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Subgroups and normal subgroups of dihedral group up to isomorphism

Abstract: In this paper, we count the number of subgroups in a dihedral group from D_3 to D_8 and then evaluate the number of subgroups in a generalized way by using basic geometry, group theory, and number theory. We prove by a different approach that the total number of subgroups in a dihedral group is $\tau(n) + \sigma(n)$, where $\tau(n)$ is the number of positive divisors of a positive integer n, and $\sigma(n)$ is the sum of positive divisors of n. Further, we investigate the number of normal subgroups of D_8 and D_9 and the structure of those normal subgroups up to isomorphism.

Keywords: Subgroup, normal subgroup, dihedral group

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1 Introduction

In [4] a dihedral group D_{2n} is the group of symmetries of a regular n-gon. A regular n-gon can be rotated or reflected to get back to the n-gon. The group D_{2n} is the group generated by the reflection y and the rotation x by the angle $\frac{2\pi}{n}$.

The set of rotations forms a cyclic group of order n, given in [5]. There is a much less spectacular way to think about D_n. Namely, let x denote the rotation by $\frac{2\pi}{n}$, and let y be a reflection in D_n. We know that $x^n = e$ (the identity element) and $y^2 = e$. Further, we can deduce from our discussion of composition of rotations and reflections above that $yx = x^{n-1}y$. We say that x and y are generators of D_n and the equations $x^n = e = y^2$, $yx = x^{n-1}y$ are relations for these generators. With this in mind, the dihedral groups can be thought of just as the abstract group

$$D_{2n} = \{x^i y^j \mid i = 1, \dots, n-1,\ x^n = e,\ y^2 = e,\ yx = x^{n-1}y\}.$$

2 Properties of dihedral group

1. D_n is a non-Abelian group of order $2n$, $n \geq 3$ [6].

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2. The number of elements of order two in D_n is n if n is odd (n reflections) and $n+1$ if n is even (n reflections, 1 rotation) [3].
3. The number of elements of order $k \neq 2$ in D_n is $\phi(k)$, provided that $k|n$ (where $\phi(k)$ is the Euler function) [4].
4. The largest possible order of any element in D_n is n [4].
5. For each $k|n$, D_n has a cyclic group of order k [3].
6. If $k|n$, then D_k is isomorphic to a subgroup of D_n [6].
7. $\mathbb{Z}_n$ is isomorphic to the set of all rotations of D_n [6].
8. The class equation of D_n is $o(D_n) = 2.n = 1 + n + 2 + 2 + \cdots \frac{n-1}{2}$ times if n is an odd number, [4].
9. The class equation of D_n is $o(D_n) = 2.n = 1 + 1 + \frac{n}{2} + \frac{n}{2} + 2 + 2 + 2 + \cdots \frac{n-2}{2}$ times if n is an even number [4].

3 Subgroups of dihedral group

3.1 Structures of subgroups of D_3 to D_8

There are two types of symmetries of a regular polygon, the rotational symmetry and the line symmetry. The rotational symmetries will be denoted by some power of x, whereas the line symmetries will be denoted by power of x times y, and when the vertices of the regular polygon are in their original circular order, that is represented by e.

Now we will first try to identify the subgroups of some dihedral group by the trial-and-error method because we should be clear about the regular polygons, the elements in D_n, the operation table, and the lattices of the subgroups for D_n.

