Clifford Algebras and Spinors

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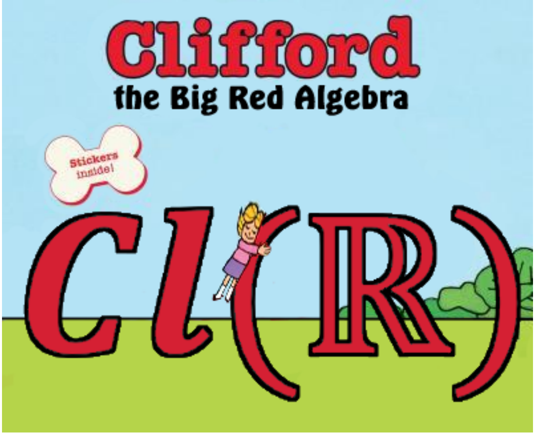
Why bother? (Motivation)

I prefer passing this course to the alternative.

Jest aside, the discussion of physics in the context of geometry has paved the way to great generalization of principles, allowing us to unify various descriptions of physics.

Clifford algebras provide an excellent generalization of rotations, spin, and even more general conformal mappings/transformations (think of the Möbius transformations from the problem sets).

What are Clifford algebras?



Presentation over!

What are Clifford algebras?

Algebras over vector spaces equipped with symmetric bilinear products (e.g. $Q(v, w) = v \cdot w$)

Consider $\mathbb{R}^2 = \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2\} = \{x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \,|x_1, x_2 \in \mathbb{R}\}$, equipped with $Q(\mathbf{r}_1, \mathbf{r}_2) = \mathbf{r}_1 \cdot \mathbf{r}_2$
Goal: Match the definition of "length" $|\mathbf{r}|^2 = Q(\mathbf{r}, \mathbf{r})$

$$\mathbf{rr} = (x_1\mathbf{e}_1 + x_2\mathbf{e}_2)^2 = x_1^2\mathbf{e}_1^2 + x_2^2\mathbf{e}_2^2 + x_1x_2(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1)$$

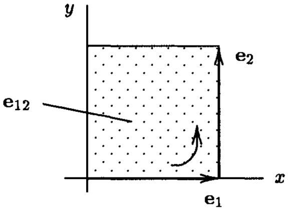
The matching provides the rule $\{\mathbf{e}_i, \mathbf{e}_j\} = 2\delta_{ij}$, or more explicitly:

$$\boxed{\begin{array}{c} \mathbf{e}_1^2 = \mathbf{e}_2^2 = 1 \\ \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 \end{array}}$$

Bivectors and Grassmann Algebras I

Let us play with our new product:
$\mathbf{r}_1\mathbf{r}_2 = (x_1\mathbf{e}_1 + x_2\mathbf{e}_2)(y_1\mathbf{e}_1 + y_2\mathbf{e}_2) = (x_1y_1 + x_2y_2) + (x_1y_2 - x_2y_1)\mathbf{e}_{12}$

It decomposes into an symmetric and anti-symmetric part: a scalar and a bivector



A visual depiction of a bivector.

Bivectors and Grassmann Algebras II

Anti-symmetric portion motivates the Grassmann/exterior product: $\mathbf{r}_1 \wedge \mathbf{r}_2 = (x_1 y_2 - x_2 y_1)\mathbf{e}_{12}$

The product of two vectors can be defined as an inner and exterior product:

$$\boxed{\begin{array}{c}\mathbf{r}_1 \cdot \mathbf{r}_2 = \frac{1}{2}(\mathbf{r}_1\mathbf{r}_2 + \mathbf{r}_2\mathbf{r}_1) \\ \mathbf{r}_1 \wedge \mathbf{r}_2 = \frac{1}{2}(\mathbf{r}_1\mathbf{r}_2\text{-}\mathbf{r}_2\mathbf{r}_1)\end{array}}$$

We may define the wedge product more generally:

$$\bigwedge_{i=1}^{n} \mathbf{r}_i \equiv \mathbf{r}_1 \wedge \cdots \wedge \mathbf{r}_n = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\sigma} \mathbf{r}_{\sigma(1)} \mathbf{r}_{\sigma(2)} \cdots \mathbf{r}_{\sigma(n)}$$

