HENDRIK LORENTZ & HIS NEW BEST FRIENDS

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WHAT SYMMETRY TRANSFORMATIONS ARE ALLOWED?

We want to preserve

$$ds^2 = (cdt)^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

which implies

$$\eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx'^\mu dx'^\nu$$

We then have the constraint for $x^\mu = f^\mu(x^\nu)$, $\det(\partial f^\mu / \partial x^\nu) = \pm 1$ and

$$\eta_{\mu\nu} \frac{\partial f^\nu}{\partial x^\rho} \frac{\partial^2 f^\mu}{\partial x^\sigma \partial x^\alpha} = 0$$

This implies all second derivatives vanish. Thus,

$$f^\mu(x^\nu) = L^\mu_\nu x^\nu + a^\nu$$

with $\det(L) = \pm 1$

SUBGROUPS & COSETS

In general, we expect spacetime to be invariant under boosts, rotations, space translations, and time translations. This leads to a total of 10 independent parameters known as the Poincaré group. If we only consider the inhomogenous transformations, we are left with the 6 parameter Lorentz group with four disconnected cosets.

Coset	Rep.		
L	$\text{diag}(1,1,1,1)$	$\det = 1$	$\Lambda_{00} \geq 1$
PL	$\text{diag}(1,-1,-1,-1)$	$\det = -1$	$\Lambda_{00} \geq 1$
TL	$\text{diag}(-1,1,1,1)$	$\det = -1$	$\Lambda_{00} \leq -1$
PTL	$\text{diag}(-1,-1,-1,-1)$	$\det = 1$	$\Lambda_{00} \leq -1$

The coset containing the identity is a subgroup called the proper Lorentz group.

COMPACTNESS

The proper Lorentz group is a non-compact Lie group. To see this we note that $\Lambda_{00} \geq 1$ implies that this coefficient is undbounded in Euclidean space. Thus the Lie group is not compact.

Theorem: A faithful finite dimensional representation of a non-compact Lie group cannot be unitary.

The unbounded parameter boost parameter β. We can use the relationship

$$\tanh \beta = \frac{v}{c}$$

to express this now as a bound parameter, but it is still not closed in Euclidean space

FUNDAMENTAL REPRESENTATION

To find the generators we start by looking for elements close to the identity

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$$

We now search for generators such that

$$x'^\mu = (1 - \frac{1}{2} i\omega^{\lambda\rho} J_{\lambda\rho})^{\mu\nu} x_\nu$$

which yields

$$(J_{\lambda\rho})^\mu{}_\nu = i(\delta^\mu_\lambda g_{\rho\nu} - g_{\lambda\nu}\delta^\mu_\rho)$$

FUNDAMENTAL REPRESENTATION

$$J_{01} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\exp(-i\beta J_{01}) = \begin{pmatrix} \cosh\beta & -\sinh\beta & 0 & 0 \\ -\sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\exp(-i\theta J_{23}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}$$

ALGEBRA

$$[J_{\mu\nu}, J_{\lambda\rho}] = -i\left(g_{\mu\lambda}J_{\nu\rho} - g_{\mu\rho}J_{\nu\lambda} - g_{\nu\lambda}J_{\mu\rho} + g_{\nu\rho}J_{\mu\lambda}\right)$$

$$\mathscr{J}_1 = J_{23} \qquad \mathscr{J}_2 = J_{31} \qquad \mathscr{J}_3 = J_{12}$$

$$\mathscr{K}_1 = J_{01} \qquad \mathscr{K}_2 = J_{02} \qquad \mathscr{K}_3 = J_{03}$$

$$[\mathscr{J}_i, \mathscr{J}_j] = i\epsilon_{ijk}\mathscr{J}_k \qquad [\mathscr{K}_i, \mathscr{K}_j] = -i\epsilon_{ijk}\mathscr{J}_k \qquad [\mathscr{J}_i, \mathscr{K}_j] = i\epsilon_{ijk}\mathscr{K}_k$$