The set of subgroups of D_3, D_4, D_5, D_6, D_7, and D_8 are $\{D_3, \{x, x^2, e\}, \{x^2y, e\}, \{e\}\}$, $\{D_4, \{x^2, y, x^2y, e\}, \{e, x, x^2, x^3\}, \{x^2, xy, x^3y, e\}, \{x^2y, e\}, \{y, e\}, \{y, e\}, \{e, y^2\}, \{xy, e\}, \{x^3y, e\}, \{e\}\}$, $\{D_5, \{e, x, x^2, x^3, x^4\}, \{y, e\}, \{xy, e\}, \{x^2y, e\}, \{x^3y, e\}, \{x^4y, e\}, \{e\}\}$, $\{D_6, \{x^3, x^4, y, x^2y, x^4y, e\}$, $\{x, x^2, x^3, x^4, x^5, e\}$, $\{x^2, x^4, xy, x^3y, x^5y, e\}$, $\{x^3, y, x^3y, e\}$, $\{x^3, xy, x^4y, e\}$, $\{x^3, x^2y, x^5y, e\}$, $\{x^2, x^4, e\}$, $\{y, e\}$, $\{x^3y, e\}$, $\{x^3, e\}$, $\{xy, e\}$, $\{x^4y, e\}$, $\{x^2y, e\}$, $\{x^5y, e\}$, $\{e\}\}$, $\{D_7, \{x, x^2, x^3, x^5, x^6, e\}$, $\{y, e\}$, $\{xy, e\}$, $\{x^2y, e\}$, $\{x^3y, e\}$, $\{x^4y, e\}$, $\{x^5y, e\}$, $\{x^6y, e\}$, $\{e\}\}$, $\{D_8, \{x^2, x^4, x^6, y, x^2y, x^4y, x^6y, e\}$, $\{x, x^2, x^3, x^4, x^5, x^6, x^7, e\}$, $\{x^2, x^4, x^6, xy, x^3y, x^5y, x^7y, e\}$, $\{x^4, y, x^4y, e\}$, $\{x^4, x^2y, x^6y, e\}$, $\{x^2, x^4, x^6, e\}\{x^4, xy, x^5y, e\}$, $\{x^4, x^3y, x^7y, e\}$, $\{y, e\}$, $\{x^4y, e\}$, $\{x^2y, e\}$, $\{x^6y, e\}$, $\{x^4, e\}$, $\{xy, e\}$, $\{x^5y, e\}$, $\{x^3y, e\}$, $\{x^7y, e\}$, $\{e\}\}$, respectively.

Knowing the sets of subgroups of D_3 to D_8, it is trivial to find families of subgroups of a dihedral group associated with a regular polygon with a smaller number of edges. However, when it comes to find the number of subgroups for a Dihedral group associated with a regular polygon with a large number of sides, then its quite tiresome. So we provide a method to find the number of subgroups of a dihedral group associated with a regular polygon with any number of sides. Note that D_n always contain the subgroups D_n and $\{e\}$ and the subgroups spanned by x^a, $a \in \mathbb{N}$. If k is coprime to n (the number of sides of the regular n-gon), then no subgroups can be spanned by x^k. Modular arithmetic demonstrates that a relatively prime number generates every number contained in the set created by mod(n); therefore each subgroup corresponds to a factor of n.

Let us investigate the subgroups for D_4 and D_8. Note that the factors of 4 are 1, 2, and 4. The subgroups of D_4 are as follows: D_4, $\{x^2, y, x^2y, e\}$, $\{e, x, x^2, x^3\}$, $\{x^2, xy, x^3y, e\}$, $\{x^2y, e\}$, $\{y, e\}$, $\{y, e\}$, $\{e, y^2\}$, $\{xy, e\}$, $\{x^3y, e\}$, $\{e\}$. Now we will first break these subgroups into two groups, the subgroups that contain only rotations and the subgroups that contain reflections.

Looking at the three subgroups that contain rotations of the square, x will span the subgroup only containing rotations generated by a $\frac{\pi}{2}$ clockwise rotation, x^2 will span the subgroup of rotations generated by a π clockwise rotation, and x^4 or e will generate the last subgroup generated by a 2π clockwise rotation. Thus we can conclude that the number of subgroups of D_4 that only contain rotations is equivalent to the number of factors of 4.

