle \oplus |J=0\rangle$$

dim 2 x 2 = 3 + 1

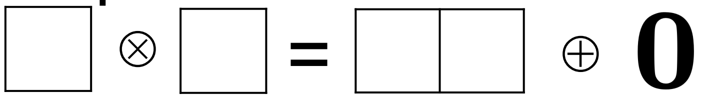
Reducible Representation

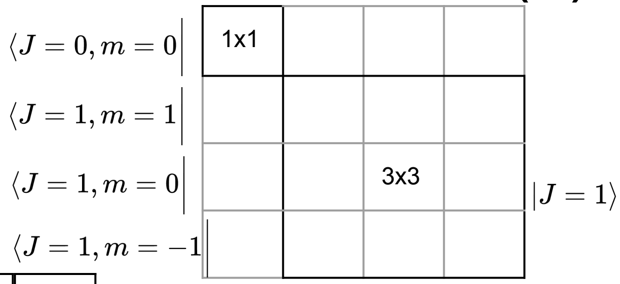
Irreducible Representation

Irreducible representation of su(2)



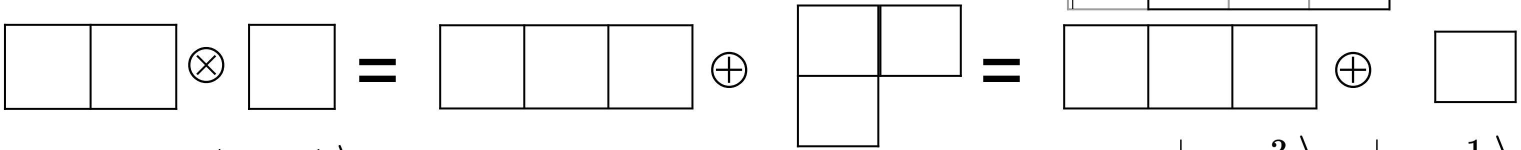
Decomposition of reducible representation of su(2)





$$\left|J=\frac{1}{2}\right\rangle \otimes \left|J=\frac{1}{2}\right\rangle = |J=1\rangle \oplus |J=0\rangle$$

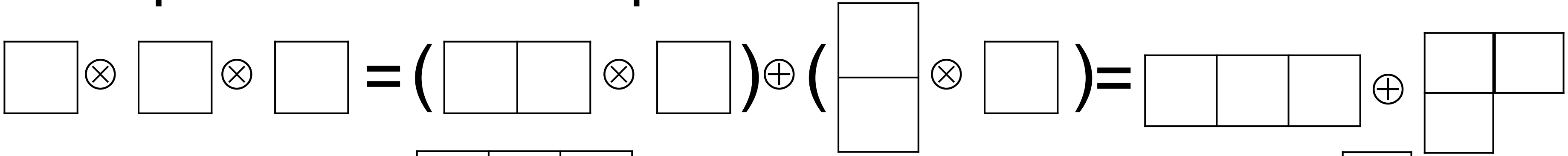
dim 2 x 2 = 3 + 1

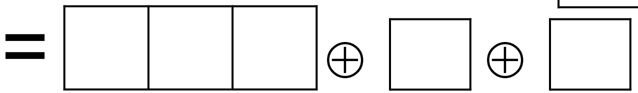


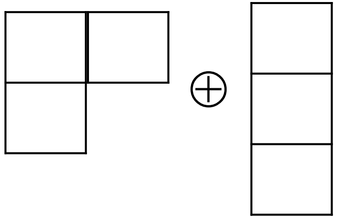
$$|J=1\rangle \otimes \left|J=\frac{1}{2}\right\rangle \qquad \left|J=\frac{3}{2}\right\rangle \oplus \left|J=\frac{1}{2}\right\rangle$$