Clifford product defines 3 distinct spaces: $\mathcal{C}\ell(\mathbb{R}^2) \cong \mathbb{R} \oplus \mathbb{R}^2 \oplus \Lambda^2\mathbb{R}^2$

Even More Decompositions

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_1$
$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

However, note the additional even-odd ($\mathbb{Z}_2$) gradation:

$$\mathcal{C}\ell(\mathbb{R}^2) \equiv \mathcal{C}\ell_2 \cong \mathcal{C}\ell_2^+ \oplus \mathcal{C}\ell_2^- \cong (\mathbb{R} \oplus \overset{2}{\Lambda}\mathbb{R}^2) \oplus (\mathbb{R}^2)$$

Trivial group is a (only other) subgroup of $\mathbb{Z}_2$:even subspace forms a subalgebra. Let's explore the even subalgebra of $\mathcal{C}\ell_2$:

$$\mathcal{C}\ell_2^+ = \{x + y\mathbf{e}_{12} | x, y \in \mathbb{R}\}$$

Note that $\langle 1, \mathbf{e}_{12} \rangle \mathbf{e}_{12} = \langle \mathbf{e}_{12}, -1 \rangle$ is a rotation of the $\{1, \mathbf{e}_{12}\}$ subspace by $\pi/2$. This is identical to the action of the imaginary unit $i \in \mathbb{C}$.

Complexities That Make The Head Spin I

In fact, $\mathcal{C}\ell_2^+$ algebraically isomorphic to $\mathbb{C}$.

Complex conjugation is an *involution* (self-inverse operation) that allows us to define the inverse of an element: $z^{-1} = z/(z\bar{z})$. We may define involutions on the Clifford Algebra, acting on $u \in \mathcal{C}\ell_2$:

grade involution	$\hat{u} = \langle u\rangle_0 - \langle u\rangle_1 + \langle u\rangle_2$
reversion	$\tilde{u} = \langle u\rangle_0 + \langle u\rangle_1 - \langle u\rangle_2$
Clifford-conjugation	$\overline{u} = \langle u\rangle_0 - \langle u\rangle_1 - \langle u\rangle_2$

Grade involutions change the sign of odd graded elements, reversions reverse the order of all the multiplied basis vectors, and Clifford-conjugations compose both of the above.

In the isomorphism above, complex conjugation is achieved by restricting reversion or Clifford-conjugation to $\mathcal{C}\ell_2^+$.

Complexities That Make The Head Spin II

Consider the action of $\mathcal{C}\ell_2^+$ on $\mathbf{r} \in \mathcal{C}\ell_2^- = \{x_1\mathbf{e}_1 + x_2\mathbf{e}_2 | \, x_1, x_2 \in \mathbb{R}\}$:

$$\mathbf{r}(a + b\,\mathbf{e}_{12}) = (\sqrt{a^2 + b^2})\exp(-\mathbf{e}_{12}\arctan b/a)\,\mathbf{r} = (\sqrt{a^2 + b^2})\,\mathbf{r}\exp(\mathbf{e}_{12}\arctan b/a)$$

For even graded elements of unit norm, the action on the space of vectors is that of a rotation.
With the isomorphism $\mathcal{C}\ell_2^+ \cong \mathbb{C}$, we obtain a representation of rotations of the plane ($SO(2)$) in the form of unit complex numbers: $SO(2) \cong U(1)$.

Complexities That Make The Head Spin III

Setting $a^2 + b^2 = 1$, and $\tan\theta = b/a$, we note that the rotation can also be expressed as a similarity transform:

$$\mathbf{r}(\cos\theta + \sin\theta\,\mathbf{e}_{12}) = \mathbf{s}^{-1}\,\mathbf{r}\,\mathbf{s}, \quad \mathbf{s} = \exp\left(\mathbf{e}_{12}\theta/2\right)$$

There are two vectors corresponding to the same rotation: $\mathbf{s}$ and $-\mathbf{s}$. This is a double covering of the rotations of $\mathbb{R}^2$, providing a description of the spin group of 2 dimensions:

$$\textbf{Spin}(2) = \{s \in \mathcal{C}\ell_2^+ \,|\, s\bar{s} = 1\}, \quad \textbf{Spin}(2)/\{\pm 1\} \cong SO(2) \qquad (1)$$

The Schrödinger-Pauli Equation; A Review

For an electron in an EM field, the Schrödinger equation reads:

$$i\partial_t\psi = \frac{1}{2m}(-i\nabla - e\vec{A})^2\psi - e\phi\psi$$

This generates a differential operator associated with the generalized momentum $\vec{\pi} = -i\nabla - e\vec{A}$, where $[\pi_i, \pi_j] = i\epsilon_{ijk}eB_k$.