We will now find the subgroups that contain rotations and reflections. The subgroup spanned by x and y produces the entire group D_n. The subgroup spanned by x^2 and y produces $\{x^2, y, x^2y, e\}$. The subgroup spanned by x^2 and xy produces $\{x^2, xy, x^3y, e\}$. The subgroups spanned by e and each reflection are $\{y, e\}$, $\{xy, e\}$, $\{x^2y, e\}$, and $\{x^3y, e\}$. By considering all these we are able to find that the number of subgroups of D_4 is equal to $3 + 1 + 2 + 4$. This is equal to the number of factors of 4 plus each factor of 4.

Now we will explore D_8, where there are a total of 19 subgroups. So we will examine two types of subgroups, namely the subgroups that only contain rotations and the subgroups that have reflections. Identifying how each subgroup of D_8 is generated will reveal the formula for the number of subgroups of D_8.

In D_8 the only subgroups that contain only the rotations are $\{x, x^2, x^3, x^4, x^5, x^6, x^7, e\}$, $\{x^2, x^4, x^6, e\}$, $\{x^4, e\}$, and $\{e\}$. The subgroup $\{x, x^2, x^3, x^4, x^5, x^6, x^7, e\}$ represents the subgroup of rotations spanned by x, $\{x^2, x^4, x^6, e\}$ is the subgroup of rotations spanned by x^2, $\{x^4, e\}$ is the subgroup spanned by x^4, and $\{e\}$ is the subgroup spanned by e. So we can conclude that D_8 has four subgroups that only contain rotations. Notice that 8 has four factors of 1, 2, 4, and 8.

The subgroups that contain both rotations and reflections are D_8, $\{x^2, x^4, x^6, y, x^2y, x^4y, x^6y, e\}$, $\{x^2, x^4, x^6, xy, x^3y, x^5y, x^7y, e\}$, $\{x^4, y, x^4y, e\}$, $\{x^4, x^2y, x^6y, e\}$, $\{x^4, xy, x^5y, e\}$, $\{x^4, x^3y, x^7y, e\}$, $\{y, e\}$, $\{x^4y, e\}$, $\{x^2y, e\}$, $\{x^6y, e\}$, $\{xy, e\}$, $\{x^5y, e\}$, $\{x^3y, e\}$, $\{x^7y, e\}$. The subgroup $\{x^2, x^4, x^6, y, x^2y, x^4y, x^6y, e\}$ is spanned by x^2 and y, and $\{x^2, x^4, x^6, xy, x^3y, x^5y, x^7y, e\}$ is spanned by x^2 and xy. Notice that x^2 span two subgroups that contains reflections. The subgroup $\{x^4, y, x^4y, e\}$ is spanned by x^4 and y, $\{x^4, xy, x^5y, e\}$ is the subgroup generated by x^4 and xy, $\{x^4, x^2y, x^6y, e\}$ is the subgroup spanned by x^4 and x^2y, and $\{x^4, x^3y, x^7y, e\}$ is the subgroup spanned by x^4 and x^3y. Therefore x^4 spans four subgroups that contain reflections. The remaining subgroups that contain reflections and identity are $\{y, e\}$, $\{xy, e\}$, $\{x^2y, e\}$, $\{x^3y, e\}$, $\{x^4y, e\}$, $\{x^5y, e\}$, $\{x^6y, e\}$, and $\{x^7y, e\}$. Notice that x^8 spans eight subgroups that contain reflections. So we can conclude that D_8 is equal to 4 + 1 + 2 + 4 + 8, which is a total of 19 subgroups. Hence the total number of subgroups of D_8 is equivalent to the number of factors of 8 plus all factors of 8.

By the above explanation, a question arises if for any dihedral group does the number of subgroups is equivalent to the number of factors of the number of sides of the regular polygon plus all factors of the number of sides of the regular polygon. So we need to verify that for any dihedral group associated with a regular polygon.

