Spontaneous Symmetry Breaking in $O(n)$ and $SU(n)$

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Physics 251: Group Theory

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Physics
UC Santa Cruz

June 16, 2023

Introduction

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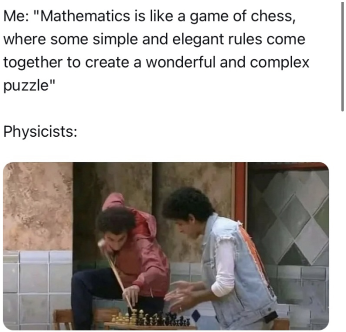
Basic outline of the topics that will be presented:

- $O(n)$
- $SU(n)$
- Explicit Examples



$\mathbf{O(n)}$

- Characterize the generators and Lie algebra of the group
- Vector Representation
- 2nd-rank symmetric tensor

$O(n)$ has $\frac{1}{2}n(n-1)$ generators, and they can be represented by;

$$L_{ij} = X_i \frac{\partial}{\partial X_j} - X_j \frac{\partial}{\partial X_i}, \; i, j = 1, ..., n.$$

The commutation relation follows as,

$$[L_{ij}, L_{kj}] = \delta_{jk} L_{il} + \delta_{il} L_{jk} - \delta_{ik} L_{jl} - \delta_{jl} L_{ik}.$$

$\mathbf{O(n)}$: Vector Representation

$O(n)$: Vector Representation Continued...

The transformation law is given by,

$$\phi_i \to \phi_i + \epsilon_{ij}\phi_j,\ \ \epsilon_{ij} = -\epsilon_{ji}$$

The covariant derivative $D_\mu \phi$ is defined as,

$$\partial_\mu \phi_i - g W^\mu_{ij} \phi_j,$$

where W^μ_{ij} is the vector gauge boson. From the transformation law, we can write down a generic invariant potential (μ, λ are real and $\lambda > 0$),

$$V(\phi) = -\frac{1}{2}\mu^2 \phi_i \phi_i + \frac{1}{4}\lambda \left(\phi_i \phi_i\right)^2 .$$

The minimum of the potential is given by,

$$\frac{\partial V}{\partial \phi_i} = (-\mu^2 + \lambda\phi_j\phi_j)\phi_i = 0,\ i = 1, ..., n.$$

The solution is given by $\phi_j\phi_j = \mu^2/\lambda$. We are free to pick the components of the vector, so we can choose a representation where there are $n-1$ zero components, and 1 non-zero component, i.e., $\phi = (0, 0, ..., \mu^2/\lambda)$. There is then a subgroup that maintains the $O(n)$ type symmetry, but there are $n-1$ components, so the subgroup has a symmetry of $O(n-1)$.

Note that this symmetry breaking is ultimately because the invariant potential depends only on the magnitude of the vector, which is why we can pick any component representation that we wish. Suppose that we had two vector representations in our theory, then the potential can only depend one $\phi_i^i\phi_1^i$, $\phi_2^j\phi_2^j$, and $\phi_i^i\phi_1^i\phi_2^j\phi_2^j$, i.e., $|\vec{\phi}_1|$, $|\vec{\phi}_2|$, and $|\vec{\phi}_1 \cdot \vec{\phi}_2|$. If we choose the first vector in the above way, and we choose the second vector to have two non-zero components, this will satisfy the above allowed terms, which means that the symmetry will be reduced from $O(n)$ to $O(n-2)$.

$\rightarrow$ Thus we can easily break the symmetry to any order we wish.

$$\underline{\mathbf{O(n):\ 2nd-rank\ Symmetric\ Tensor\ Representation}}$$

$O(n)$: 2nd-rank Symmetric Tensor Representation

The infinitesimal transformation law is given by,

$$\phi_{ij} \to \phi_{ij} + \epsilon_{ik}\phi_{kj} + \epsilon_{jk}\phi_{ik},\ \epsilon_{ij} = -\epsilon_{ji}$$

The covariant derivative $D_\mu\phi$ is defined as,

$$\partial_\mu\phi_{ij} - gW^\mu_{ik}\phi_{kj} - gW^\mu_{jk}\phi_{ik},$$

From the transformation law, we can write down a generic invariant potential ($\mu, \lambda_1, \lambda_2$ are real), with $\phi_{ij} = \phi_{ji}, Tr(\phi) = 0$,

$$V(\phi) = -\frac{1}{2}\mu^2\phi_{ij}\phi_{ij} + \frac{1}{4}\lambda_1\left(\phi_{ij}\phi_{ji}\right)^2 + \frac{1}{4}\lambda_2\left(\phi_{ij}\phi_{jk}\phi_{kl}\phi_{li}\right),$$