This motivates the definition of the Pauli matrices,

$$\boxed{\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},}$$

employed to make the spin of the electron manifest

$$i\partial_t\psi = \frac{1}{2m}(\vec{\pi}\cdot\vec{\pi} - e(\vec{\sigma}\cdot\vec{B}))\psi - e\phi\psi$$

Taking a Step "Up" I

Noting the identity $(\vec{\sigma} \cdot \vec{B})^2 = B^2 I$ (or doing the homework), we regard $\{\sigma_i\}$ an orthonormal basis for $\mathbb{R}^3$, inviting an exploration of the Pauli-Schrödinger equation in the language of $\mathcal{C}\ell(\mathbb{R}^3) \equiv \mathcal{C}\ell_3$.

Recall that $\mathbf{r}_1 \mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_2 + \mathbf{r}_1 \wedge \mathbf{r}_2$ in $\mathcal{C}\ell_2$. This generalizes across all finite dimensions (and bilinear signatures). A terse summary of Pauli's achievement is found in his recognition of the underlying Clifford structure:

$$\boldsymbol{\pi}^2 = \vec{\pi} \cdot \vec{\pi} + \vec{\pi} \wedge \vec{\pi} = \pi^2 - eB$$

A deeper explanation entails the exploration of $\mathcal{C}\ell_3$, along with the *real* algebra generated by the Pauli matrices, $\mathrm{Mat}(2, \mathbb{C})$.

Taking a Step "Up" II

$\mathcal{C}\ell_3$ constitutes of scalars (a_0), vectors ($a_1^i \mathbf{e}_i$), bivectors ($a_2^{ij}\mathbf{e}_i\mathbf{e}_j$), and volume elements ($a_3\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = a_3\mathbf{e}_{123}$). Since we simply require $\{\mathbf{e}_i, \mathbf{e}_j\} = 2\delta_{ij}$, with a necessity for 3 basis vectors, the Pauli matrices may be identified with these elements.

As an aside, we may explore the even subalgebra of $\mathcal{C}\ell_3$, $\mathcal{C}\ell_3^+ \cong \mathbb{R} \oplus \Lambda^2\mathbb{R}^3$. Note that $\Lambda^2\mathbb{R}^3$ is spanned by $\{\mathbf{e}_{12}, \mathbf{e}_{23}, \mathbf{e}_{31}\}$, where each element squares to -1, while $\mathbf{e}_{ik}\mathbf{e}_{kj} = -\mathbf{e}_{kj}\mathbf{e}_{ik} = \mathbf{e}_{ij}$. Replacing them with $\{i, j, k\}$, we note that we have obtained the structure of the quaternions, $\mathbb{H}$.

While this is no rigorous proof, it is fairly clear that $\mathcal{C}\ell_3^+ \cong \mathbb{H}$. In fact, $\mathcal{C}\ell_3^- \cong \mathbb{R} \oplus \Lambda^3\mathbb{R}^3 = \{a + \mathbf{e}_{123}b |\, a, b \in \mathbb{R}\} \cong \mathbb{C}$, where $\mathbf{e}_{123}^2 = -1$. With a few additional arguments, one can demonstrate that $\mathcal{C}\ell_3 \cong \mathbb{C} \otimes \mathbb{H}$.