Suppose D_n is a dihedral group, where $n \in \mathbb{N}$. Then by the fundamental theorem of arithmetic we have the prime factorization $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ of $n > 1$. So the positive divisors of n are of the form $d = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, where $0 \le a_1 \le k_i$, where $i = 1, 2, \dots, r$. Therefore d spans the subgroup of rotations $x^d, x^{2d}, \dots, e$, a subgroup of D_n, from which we can calculate the number of subgroups of D_n.

3.2 Counting technique for subgroups of a dihedral group in generalized form

Let us verify that the total number of subgroups in D_n is equal to $\tau(n) + \sigma(n)$, where $\tau(n)$ is the number of positive divisors of n, and $\sigma(n)$ is the sum of the positive divisors of n [1].

Lemma 1. *Total number of subgroups in D_n is equal to $\tau(n) + \sigma(n)$.*

Proof. The proof has be done by Cavior [2], but here we give it by a different approach. By definition, $\tau(n)$ denotes the number of positive divisors of a positive integer n. Let d and n be positive integers such that d is a divisor of n, which implies that there exists a positive integer $m = \frac{n}{d}$ such that x^d spans the closed set $\{x^d, x^{2d}, \dots, x^n\}$ of rotations. Now this set is a subgroup as it contains all the inverses for each element [7]. Hence this clarifies that every power of x that is a divisor of n spans a subgroup of rotations, any multiple of d that is not a divisor of n spans the same subgroup as x^d, and any power of x that is coprime to n spans the same subgroup as x. So we can conclude that the number of subgroups of D_n that only contain rotations is equivalent to the number $\tau(n)$ of divisors of n.

We now claim that $\sigma(n)$ represents the number of subgroups that contain reflections. Let q, n, and d be positive integers such that d is a divisor of n, $\sigma(n) = q + d$, and $0 \le a_i \le d$, where $a_i = 1, 2, \dots, d$. The subgroups spanned by x^d and $x^{a_i}y$ are as follows; $\{x^d, x^2 d, \dots, e, y, x^d y, x^{2d} y, \dots\}$, $\{x^d, x^2 d, \dots, e, xy, x^{d+1} y, \dots\}$, and $\{x^d, x^{2d}, \dots, e, x^{d-1} y, \dots\}$. Each subgroup spanned by x^d and $x^{a_i}y$ contains a particular element of the set $\{y, xy, x^2 y, \dots, x^{d-1} y\}$. This set has d elements, which implies that x^d and $x^{a_i}y$ span d subgroups that contain reflections. Hence the number of subgroups containing reflections is equivalent to the sum of the divisors of n.

From the above it is clear that the number of subgroups of D_n is $\tau(n) + \sigma(n)$. □

4 Normal subgroups in D_8 and their structures up to isomorphism

Consider $D_8 = \{x_0, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, x^2y, x^3y, x^4y, x^5y, x^6y, x^7y\}$, where x is the rotation at $\frac{\pi}{4}$ angle, y is the reflection about the line of symmetries, and x_0 is the rotation at 0 angle and is the identity element in D_8.

The possible orders of elements in D_8 are 1, 2, 4, 8, and 16. The number of elements of orders 1, 2, 4, 8, and 16 in D_8 are 1, 9, 2, 4, and 0, respectively. The subgroup of order 1 in D_8 is isomorphic to $\mathbb{Z}_1$ and is unique. So it is normal in D_8.

The subgroup of order 2 in D_8 is isomorphic to $\mathbb{Z}_2$, and there are 9 such subgroups. The structures of those subgroups are $H_1 = \{x_0, y\}$, $H_2 = \{x_0, xy\}$, $H_3 = \{x_0, x^2y\}, \ldots, H_8 = \{x_0, x^7y\}$, $H_9 = \{x_0, x^4\}$. Here $Cl(y) = \{y, x^2y, x^4y, x^6y\}$, which is not a subset of H_1, H_3, H_5, H_7, respectively. Similarly, $Cl(xy) = \{xy, x^3y, x^5y, x^7y\}$ is not a subset of H_2, H_4, H_6, H_8, respectively. So $H_1, H_2, \ldots, H_8$ are not normal in D_8. However, $Cl(x^4) = \{x^4\} \subset H_9$ and $Cl(x_0) = \{x_0\} \subset H_9$. Therefore H_9 is normal in D_8.