dim 3 x 2 = 4 + 2

Example: 3 electrons with ½ spin states





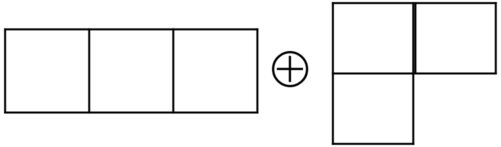


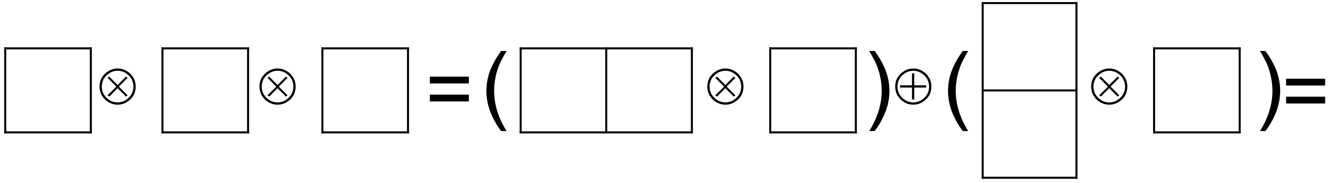
dim

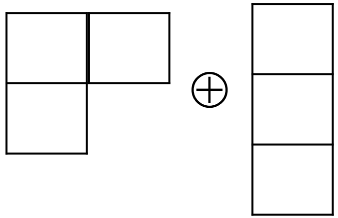
2 x 2 x 2 = 4 + 2 + 2

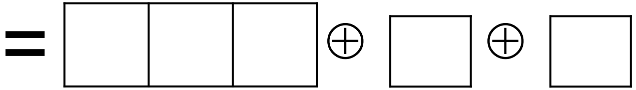
$$\left|j_1=\frac{1}{2}\right\rangle \otimes \left|j_2=\frac{1}{2}\right\rangle \otimes \left|j_3=\frac{1}{2}\right\rangle = \left|J=\frac{3}{2}\right\rangle \oplus \left|J=\frac{1}{2}\right\rangle \oplus \left|J=\frac{1}{2}\right\rangle$$

Example: 3 electrons with ½ spin states





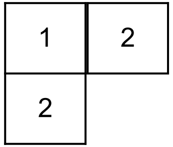




dim

2 x 2 x 2 = 4 + 2 + 2

$$\left|j_1=\frac{1}{2}\right\rangle \otimes \left|j_2=\frac{1}{2}\right\rangle \otimes \left|j_3=\frac{1}{2}\right\rangle = \left|J=\frac{3}{2}\right\rangle \oplus \left|J=\frac{1}{2}\right\rangle \oplus \left|J=\frac{1}{2}\right\rangle$$

$\lambda_2 =$ 

$$\phi_a(\uparrow)\phi_b(\uparrow)\phi_c(\uparrow) \qquad \left|J=\frac{3}{2}, m=\frac{3}{2}\right\rangle$$

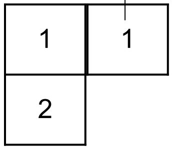
$$(\phi_a(\uparrow)\phi_b(\uparrow)\phi_c(\downarrow)+\phi_a(\uparrow)\phi_b(\downarrow)\phi_c(\uparrow)+\phi_a(\downarrow)\phi_b(\uparrow)\phi_c(\uparrow))/3 \qquad \left|J=\frac{3}{2}, m=\frac{1}{2}\right\rangle$$

$$(\phi_a(\uparrow)\phi_b(\downarrow)\phi_c(\downarrow)+\phi_a(\downarrow)\phi_b(\downarrow)\phi_c(\uparrow)+\phi_a(\downarrow)\phi_b(\uparrow)\phi_c(\downarrow))/3 \qquad \left|J=\frac{3}{2}, m=-\frac{1}{2}\right\rangle$$