Following Ling-Fong Li, the condition for the minimum of the potential is given by:

$$\frac{\partial V}{\partial \phi_i} = -\mu^2 \phi_i + \lambda_1 \left(\sum_{j=1}^{n} \phi_j^2 \right) \phi_i + \phi_2 \phi_i^3 - g = 0, \; i = 1, ..., n$$

with $\sum_i \phi_i = 0$, where g is a Lagrange multiplier. While it is possible to solve the above equation (See Ling-Fong Li appendix B), for our purposes it is not necessary as we can simply be content knowing that a third order polynomial has at most 3 solutions, ϕ_1, ϕ_2, ϕ_3 which can be written in the form

$$\phi = \begin{pmatrix} \phi_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \phi_1 & & & & & & \\ & & & \phi_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \phi_2 & & & \\ & & & & & & \phi_3 & & \\ & & & & & & & \ddots & \\ & & & & & & & & \phi_3 \end{pmatrix}$$

There are $n_1, n_2,$ and n_3 of each ϕ_i respectively, under the constraint that $n_1 + n_2 + n_3 = n$. Thus, in general, we can say that the symmetry is broken along $O(n) \to O(n_1) \times O(n_2) \times O(n_3)$.

However, doing the calculation much more carefully, there are two cases to consider:

$\underline{\lambda_1 > 0,\ \lambda_2 > 0:}$

$O(n) \to O(n_1) \times O(n - n_1)$, where $n_1 = \frac{1}{2}n$ if n is even or $n_1 = \frac{1}{2}(n+1)$ if n is odd.

$\underline{\lambda_1 > 0,\ \lambda_2 < 0:}$

$O(n) \to O(n-1)$

$\mathbf{SU(n)}$

- Characterize the generators and the Lie algebra of the group
- Vector Representation
- 2nd-rank Symmetric Tensor
- Toy Model with n = 3

$SU(n)$ has $n^2 - 1$ generators that obey the relation $U^i_j = \left(U^j_i\right)^\dagger$, and have a commutation relation of the form:

$$\left[U^j_i, U^l_k\right] = \delta^k_j U^l_i - \delta^l_i U^j_k$$

$\mathbf{SU(n)}$: Vector Representation

The infinitesimal transformation law is given by,

$$\psi_i \to \psi_i + i\epsilon_i^j \psi_j,\ \psi^i = (\psi_i)^*,\ \epsilon_i^j = \left(\epsilon_i^j\right)^*$$

The covariant derivative $D_\mu \psi$ is defined as,

$$\partial_\mu \psi_i - igW_{\mu i}^j \psi_j,$$

where $W_{\mu i}^j$ is the vector gauge boson. From the transformation law, we can write down a generic invariant potential (μ, λ are real),

$$V(\psi) = \frac{1}{2}\mu^2 \psi_i \psi^i + \frac{1}{4}\lambda \left(\psi_i \psi^i\right)^2.$$

We can find the minimum of this potential,

$$\frac{\partial V}{\partial \psi_i} = \left(-\mu^2 + \lambda \ \psi_k \psi^k\right) \psi_i = 0, \ i = 1, ..., n$$

Solving for the new minimum we get, $\psi_k \psi^k = \mu^2/\lambda$. We are free to pick the components of the vector, so we can choose a representation where there are $n-1$ zero components, and 1 non-zero component, i.e., $\psi = (0, 0, ..., \mu^2/\lambda)$. There is then a subgroup that maintains the $SU(n)$ type symmetry, but there are $n-1$ components, so the subgroup has a symmetry of $SU(n-1)$.

$SU(n)$: Vector Representation Continued...

We may then ask ourselves how many massive gauge bosons will there be due to the symmetry breaking of $SU(n) \to SU(n-1)$? Consider some group G with subgroup H. The dimension of the quotient group, G/H, will tell us exactly this property $(\dim(G/H) = \dim(G) - \dim(H))$. Let $G = SU(n)$ and $H = SU(n-1)$:

$$\dim(SU(n)/SU(n-1)) = \dim(SU(n)) - \dim(SU(n-1))$$
$$\dim(SU(n)/SU(n-1)) = n^2 - 1 - \left(n^2 - 1\right)^2 + 1 = 2n - 1$$

Thus, there will be $2n-1$ massive gauge bosons after the breaking from $SU(n)$ to $SU(n-1)$, which tells us how many generators of the original symmetry are broken.

A quick example:
Consider $SU(3) \rightarrow SU(2)$, there will be $2 \cdot 3 - 1 = 5$ massive bosons, 3 generators will be unbroken: $SU(2)$ symmetry.

