The Lorentz and Poincaré groups in relativistic field theory

Eric Shahly

June 2019
The orthogonal groups $O(n)$, which preserve the norm of a vector in $\mathbb{R}^n$, can be generalized to groups $O(m,n)$ which preserve an indefinite metric $g = \mathbb{1}_{m,n} = \begin{bmatrix} \mathbb{1}_m & 0 \\ 0 & -\mathbb{1}_n \end{bmatrix}$.

The defining relation for $\Lambda \in O(m,n)$ is

$$\Lambda^T g \Lambda = g$$

For $\vec{x}, \vec{y} \in \mathbb{R}^{m+n}, \Lambda \in O(m,n)$, $\vec{y} = \Lambda \vec{x}$ implies

$$y_1^2 + \ldots + y_m^2 - y_{m+1}^2 - \ldots - y_{m+n}^2 = x_1^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_{m+n}^2$$
In special relativity we are interested in Lorentz transformations which preserve the "lengths" of four-vectors in Minkowski space,

\[ X^\mu X_\mu = g_{\mu\nu} X^\mu X^\nu = (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 \]

where \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) is the Minkowski metric.

Equivalently, a Lorentz transformation \( \Lambda \) must satisfy \( \Lambda^T g \Lambda = g \). That is, \( \Lambda \in \text{O}(1,3) \).
The Lorentz group \( O(1,3) \)

- The defining relation \( \Lambda^T g \Lambda = g \) implies that \( |\det \Lambda| = 1 \) and \( |\Lambda^0_0| \geq 1 \) for any \( \Lambda \in O(1,3) \).

- \( O(1,3) \) has 4 connected components corresponding to different possible signs of \( \det \Lambda \) and \( \Lambda^0_0 \).

- The component connected to the identity is a subgroup, often called the proper orthochronous Lorentz group \( SO(1,3)^+ \).

- We can think of the 4 components as a group:
  \[ \{1, P, T, PT\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2. \]
To construct a relativistic field theory, we would like our equations of motion for a particular field to hold true in all reference frames. This means the action $S = \int \mathcal{L}(x) d^4x$ should be invariant with respect to any transformation $\Lambda \in SO(1, 3)^+$.

The Lagrangian density $\mathcal{L}$ transforms like a Lorentz scalar:

$$\mathcal{L}(x) \rightarrow \mathcal{L}(\Lambda^{-1}x)$$
Consider a multiplet field $\Psi_a$ with $n$ components. It must transform as $\Psi_a(x) \rightarrow M_{ab}(\Lambda) \Psi_b(\Lambda^{-1}x)$, where $M_{ab}(\Lambda)$ is an $n \times n$ matrix. This requires us to construct an $n$-dimensional representation of $SO(1, 3)^+$. 
Linearizing the defining relation about the identity, we find that $x \in \mathfrak{so}(1, 3)$ must satisfy $x^T g + gx = 0$. $x$ is a $4 \times 4$ matrix, and this relation imposes 10 conditions on the elements of $x$. We can construct a basis $a_1, \ldots, a_6$ of the Lie algebra consisting of matrices $J_i$ and $K_i$ for $i = 1, 2, 3$. 
The Lie algebra $\mathfrak{so}(1, 3)$

\[
J_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix} \quad J_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
J_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

The $J_i$ are clearly the generators of the SO(3) subgroup. The corresponding one-parameter subgroups are finite spatial rotations.
The Lie algebra $\mathfrak{so}(1, 3)$

\[ K_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ K_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \]

The $K_i$ are the generators of Lorentz boosts.
In this basis, the algebra looks like:

\[ [J_i, J_j] = \epsilon_{ijk} J_k \]

\[ [J_i, K_j] = \epsilon_{ijk} K_k \]

\[ [K_i, K_j] = -\epsilon_{ijk} J_k \]

for \( i, j = 1, 2, 3 \)
The Lie algebra $\mathfrak{so}(1, 3)$

- We can consider the complexification of the Lie algebra, denoted by $\mathfrak{so}(1, 3)_\mathbb{C}$, in order to construct a basis that is equivalent to a direct sum of Lie algebras (that is, the complexification $\mathfrak{so}(1, 3)_\mathbb{C}$ is semi-simple).