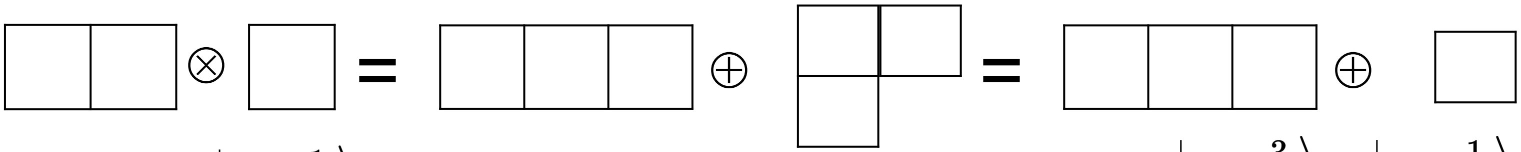
$$\phi_a(\downarrow)\phi_b(\downarrow)\phi_c(\downarrow) \qquad \left|J=\frac{3}{2}, m=-\frac{3}{2}\right\rangle$$

$$\psi_{\lambda_2} \quad \frac{1}{3}(\phi_a(\uparrow)\phi_b(\downarrow)\phi_c(\downarrow)-\phi_a(\downarrow)\phi_b(\uparrow)\phi_c(\downarrow))$$
$$\frac{1}{3}(\phi_a(\uparrow)\phi_b(\downarrow)\phi_c(\downarrow)-\phi_a(\downarrow)\phi_b(\downarrow)\phi_c(\uparrow))$$

$$\psi_{\lambda_1} \quad \frac{2}{3}(\phi_a(\uparrow)\phi_b(\uparrow)\phi_c(\downarrow)-\phi_a(\downarrow)\phi_b(\uparrow)\phi_c(\uparrow))$$
$$\frac{1}{3}(\phi_a(\uparrow)\phi_b(\downarrow)\phi_c(\uparrow)-\phi_a(\uparrow)\phi_b(\downarrow)\phi_c(\uparrow))$$

$\lambda_1 =$ 

Decomposition of reducible representation of su(2)

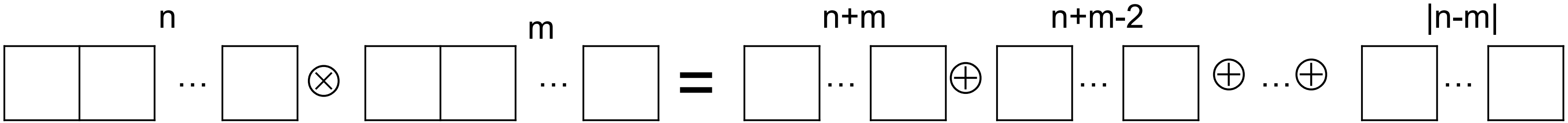


$$|J=1\rangle \otimes \left|J=\frac{1}{2}\right\rangle \qquad\qquad \left|J=\frac{3}{2}\right\rangle \oplus \left|J=\frac{1}{2}\right\rangle$$

dim 3 x 2 = 4 + 2

$$|J=1\rangle \otimes |J=1\rangle \qquad\qquad |J=2\rangle \oplus |J=1\rangle \oplus |J=0\rangle$$

dim 3 x 3 = 5 + 3 + 1



$$\left|J=\frac{n}{2}\right\rangle \otimes \left|J=\frac{m}{2}\right\rangle \qquad \left|J=\frac{n+m}{2}\right\rangle \oplus \left|J=\frac{n+m}{2}-1\right\rangle \oplus \ldots \oplus \left|J=\frac{|n-m|}{2}\right\rangle = \sum_{i=|n-m|}^{n+m} \left|J=\frac{i}{2}\right\rangle$$

dim (n+1) x (m+1) = (n+m+1) + (n+m-2) +...+ (|n-m|+1)

Clebsch-Gordan series $$(2j_1+1)\otimes(2j_2+1) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \oplus(2j+1)$$