↑ $SO(3)$ subalgebra

ALGEBRA

$$\mathscr{N}_i^{\pm} = \frac{1}{2}(\mathscr{J}_i \pm i\mathscr{K}_i)$$

$$[\mathscr{N}_i^{+}, \mathscr{N}_j^{+}] = i\epsilon_{ijk}\mathscr{N}_k^{+} \qquad [\mathscr{N}_i^{-}, \mathscr{N}_j^{-}] = i\epsilon_{ijk}\mathscr{N}_k^{-} \qquad [\mathscr{N}_i^{+}, \mathscr{N}_j^{-}] = 0$$

$SU(2)$ subalgebra $\qquad$ $SU(2)$ subalgebra

Though the proper Lorentz group shares the same algebra as $SU(2) \times SU(2)$, they are not the same group. Locally, the Lorentz group is isomorphic to $SL(2, \mathbb{C})$.

REPRESENTATION THEORY

We can extend the enumeration on representations of $SU(2)$ to the Lorentz group. As we would label a representation of $SU(2)$, j, with it's $(2j+1)$ degrees of freedom, we can label a representation of the Lorentz group (j_1, j_2) by the corresponding representations of j_1 and j_2 in $SU(2)$.

Useful representations:

- $(0,0)$ the trivial representation
- $(\frac{1}{2},0)$ the Weyl representation used to describe neutrinos
- $(\frac{1}{2},0)\bigoplus(0,\frac{1}{2})$ Dirac representation used to describe charged leptons
- $(\frac{1}{2},\frac{1}{2})$ the defining representation of $SO(3,1)$
- $(1,0)\bigoplus(0,1)$ tensor representation used to describe electromagnetism

REPRESENTATION THEORY

Theorem: The complex conjugate of the (j_1, j_2) representations of the Lorentz algebra is the (j_2, j_1) representation.

Representations of the form (j_1, j_1) and $(j_1, j_2) \bigoplus (j_2, j_1)$ are real

EXTENDING THE ALGEBRA

In the differential representation

$$J_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

The algebra can be extended to include the generators for translations

$$P_\mu = -i\partial_\mu$$

$$[J_{\mu\nu}, J_{\lambda\rho}] = -i\left(g_{\mu\lambda}J_{\nu\rho} - g_{\mu\rho}J_{\nu\lambda} - g_{\nu\lambda}J_{\mu\rho} + g_{\nu\rho}J_{\mu\lambda}\right)$$

$$[J_{\mu\nu}, P_\rho] = -i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu)$$

EXTENDING THE ALGEBRA

Further, we can add dilations

$$x'^{\mu} = e^{\lambda} x^{\mu}$$

Their generator

$$D = -x^{\mu} \partial_{\mu}$$

$$[D, P_{\mu}] = i P_{\mu}$$

EXTENDING THE ALGEBRA

Additionally, we can include special conformal transformations of the form

$$x'^{\mu} = \frac{x^{\mu} - x^2 b^{\mu}}{1 - 2b \cdot x + x^2 b^2}$$

The corresponding generators

$$K_{\mu} = -i(x^2 \partial_{\mu} - 2x_{\mu} x^{\nu} \partial_{\nu})$$

$$[J_{\mu\nu}, K_{\rho}] = -i(g_{\nu\rho} K_{\mu} - g_{\mu\rho} K_{\nu})$$

$$[P_{\mu}, K_{\nu}] = -2i(J_{\mu\nu} - g_{\mu\nu} D)$$

$$[D, K_{\mu}] = -iK_{\mu}$$

CONFORMAL ALGEBRA

Together these new generators create the conformal algbra. We can extend these arguments to d dimensional Minkowski space and index our new generators with indices ranging from -1 to d

$$J_{MN} = \begin{cases} J_{\mu\nu}, & \mu, \nu = 0, ..., d-1 \\ J_{-1,d} = D \\ J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu), \mu = 0, ..., d-1 \\ J_{d,\mu} = \frac{1}{2}(P_\mu - K_\mu), \mu = 0, ..., d-1 \end{cases}$$

$$[J_{MN}, J_{PQ}] = -i\left(g_{MP}J_{NQ} - g_{MQ}J_{NP} - g_{NP}J_{MQ} + g_{NQ}J_{MP}\right)$$

Isomorphic to $\mathfrak{so}(2, d)$

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