The subgroups in D_8 of order 4 up to isomorphism are $\mathbb{Z}_4$ and $D_2 \cong K_4$. The subgroup of D_8 isomorphic to $\mathbb{Z}_4$ is unique, and hence it is normal in D_8. The subgroups in D_8 isomorphic to D_2 are $H_1' = \{x_0, x^4, y, x^4y\}$, $H_2' = \{x_0, x^4, xy, x^5y\}$, $H_3' = \{x_0, x^4, x^2y, x^6y\}$, $H_4' = \{x_0, x^4, x^3y, x^7y\}$. Here $Cl(y)$ is not a subset of H_1' and H_3', and $Cl(xy)$ is not a subset of H_2' and H_4'. Therefore H_1', H_2', H_3', and H_4' are not normal in D_8.

The subgroups in D_8 of order 8 in D_8 up to isomorphism are $\mathbb{Z}_8$ and D_4. Here the subgroup of D_8 isomorphic to $\mathbb{Z}_8$ is unique, and hence it is normal in D_8. There are two subgroups isomorphic to D_4, $G_1 = \{x_0, x^2, x^4, x^6, y, x^2y, x^4y, x^6y\}$ and $G_2 = \{x_0, x^2, x^4, x^6, xy, x^3y, x^5y, x^7y\}$, and both have the index 2. Therefore both G_1 and G_2 are normal in D_8.

So there are 7 normal subgroups of D_8.

5 Normal subgroups of D_9 and their structures up to isomorphism

To find the normal subgroups of D_9 and their structures up to isomorphism, we first check the possible orders of the subgroups of D_9, that is, 1, 2, 3, 6, 9, and 18, and the number of elements of orders 1, 2, 3, 6, 9, and 18 are 1, 9, 2, 0, 6, and 0, respectively.

The subgroups of order 1, 2, 3, 6, 9, and 18 are isomorphic to $\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, D_3, $\mathbb{Z}_9$, and D_9, respectively. The number of subgroups isomorphic to $\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, D_3, $\mathbb{Z}_9$, and D_9 are 1, 9, 1, 3, 1, and 1, respectively. Since subgroups of D_9 isomorphic to $\mathbb{Z}_1$, $\mathbb{Z}_3$, $\mathbb{Z}_9$, and D_9 are unique, they are normal. So till now we have four normal subgroups of D_9. Let us further investigate other normal subgroups.

Now we check whether the subgroups of D_9 isomorphic to $\mathbb{Z}_2$ are normal. The subgroups isomorphic to $\mathbb{Z}_2$ are $H_1 = \{e, y\}$, $H_2 = \{e, xy\}$, $H_3 = \{e, x^2y\}, \ldots, H_9 = \{e, x^8y\}$. Clearly, $Cl(y) = \{y, x^2y, x^4y, x^6y, x^8y, xy, x^3y, x^5y, x^7y\}$ is not a subset of $H_1, H_2, \ldots, H_9$. So we can conclude that there are no normal subgroups of D_9 isomorphic to $\mathbb{Z}_2$.

Now we further check whether the subgroups of D_9 isomorphic to D_3 are normal. There are a total of three subgroups of D_9 isomorphic to D_3: $H_1' = \{x^3, x^6, e, y, x^3y, x^6y\}$, $H_2' = \{x^3, x^6, e, xy, x^4y, x^7y\}$, $H_3' = \{x^3, x^6, e, x^2y, x^5y, x^8y\}$. Clearly, $Cl(y) = \{y, xy, x^2y, x^3y, x^4y, \ldots, x^8y\}$ is not a subset of H_1', H_2', and H_3'. So we can conclude that there are no normal subgroups of D_9 isomorphic to D_3. Hence D_9 has four normal subgroups.

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