- Define a new basis through complex linear combinations of the original basis vectors:

\[
\begin{align*}
\vec{J}_+ &\equiv \frac{1}{2}(\vec{J} + i\vec{K}) \\
\vec{J}_- &\equiv \frac{1}{2}(\vec{J} - i\vec{K})
\end{align*}
\]
This new basis satisfies the following commutation relations:

\[ [J^i_+, J^j_+] = \epsilon^{ijk}J^k_+ \]

\[ [J^i_-, J^j_-] = \epsilon^{ijk}J^k_- \]

\[ [J^i_+, J^j_-] = 0 \]

In this basis, it is clear that \(\mathfrak{so}(1, 3)_\mathbb{C} \cong \mathfrak{su}(2)_\mathbb{C} \bigoplus \mathfrak{su}(2)_\mathbb{C}\). At the group level we have \(\text{SO}(1, 3)^+ \cong \text{SL}(2, \mathbb{C})/\mathbb{Z}_2\).
The previous result implies we can characterize a representation of \( \text{SO}(1, 3)^+ \) by a pair of half-integer numbers, \((s_1, s_2)\) where \(s_1, s_2 = 0, \frac{1}{2}, 1, \ldots\)

The dimension of this representation is \((2s_1 + 1)(2s_2 + 1)\).

These representations define the possible types of fields which can be described by our relativistically invariant field theory.
Finite dimensional reps. of $\text{SO}(1, 3)^+ $

- $(0, 0) \rightarrow$ scalar fields, $\Phi$
- $(\frac{1}{2}, 0) \rightarrow$ left chiral Weyl spinor, $\psi_L$
- $(0, \frac{1}{2}) \rightarrow$ right chiral Weyl spinor, $\psi_R$
- $(\frac{1}{2}, \frac{1}{2}) \rightarrow$ four-vector, $V^\mu$
Note that there are two different 2-dimensional representations which are appropriate for describing the transformations of a spin $\frac{1}{2}$ field: \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\).

These representations describe Weyl fermions $\psi_L$ and $\psi_R$ of left and right chirality respectively. $\psi_L$ and $\psi_R$ are fundamentally different degrees of freedom - see neutrinos, and electroweak theory.

Under infinitesimal Lorentz transformations,

\[
\psi_L \rightarrow (1 - i\vec{\theta} \cdot \vec{\sigma}/2 - \vec{\zeta} \cdot \vec{\sigma}/2)\psi_L
\]
\[
\psi_R \rightarrow (1 - i\vec{\theta} \cdot \vec{\sigma}/2 + \vec{\zeta} \cdot \vec{\sigma}/2)\psi_R
\]

In the massless limit, a state of definite chirality is also a state of definite helicity.
In QED, one often uses four component Dirac spinors which mix $\psi_L$ and $\psi_R$.

This corresponds to the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation. The generators are

$$S^{0i} = -\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$S^{ij} = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

for $i, j = 1, 2, 3$. 

Eric Shahly

The Lorentz and Poincaré groups in relativistic field theory
One can show that the Klein-Gordon equation, 
\[(\partial^2 + m^2)\Phi(x) = 0\] is invariant under the transformation 
\[\Phi(x) \rightarrow \Phi(\Lambda^{-1}x)\]

Similarly, the Dirac equation \[(i\gamma^\mu \partial_\mu - m)\psi(x) = 0\] is invariant under the transformation \[\psi(x) \rightarrow \exp(-\frac{i}{2} \omega_{\mu\nu} S^{\mu\nu})\psi(\Lambda^{-1}x)\]

And, the Weyl equations \[i\sigma^\mu \partial_\mu \psi_L = 0\] and \[i\sigma^\mu \partial_\mu \psi_R = 0\] are invariant under transformations of \[\psi_L\] and \[\psi_R\] under the 2-dimensional fundamental / anti-fundamental reps. of SU(2) respectively.
Important note: SO(1,3) is non-compact. This means that the finite-dimensional representations are not unitary - we can see this by looking at the boost component of the general transformation. In a quantum theory we need unitary representations. Basis states of the Hilbert space of a quantum field theory transform under unitary \emph{infinite}-dimensional representations of the full Poincaré group.
The Poincaré group is also known as the inhomogeneous Lorentz group, or ISO(1,3). This group contains all of SO(1,3) as a subgroup, with 4 additional generators $P^\mu$ that generate translations in spacetime.