Taking a Step "Up" III

Given $\mathbf{e}_i \simeq \sigma_i$, investigate the effect of involutions on an arbitrary element of $\mathrm{span}\{I, \sigma_1, \sigma_2, \sigma_3\} \cong \mathrm{Mat}(2, \mathbb{C})$:

grade involution	$\hat{u} = \langle u \rangle_0 - \langle u \rangle_1 + \langle u \rangle_2 - \langle u \rangle_3$
reversion	$\tilde{u} = \langle u \rangle_0 + \langle u \rangle_1 - \langle u \rangle_2 - \langle u \rangle_3$
Clifford-conjugation	$\overline{u} = \langle u \rangle_0 - \langle u \rangle_1 - \langle u \rangle_2 + \langle u \rangle_3$

If $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

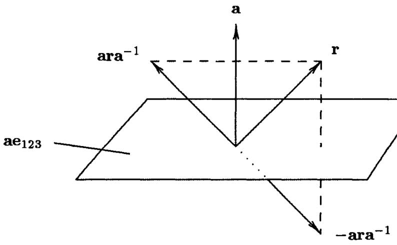
$$\hat{u} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

with $\tilde{u} \to u^\dagger$, and for a non-singular u, $\bar{u} \to u^{-1} \det u$ and $\hat{u} \to (u^{-1} \det u)^\dagger$.

Spinning in 3D I

Generalize our picture of rotations to higher dimensions - we need an axis and an angle in 3D, unlike the case of 2D.
Let us rotate the vector $\mathbf{r}$ about the axis $\mathbf{a}/|\mathbf{a}|$ by some angle $\alpha = |\mathbf{a}|$.

Recall that $2\mathbf{a}\cdot\mathbf{r} = \mathbf{ar} + \mathbf{ra}$. This implies that
$-\mathbf{ara}^{-1} = \mathbf{r} - 2(\mathbf{r}\cdot\mathbf{a})\mathbf{a}^{-1} = \mathbf{r} - 2(\mathbf{r}\cdot\mathbf{a})\mathbf{a}/\mathbf{a}^2$



Rotating around $\mathbf{a}$ by reflecting.

Spinning in 3D II

$\mathbf{ae}_{123} = a_1\mathbf{e}_{23} + a_2\mathbf{e}_{31} + a_3\mathbf{e}_{12}$ gives us the bivector dual to $\mathbf{a}$, which forms the plane of rotation. Here, we may replace the bivectors with $\mathbf{e}_{ij} \simeq \sigma_{ij} = i\epsilon_{ijk}\sigma_k$.

From our exercises with Pauli matrices, we recognize that:

$$\exp\left(\mathbf{ae}_{123}\right) \simeq \exp\left(i\alpha\hat{a}\cdot\vec{\sigma}\right) = I\cos\alpha + i\hat{a}\cdot\vec{\sigma}\sin\alpha \simeq \cos\alpha + \mathbf{e}_{123}\frac{\mathbf{a}}{|\mathbf{a}|}\sin\alpha$$

This provides our formalism for rotations in 3D:

$$\mathbf{ara}^{-1} = \mathbf{srs}^{-1}, \quad \mathbf{s} = \exp\left(\frac{1}{2}\mathbf{ae}_{123}\right) \tag{2}$$

Spinning in 3D III

Note that $\mathbf{s}^{-1}$ is obtained either by changing the signs of all vectors, while leaving the trivector (pseudoscalar) unchanged, or vice-versa. In either case, the sign of all bivectors must flip. The first case corresponds to a reversion, while the second, Clifford-conjugation. As such, we require that for $\mathbf{s} \in \mathcal{C}\ell_3$ to be a rotation, we require $\mathbf{s}\tilde{\mathbf{s}} = \mathbf{s}\bar{\mathbf{s}} = 1$. Identifying $\pm\mathbf{s}$ as the same rotation, we find:

$$\mathbf{Spin}(3) = \{\mathbf{s} \in \mathcal{C}\ell_3 | \, \mathbf{s}\tilde{\mathbf{s}} = 1, \mathbf{s}\bar{\mathbf{s}} = 1\}$$

In our Hermitian basis for $\mathrm{Mat}(2, \mathbb{C})$, this becomes: $s^\dagger s = I, \det s = 1$, providing the isomorphism:

$$SU(2) \cong \{u \in \mathrm{Mat}(2, \mathbb{C}) | \, u^\dagger u = I, \det u = 1\} \cong \mathbf{Spin}(3) \qquad (3)$$

Spinors, Ideals, and Spinor Operators I

Since a Pauli spinor is well represented as an element of $\mathbb{C}^2$, we may promote it to a subspace of $\mathrm{Mat}(2,\mathbb{C})$:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2 \to \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} \in \mathrm{Mat}(2,\mathbb{C})f \subset \mathrm{Mat}(2,\mathbb{C}),$$

where $f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a *primitive idempotent* in $\mathrm{Mat}(2,\mathbb{C})$.