Selection Rules

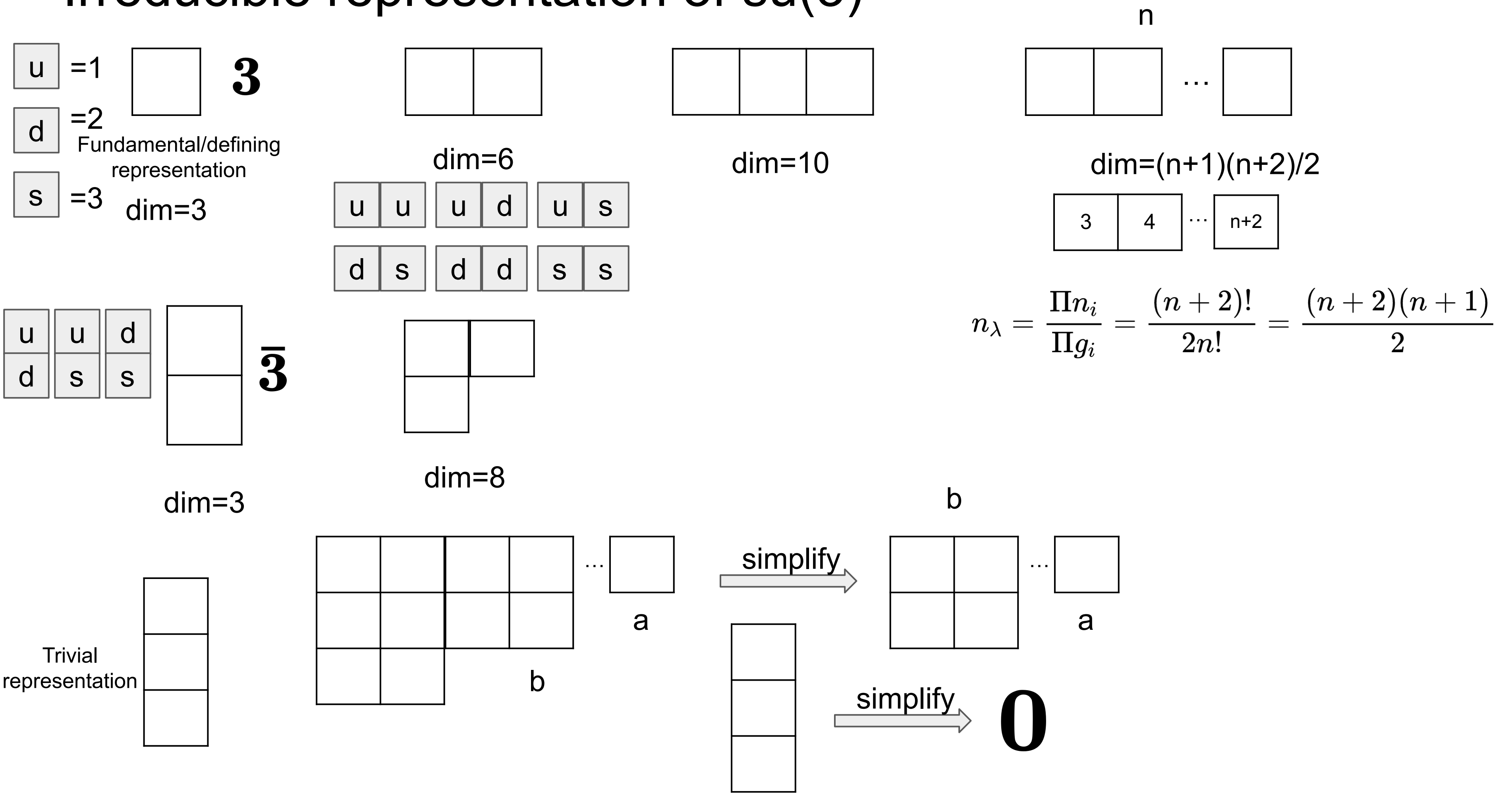
Clebsch-Gordan series $$(2j_1+1)\otimes(2j_2+1)=\sum_{j=|j_1-j_2|}^{j_1+j_2}\oplus(2j+1)$$

$$|j_1\,m_1\rangle\otimes|j_2\,m_2\rangle=\sum_{j=|j_1-j_2|}^{j_1+j_2} CG\,|j\,m\rangle$$

$$|m_1\,m_2\rangle=\sum_{j=|j_1-j_2|}^{j_1+j_2}|j\,m\rangle\langle j\,m|m_1\,m_2\rangle$$

$$|j_1-j_2|\le j\le j_1+j_2$$

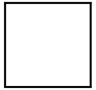
Irreducible representation of su(3)



$$n_\lambda = \frac{\Pi n_i}{\Pi g_i} = \frac{(n+2)!}{2n!} = \frac{(n+2)(n+1)}{2}$$