Going forward, suppose that we had two vectors in our potential, as in the $O(n)$ case we would break the symmetry from $SU(n)$ to $SU(n-2)$. Meaning that to fully break the symmetry we would need $n-1$ vector representations.

$$\underline{\mathbf{SU(n): \ 2nd - rank \ Symmetric \ Tensor \ Representation}}$$

The infinitesimal transformation law is given by,

$$\psi_{ij} \to \psi_{ij} + i\epsilon_i^k \psi_{kj} + i\epsilon_j^k \psi_{ik},\ \psi_{ij} = \left(\psi^{ij}\right)^*,\ \psi_{ij} = \psi_{ji}$$

The covariant derivative $D_\mu \psi$ is defined as,

$$\partial_\mu \psi_{ij} - igW_{\mu i}^l \psi_{lj} - igW_{\mu j}^l \psi_{il},$$

where $W_{\mu i}^j$ is the vector gauge boson. From the transformation law, we can write down a generic invariant potential ($\mu, \lambda_1, \lambda_2$ are real as before),

$$V(\psi) = \frac{1}{2}\mu^2 \psi_{ij}\psi^{ij} + \frac{1}{4}\lambda_1 \left(\psi_{ij}\psi^{ij}\right)^2 + \frac{1}{4}\lambda_2 \left(\psi_{ij}\psi^{jk}\psi_{kl}\psi^{li}\right).$$

We can find the minimum of this potential,

$$\frac{\partial V}{\partial \psi_{ij}} = -\mu^2 \psi^{ij} + \frac{1}{2}\lambda_1 \left(\psi_{lm}\psi^{lm}\right)\psi_{ij} + \frac{1}{2}\lambda_2 \left(\psi^{jk}\psi_{kl}\psi^{li}\right) = 0,$$

where $i, j = 1, ..., n$. Following Ling-Fong Li, we define the Hermitian matrix X defined such that $X_l^k \equiv \psi_{lm}\psi^{mk}$. This allows us to write the above minimum condition as:

$$-\mu^2\psi^{ij} + \lambda_1(X_l^l)\psi^{ij} + \lambda_2(X_l^j)\psi^{li} = 0.$$

Because X is Hermitian, we can always diagonalized via a unitary transformation so we can write,

$$\left[-\mu^2 + \lambda_1 \sum_{k=1}^{n} X_k + \lambda_2 X_i\right] \psi^{ij} = 0,\ j = 1, ..., n$$

For $\lambda_2 > 0$:

The solution to this equation is given by $X = \alpha^2 I_{n\times n}$, where $\alpha^2 = \mu^2/(\lambda_1 n + \lambda_2)$. From Appendix C of Ling-Fong Li, it can be shown that the form of ψ is given by $\psi_{ij} = \alpha\delta_{ij}$, i.e., $\psi = \alpha I_{n\times n}$. But what specific symmetry is this? It is $O(n)$ because $\psi \to U^T\psi U = U^T U\psi = \psi$ if U is orthogonal. So in the case of the $\lambda_2 > 0$, the symmetry breaks from $SU(n) \to O(n)$.

For $\lambda_2 < 0$:
The solution for X is given by,

$$X = d^2 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix},\ d^2 = \frac{\mu^2}{\lambda_1 + \lambda_2},\ \lambda_1 + \lambda_2 > 0$$

which gives ψ to be $\psi = d \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$.

Mimicking the vector representation, this is clearly an explicit $SU(n-1)$ symmetry. So in this case, the symmetry breaks from $SU(n) \to SU(n-1)$.

$\underline{\mathbf{SU(n): \ Toy\ Model\ of\ SU(3) \to SU(2)}}$

Consider the following Lagrangian for a complex scalar field in the fundamental representation of a global $SU(3)$ symmetry:

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \left(\phi^\dagger \phi - \frac{1}{2} v^2\right)^2 .$$

The potential is $V(\phi) = \left(|\phi|^2 - \frac{1}{2}v^2\right)^2$. The minimum of the potential gives a vev of $\phi = \frac{1}{\sqrt{2}}(0, 0, v)^T$. Clearly the first two component have $SU(2)$ symmetry and the last component has broken the $SU(3)$ into $SU(2)$. We know that from our simple calculation of the dimension of G/H, that there should be $2n - 1$ massive bosons in the end (5 in our case) and 3 generators will remain unbroken. But which ones are they?

Note that the generators of $SU(3)$ are the Gell-Mann matrices, λ^a for $a = 1, 2, ..., 8$. The unbroken generators are those that satisfy the following condition: $\lambda^a \langle\phi\rangle = (0, 0, 0)^T$. In our case, $a = 1, 2, 3$ satisfy this relation, explicitly these generators are:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\underline{\mathbf{SU(n): \ A \ more \ interesting \ Toy \ Model \ of \ SU(3) \to SU(2)}}$

Consider the following gauged version of the previous Lagrangian,

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - \left(\phi^\dagger \phi - \frac{1}{2} v^2\right)^2 - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a,$$

where the covariant derivative and the $F_{\mu\nu}$ tensor are defined: $D_\mu = \partial_\mu + igA^a_\mu \lambda_a$, and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc} A_{\mu b} A_{\mu c}$. Let us find the actual masses of the 5 massive particles that get created in the breaking of the $SU(3) \rightarrow SU(2)$ symmetry.