The space of spinors is then the left *ideal* of generated by f, since for $s \in S \equiv \mathrm{Mat}(2,\mathbb{C})f$, and $u \in \mathcal{C}\ell_3$, $us \in S$, once a notion of scaling is provided.

This is provided by elements of $\mathbb{F} = f\mathcal{C}\ell_3 f = \{\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \,|\, c \in \mathbb{C}\}$, the field of scalars. The scaling of $\psi \in S$ by $\lambda \in \mathbb{F}$ is provided by $\psi\lambda$, and a scalar product $\tilde{s}s \simeq \psi^\dagger\psi$ may be constructed.

Spinors, Ideals, and Spinor Operators II

To consider an active operator formulation of spinors, we consider $\Psi \in \mathcal{C}\ell_3^+$, given by $\Psi = \psi + \hat{\psi}$ for $\psi \in S \simeq u \in \mathcal{C}\ell_3 f$.

We are used to considering spinor expectations: $s_i = \psi\sigma_i\psi^\dagger = 2\langle u\mathbf{e}_i\tilde{u}\rangle_0$. In the framework of $\Psi \in \mathcal{C}\ell_3^+$, we obtain $\mathbf{s} = \Psi\mathbf{e}_3\tilde{\Psi}$, allowing us to rewrite the Pauli-Schrödinger equation:

$$i\partial_t\psi = \frac{\boldsymbol{\pi}^2}{2m}\Psi - \frac{e}{2m}\vec{B}\Psi\mathbf{e}_3 - e\phi\Psi$$

This formulation makes the quantization direction of the spin, $\mathbf{e}_3$, manifest, associated with our choice of idempotent!

A New Spin on Spacetime I

To construct spinors for spacetime, we note that our underlying vector space is now somewhat different: $\mathbb{R}^{1,3}$ or $\mathbb{R}^{3,1}$, instead of $\mathbb{R}^n$.

Focusing on $\mathbb{R}^{1,3}$, we obtain the familiar Dirac matrices, $\mathbf{e}_\mu \simeq \gamma_\mu$ that obey $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} I$, providing the identification $\mathcal{C}\ell_{1,3} \simeq \mathrm{Mat}(2, \mathbb{H})$. We may also explicitly present the corresponding double-covers of $O(1,3)$, $SO(1,3)$, and $SO_+(1,3)$, $\mathbf{Pin}(1,3), \mathbf{Spin}(1,3)$. These are obtained by choosing elements $\mathbf{s}\tilde{\mathbf{s}} = \pm 1$ in the spaces $\mathcal{C}\ell_{1,3}$ and $\mathcal{C}\ell^+_{1,3}$, while $\mathbf{Spin}_+(1,3)$ is the subset of $\mathbf{Spin}(1,3)$ with $\mathbf{s}\tilde{\mathbf{s}} = 1$.

Considering $\mathbb{R}^{3,1}$, we note that $\mathbf{e}_i^2 = -\mathbf{e}_0^2 = 1$. This, along with the anti-commutation, generates the Clifford algebra $\mathcal{C}\ell_{3,1} \simeq \mathrm{Mat}(4, \mathbb{R})$. Similarly, $SO_+(3,1)$ has a double-covering of $\mathbf{Spin}_+(3,1) = \{\mathbf{s} \in \mathcal{C}\ell^+_3 \,|\, \mathbf{s}\tilde{\mathbf{s}} = 1\}$.

A New Spin on Spacetime II

We may now exploit the same machinery that we used on the Pauli-Schrödinger equation on the Dirac equation:

$$(\gamma_\mu(i\partial^\mu - eA^\mu) - m)\psi = 0$$

Considering the spinor operators found in $\mathcal{C}\ell^+_{1,3}$, we may rewrite the Dirac equation to make the time-like quantization direction apparent:

$$(\partial\Psi\gamma_{21} - e\mathbf{A}\Psi - m\Psi\mathbf{e}_0) = 0$$

Thank you

Thank you for your patience during this presentation. It has been a fun[1] course.

[1] There's no way this material is easier than Jackson, Howie.

References

Clifford Algebras and Spinors - Lounesto, Pertti