Irreducible representation of su(3)

u
d
s

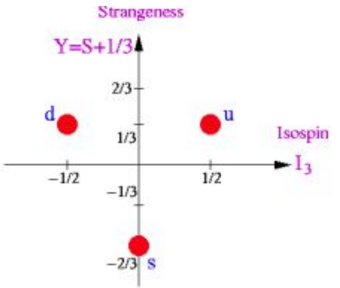


3

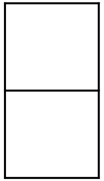
Fundamental/defining representation

dim=3

quarks	I3	Y
u	1/2	1/3
d	-1/2	1/3
s	0	-2/3



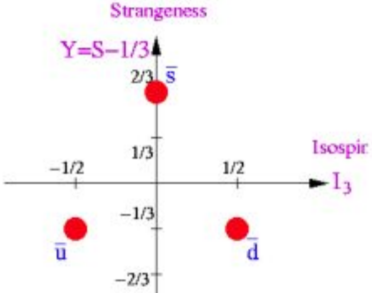
u	u	d
d	s	s

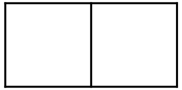


$\bar{\mathbf{3}}$

dim=3

states	I3	Y
u/d $\bar{s}$	0	2/3
u/s $\bar{d}$	1/2	-1/3
d/s $\bar{u}$	-1/2	-1/3

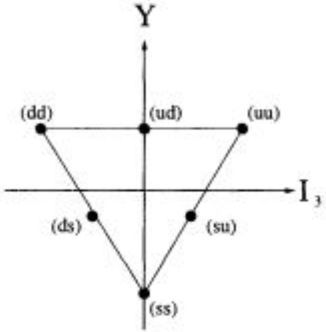


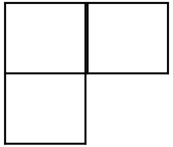


dim=6

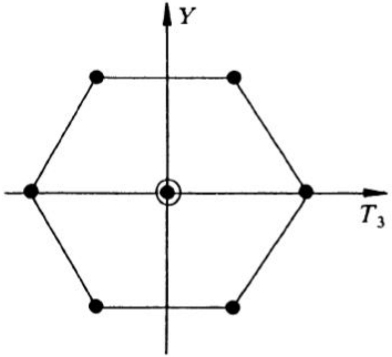
u	u	u	d	u	s
d	s	d	d	s	s

quarks	I3	Y
uu	1	2/3
ud	0	2/3
us	1/2	-1/3
ds	-1/2	-1/3
dd	-1	2/3
