Spinor and Spin Group

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SO(3) as an example

- SU(2) is the universal covering group of SO(3)

\[ \text{SO}(3) \simeq \text{SU}(2)/\mathbb{Z}_2 \]

- SU(2) transformation of 2 \times 2 matrices induces a rotation

\[ U \sigma_i U^{-1} = R_{ij} \sigma_j \Rightarrow \text{define } X = \vec{\sigma} \cdot \mathbf{x}, \quad X \rightarrow X' = \vec{\sigma} \cdot (\tilde{R} \mathbf{x}) \]

- Spinor is fundamental representation of SU(2) & a double valued rep of SO(3)
Spin group and Spinor

The properties above can be generalized.

- Spin group $\text{Spin}(n)$ is the universal covering group of $\text{SO}(n)$. It is also a double covering.
- Spinors are fundamental representations of $\text{Spin}(n)$.
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- Spin group $Spin(n)$ is the universal covering group of SO(n). It is also a double covering.
- Spinors are fundamental representations of $Spin(n)$.

We will be focusing on one example of spinor — Lorentz spinor.
\( Spin(1, 3) \cong SL(2, \mathbb{C}) \)

Extend Lorentz transformation to complex plane \( L(R) \to L(C) \)

- \( L(R) \): \( x \cdot x = (x^0)^2 - \vec{x} \cdot \vec{x}, \ x \in R^4 \)
- \( L(C) \): \( z \cdot z = (z^0)^2 - \vec{z} \cdot \vec{z}, \ z \in C^4 \)

Similar to before, define \( Z \equiv z^\mu \cdot \sigma_\mu \), where \( \sigma_\mu = (1, \vec{\sigma}) \). Then, \( det(Z) = z \cdot z \)

Now suppose we have a matrix transformation:

\[ Z' = AZB^T, \]

where \( A \) and \( B \) are complex \( 2 \times 2 \) matrices. If such transformation preserve the determinant of \( Z \), then this induce a Lorentz transformation \( L(C) \) on \( z \).
\[ \text{Spin}(1, 3) \cong SL(2, \mathbb{C}) \]

\[ \det(Z') = \det(Z) \implies \det(A)\det(B) = 1 \]

However, the pair of matrices \((A, B)\) is not unique, for example, \((cA, B/c)\) with \(c\) to be any complex number, represents the same matrix transformation. Let's choose \(c\) such that

\[ \det(cA) = \det(B/c) = 1 \]

This does not eliminate the choice of sign for \((A, B)\). Now we have a pair of \(2 \times 2\) matrices both with determinant 1, so such matrix transformation form the group \(SL(2, \mathbb{C}) \times SL(2, \mathbb{C})\).
\( \text{Spin}(1, 3) \cong SL(2, C) \)

\( SL(2, C) \times SL(2, C) \) induce a Lorentz transformation \( L(C) \)

\[ A\sigma_\nu B^T = \sigma_\mu \Lambda^\mu_\nu (A, B). \]

Now let's prove the 2 to 1 correspondence of \( SL(2, C) \times SL(2, C) \rightarrow L(C) \).

Suppose exist two pairs of matrices \((A, B)\) and \((A', B')\) such that

\[ A\sigma_\mu B^T = A'\sigma_\mu B'^T. \]

Then we have:

\[ \sigma_\mu = A^{-1}A'\sigma_\mu B'^T(B^T)^{-1} \equiv C\sigma_\mu D^T \quad (1) \]
Spin(1,3) ≃ SL(2, C)

Now, if we have $C = \lambda_1 I$, $D = \lambda_2 I$, then since $A$, $B$, $A'$, $B'$ are determinant 1, we have $\lambda_1 = \pm 1$, $\lambda_2 = \pm 1$. But in order to satisfy equation (1), we must have $\lambda_1 \lambda_2 = +1$, so $\lambda_1 = \lambda_2 = \pm 1$. Thus we have a 2 to 1 homeomorphism.

Let's introduce a lemma to show $C = \lambda_1 I$, $D = \lambda_2 I$.

- Lemma: $\tilde{\sigma}_\mu M \sigma^\mu = 2 Tr(M) I$

Times $\tilde{\sigma}_\mu$ on the left side of equation (1), we have

\[
\tilde{\sigma}_\mu \sigma^\mu = \tilde{\sigma}_\mu C \sigma^\mu D^T.
\]

Use the lemma above,

\[
4 I = 2 Tr(C) D^T \Rightarrow D = \lambda I.
\]
\[ \text{Spin}(1, 3) \simeq \text{SL}(2, \mathbb{C}) \]

Similarly we can show \( C = \lambda_1 I \). And with the argument in last slides, we have:

\[
\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \to L(\mathbb{C})
\]

is a \( 2 \to 1 \) homeomorphism.
Spin(1, 3) \simeq SL(2, C)

Since we are clear with \( SL(2, C) \times SL(2, C) \to L(C) \), let’s get back to \( L(R) \). For a pair \((A, B)\) to give real Lorentz transformation, we need to have

\[
(AXB^T)^\dagger = AXB^T
\]

for any Hermitian \( X \). By this we have,

\[
X = A^{-1} B^* X A^\dagger (B^T)^{-1}.
\]

Use the lemma again, we have

\[
A^{-1} B^* = A^* B^{-1} = \pm I.
\]

So \((A, B) \to (A, \bar{A})\) or \((A, -\bar{A})\).

\[
SL(2, C) \to L_+^\uparrow
\]

is a \( 2 \to 1 \) homeomorphism.
We know that Lie algebra $sl(2, C)$ is actually the complexification of $su(2)$, so we can imagine that the representation of $sl(2, C)$ is a tensor product of representation of $su(2)$. Let’s introduce a theorem to help us figure out the representation.

- The irrep of $sl(2, C)$ are labeled by $(s_1, s_2)$ for $s_j = 0, \frac{1}{2}, 1, \cdots$, with dimension $(2s_1 + 1)(2s_2 + 1)$.

Use this theorem, we can introduce some representations that are of most physical interest.
Representation of $SL(2, C)$

- $(0, 0)$: The trivial representation on $C$, also called scalar representation
- $(\frac{1}{2}, 0)$: Often called left-handed Weyl spinors. The representation space is $C^2$
- $(0, \frac{1}{2})$: Often called right-handed Weyl spinors. Representation space is also $C^2$, this transform independently with $(\frac{1}{2}, 0)$
- $(\frac{1}{2}, \frac{1}{2})$: Called the ”vector” representation. Transforms as $X \rightarrow \Omega X \Omega^\dagger$, and this induce a Lorentz transformation on $R^4$

Through $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are transform independently under $L_\uparrow$, they transform into each other under parity transformation. So we usually use reducible the 4 complex dimensional representation $(\frac{1}{2}, 0) + (0, \frac{1}{2})$, which is known as ”Dirac spinors”.
Spin group $Spin(n)$ is the universal covering group of $SO(n)$. It is also a double covering.

Spinors are fundamental representations of $Spin(n)$.

$Spin(1, 3) \simeq SL(2, \mathbb{C}) \simeq SO(1, 3)/\mathbb{Z}_2$
Thank you for your attention!


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