We already know the minimum of this potential from the previous example, so if we use it the only relevant terms of the Lagrangian are the covariant derivative and the potential. Plugging in our solution for ϕ we find,

$$\mathcal{L} \supset \frac{1}{2} g^2 A_\mu^a \lambda_a \begin{pmatrix} 0 & 0 & v \end{pmatrix} A_b^\mu \lambda^b \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = \frac{1}{2} g^2 A_\mu^a A_b^\mu \begin{pmatrix} 0 & 0 & v \end{pmatrix} \lambda_a \lambda^b \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

From the previous example we know that $a, b = 1, 2, 3$ get annihilated so that means we must consider only the values $a, b = 4, 5, 6, 7, 8$. These non-zero terms will be,

$$\mathcal{L} = \frac{1}{2} g^2 v^2 A_\mu^a A_b^\mu \lambda_a \lambda^b.$$

We can simplify this by using the identity,

$$\lambda_a \lambda_b = \frac{1}{2}\left([\lambda_a, \lambda_b] + \{\lambda_a, \lambda_b\}\right) = \frac{1}{2}\left(\frac{2}{3}\delta_{ab} + d^{abc}\lambda_c\right),$$

where d^{abc} take the values given below. Rewriting our Lagrangian we have, $\mathcal{L} = \frac{1}{2}g^2v^2A^a_\mu A^\mu_b \frac{1}{2}(\frac{2}{3}\delta_{ab} + d^{abc}\lambda_c)$,

$\mathcal{L} = \frac{1}{6}g^2v^2A^a_\mu A^\mu_a + \frac{1}{4}g^2v^2A^a_\mu A^\mu_b d^{abc}\lambda_c$.

The non-zero values of d^{abc} are as follows:

$d^{118} = d^{228} = d^{338} = -d^{888} = \frac{1}{\sqrt{3}}$
$d^{448} = d^{558} = d^{668} = d^{778} = -\frac{1}{2\sqrt{3}}$
$d^{146} = d^{157} = -d^{247} = d^{256} = \frac{1}{2}$
$d^{344} = d^{355} = -d^{366} = -d^{377} = \frac{1}{2}$

For $a = b \neq c$ we have $d^{448}, d^{558}, d^{668}, d^{778} = -\frac{1}{2\sqrt{3}}$ and for $a = b = c$ we have $-d^{888} = \frac{1}{\sqrt{3}}$, all other values are not relevant because they have $a < 4$ or $b < 4$.

$\underline{a = b \neq c}$:

$$\mathcal{L} = \left[\frac{1}{6}g^2v^2 - \frac{1}{4}g^2 \begin{pmatrix}0 & 0 & v\end{pmatrix} \frac{\lambda_8}{-2\sqrt{3}} \begin{pmatrix}0\\0\\v\end{pmatrix}\right] A^a_\mu A^\mu_a$$

where $\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -2\end{pmatrix}$, this gives us, $\mathcal{L} = \frac{1}{4}g^2v^2A^a_\mu A^\mu_a$, which we can read off the mass term directly, $m_A = \frac{1}{2}gv$ for $a = 4, 5, 6, 7$.

$\underline{a = b = c = 8}$:

$$\mathcal{L} = \left[\frac{1}{6}g^2v^2 - \frac{1}{4}g^2 \begin{pmatrix} 0 & 0 & v \end{pmatrix} \frac{\lambda_8}{-\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}\right] A^a_\mu A^\mu_a$$

this gives us, $\mathcal{L} = \frac{1}{3}g^2v^2A^a_\mu A^\mu_a$, which we can read off the mass term directly, $m_A = \frac{1}{\sqrt{3}}gv$ for $a = 8$.

So not only have we generated the 5 massive bosons, we also have a Lagrangian that generates different masses.

Thank you for your time and attention, there is obviously much more about symmetry breaking that I was not able to cover in this talk. Here are a few references I used heavily:

- Group theory of the spontaneously broken gauge symmetries - Ling-Fong Li (1973)
- Comment on "Group theory of the spontaneously broken gauge symmetries" - Victor Elias, Shalom Eliezer, and Arthur R. Swift (1975)
- An Introduction to Quantum Field Theory - Peskin & Schroeder
- Lecture Notes - Stefania Gori