ss	0	-2/3

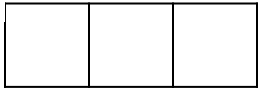




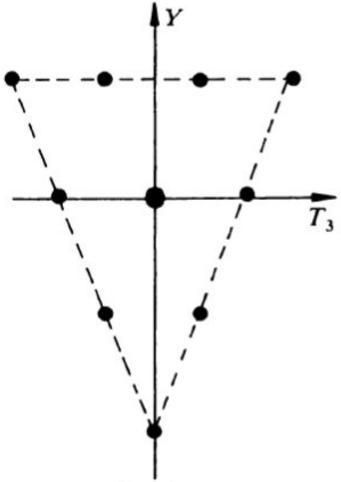
dim=8



octet

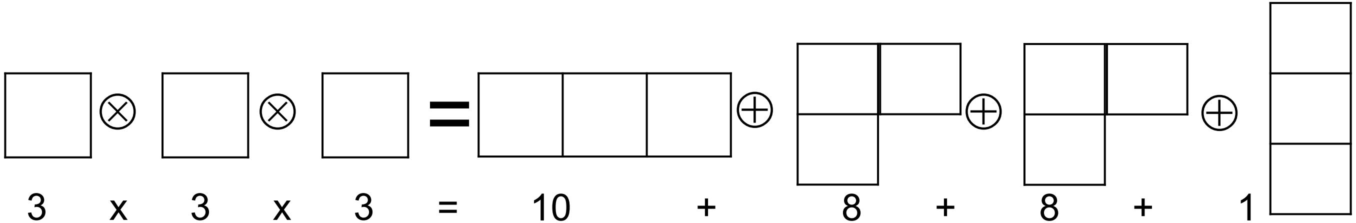


dim=10



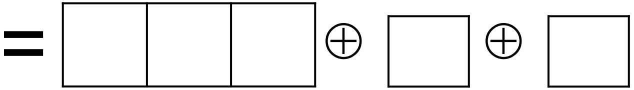
decuplet

Example: Protons— Direct product of 3 quarks



3 x 3 x 3 = 10 + 8 + 8 + 1

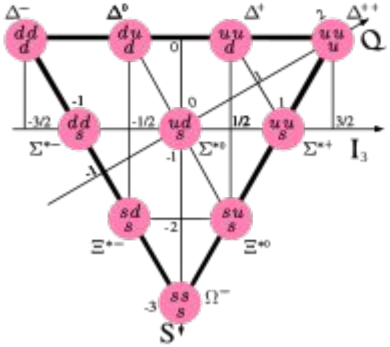
dim **spin**

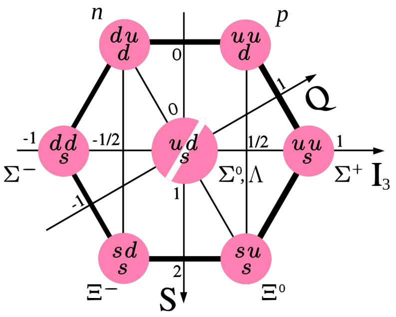


2 x 2 x 2 = 4 + 2 + 2

$$\left|j_1=\frac{1}{2}\right\rangle \otimes \left|j_2=\frac{1}{2}\right\rangle \otimes \left|j_3=\frac{1}{2}\right\rangle = \left|J=\frac{3}{2}\right\rangle \oplus \left|J=\frac{1}{2}\right\rangle \oplus \left|J=\frac{1}{2}\right\rangle$$