We can outline the strategy for classifying the infinite-dimensional unitary representations of the Poincaré group.
Consider the abelian invariant subgroup $T_4$ of ISO(1,3) (the translation group in four dimensions).

Basis vectors of our representation are built out of eigenvectors of the generators of this group ($P^\mu$), plus those of commuting operators from the Lie algebra of the little group.

Basis vectors are characterized by their eigenvalues with respect to quadratic Casimir operators: $C_1 = P_\mu P^\mu$ (eigenvalue $c_1 =$mass) and $C_2 = W_\lambda W^\lambda$, where

$$W^\lambda \equiv \epsilon^{\lambda\mu\nu\sigma} J_{\mu\nu} P_\sigma / 2$$

is called the Pauli-Lubanski vector.
Infinite dimensional reps. of the Poincaré group

Four possible cases:
- $c_1 = 0, \ p^{\mu} = 0$ (vacuum)
- $c_1 > 0$ (massive particle)
- $c_1 = 0, \ p^{\mu} \neq 0$ (massless particle)
- $c_1 < 0$

Each case leads to a different little group of the factor group $SO(1,3)$. Each irreducible unitary representation of the little group then induces an irreducible unitary representation of the full Poincaré group.
Case 1: $c_1 = 0, p^\mu = 0$

The little group is the subgroup of the factor group $ISO(1, 3)/T_4 \cong SO(1, 3)$ which leaves the test vector invariant. In this case, the little group is the full homogeneous Lorentz group $SO(1,3)$.
Case 2: \( c_1 > 0 \)

For a massive particle, we can always boost to a rest frame where \( p^\mu = (M, 0, 0, 0) \). In this case the little group is just the rotation group, \( \text{SO}(3) \). The basis vectors we will choose will satisfy:

\[
P^\mu |0\lambda\rangle = p^\mu |0\lambda\rangle
\]

\[
J^2 |0\lambda\rangle = s(s + 1) |0\lambda\rangle
\]

\[
J_3 |0\lambda\rangle = \lambda |0\lambda\rangle
\]

Here, \( 0 \) is labeling the three-momentum \( \vec{p} = 0 \).
Case 2: $c_1 > 0$

By acting on these basis vectors with elements of SO(1,3) not in the little group, we can obtain a general state $|p\lambda\rangle$. The result is a representation of the Poincaré group labeled by $(M, s)$ that is irreducible, unitary and infinite-dimensional.
Case 3: $c_1 = 0, p^\mu \neq 0$

This corresponds to a massless particle moving with some momentum $\omega$, so we can write the four-momentum as $p^\mu = (\omega, 0, 0, \omega)$. The little group is even smaller: $SO(2)$. The main difference is that in this case, the helicity $\lambda$ is Lorentz invariant.
Result:

- Massless states are identified with basis vectors of infinite-dimensional irreps. labeled by a momentum $\vec{p}$ and a helicity $\lambda$. There are two distinct possible states corresponding to either $\lambda = s$ or $\lambda = -s$ for some $s = 0, \frac{1}{2}, 1, \ldots$

- Massive states are identified with basis vectors of infinite-dimensional irreps. labeled by a mass $M$, a momentum $\vec{p}$, and a helicity $\lambda = -s, \ldots, s$ for some $s = 0, \frac{1}{2}, 1, \ldots$
References

- *Group Theory in Physics*, by Wu-Ki Tung
- *Group Theory in Physics, Vol. 2*, by J.F. Cornwell
- *The Quantum Theory of Fields, Vol. 1*, by Steven Weinberg
- *An Introduction to Quantum Field Theory*, by Michael Peskin and Daniel Schroeder