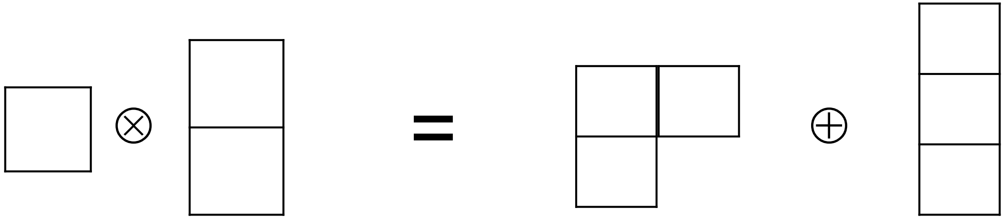
Spin 3/2 baryon decuplet





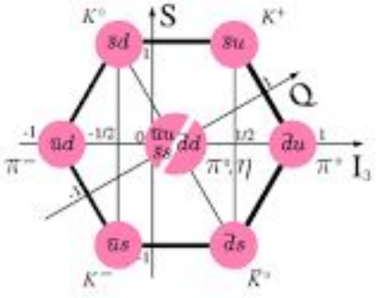
Spin 1/2 baryon octet

Example: Meson — Direct product of quark and antiquark

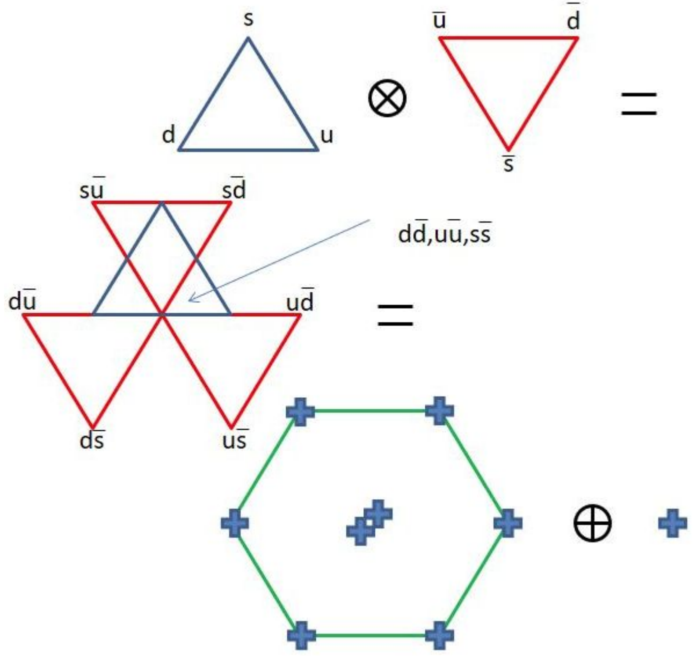


dim 3 x 3 = 8 + 1

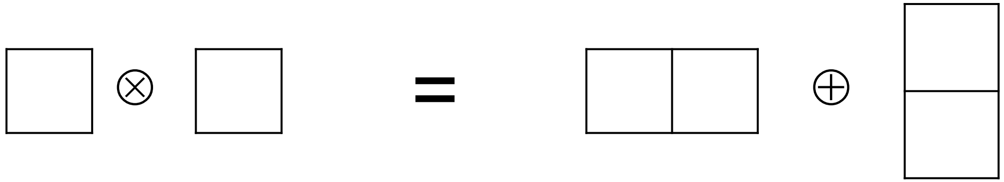
$\mathbf{3} \times \bar{\mathbf{3}}$



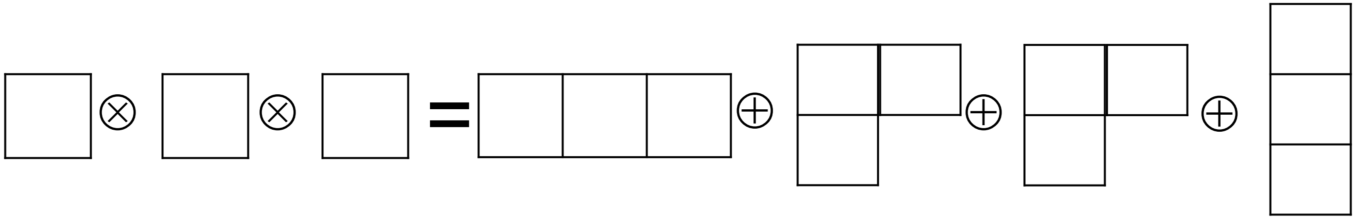
Meson octet



General Case: Decompose the direct product of irrep of SU(N)



dim N x N = N(N+1)/2 + N(N-1)/2



dim N x N x N = N(N+1)(N+2)/6 + 2N(N-1)(N+1)/3 + N(N-1)(N-2)/6

Summary of Standard Young tableaux Rules

- The number in the row doesn't decrease
- The number in the column has to increase
- # of Standard Young tableaux = dim of irreps
  - For different states (no degeneracy): Sn $n_\lambda = \frac{n!}{\Pi g_i}$
  - For irreducible representation for SU(N) $n_\lambda = \frac{N}{G} = \frac{\Pi n_i}{\Pi g_i}$

n	n+1	n+2	n+3	n+4
n-1	n	n+1	n+2	
n-2				

- Young Tableaux can construct primitive idempotent
- Decomposition of reducible representations into irreducible ones

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