Boson & Fermion Realisations of Lie Algebras

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Overview

- Refresh on Young diagrams
- Realisations of Lie algebras
- Bosonic Realisations & examples
- Fermionic Realisations & example

Young diagrams pt 1

- Lie algebras have irreps characterized by labels (quantum numbers)
- Semisimple Lie algebras: numbers of labels is equal to the rank
- For some tensor of rank t,

$$t = \lambda_1 + \ldots + \lambda_n \qquad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$$

- Different types for different Lie algebras (recall trace of O(n) tensor)

Young diagrams pt 2

- u(n): $[\lambda_1, \lambda_2, \ldots, \lambda_n]$

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0 \,.$$

- su(n) has (n-1) total labels
- Equivalence relations among these sets of labels

$$[\lambda_1, \lambda_2 \ldots, \lambda_n] \to [\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n, 0]$$

Subalgebra chains

- Characterizing the algebra with sets of quantum numbers,

$$\left| \begin{array}{ccccccc} g & \supset & g' & \supset & g" & \supset \dots \\ \downarrow & & \downarrow & & \downarrow & \\ [\lambda] & & [\lambda'] & & [\lambda''] & \end{array} \right\rangle$$

- Solved for some chains of algebras by Gel'fand and Cetlin, (canonical chains) e.g.

$$u(n) \supset u(n-1) \supset u(n-2) \supset \dots \supset u(1)$$

- For physics interest, often necessary to decompose Lie algebras into other algebras outside the canonical chain.

Subalgebra chains - Gel'fand pattern

- For the chain: $u(n) \supset u(n-1) \supset u(n-2) \supset \ldots \supset u(1)$
- Labels organised like so:

$$
\begin{array}{ccccccccccccc}
\lambda_{1,n} & & \lambda_{2,n} & & & & & & \lambda_{n-1,n} & & \lambda_{n,n} \\
& \lambda_{1,n-1} & & \cdots & & & \cdots & & & \lambda_{n-1,n-1} & \\
& & \cdots & & & & & \cdots & & & \\
& & & \lambda_{1,2} & & \lambda_{2,2} & & & & & \\
& & & & \lambda_{1,1} & & & & & &
\end{array}
$$

- Inequality relations on the labels, e.g.

$$\lambda_{1,2} \geq \lambda_{1,1} \geq \lambda_{2,2}$$

Realisations of Lie Algebras

- $E_{\alpha\beta} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ $\quad \alpha, \beta = 1, \dots, n \quad$ for $u(n)$

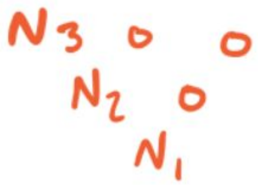
  $E_{\alpha\beta} = b_\alpha^\dagger b_\beta$ $\quad b_\alpha, b_\alpha^\dagger$ act on $|0\rangle$

  $b_\alpha |0\rangle = 0$

- Ado's theorem: any compact Lie algebra is a subalgebra of u(n)
  - Realise other algebras by taking correct combos of elements

Boson & Fermion Realisations

- Using bosonic/fermionic operators construct Lie algebras
- Only the totally (anti) symmetric parts



Action on the basis

u(2)

$$\left| \begin{matrix} \lambda_{1,2} & & \lambda_{2,2} \\ & \lambda_{1,1} & \end{matrix} \right\rangle$$

su(2)

$$\left| \begin{matrix} \lambda_{1,2} = 2J & & \lambda_{2,2} = 0 \\ & \lambda_{1,1} = M + J & \end{matrix} \right\rangle$$

$$E_{1,1} \mid J, M\rangle = (M + J) \mid J, M\rangle$$

$$E_{2,2} \mid J, M\rangle = (-M + J) \mid J, M\rangle$$

$$\frac{1}{2}\left(E_{1,1} - E_{2,2}\right) \mid J, M\rangle = M \mid J, M\ \rangle$$

Boson Realisations

$$E_{\alpha\beta} = b^\dagger_\alpha b_\beta$$

$b_\alpha, b^\dagger_\alpha$ act on $|0\rangle$

$$b_\alpha |0\rangle = 0$$

$$[b_\alpha, b^\dagger_{\alpha'}] = \delta_{\alpha\alpha'}$$

$$[b_\alpha, b_{\alpha'}] = [b^\dagger_\alpha, b^\dagger_{\alpha'}] = 0$$

Boson Realisations

$$\overbrace{\Box\,\Box\;\ldots\;\Box}^{N} \qquad [N,0,0,\ldots,0] = [N,\dot{0}] \equiv [N]$$

$\lambda_{1,n}$

$\lambda_{1,n-1}$

$\lambda_{1,n-2}$

$\ddots$

$$\overbrace{\Box\,\Box\ldots\Box}^{\lambda_{1,n}}$$

$$\underbrace{\Box\ldots\Box}_{\lambda_{1,n-1}}$$

$u(3)$

$$\begin{matrix} N_3 & & 0 & & 0 \\ & N_2 & & 0 & \\ & & N_1 & & \end{matrix}$$

$0 \leq N_1 \leq N_2$

$0 \leq N_2 \leq N_3$

Boson Realisations - u(1)

$[b^\dagger, b] = 1 \qquad [b, b] = [b^\dagger, b^\dagger] = 0$

$$\left| \begin{matrix} u(1) \\ \downarrow \\ N \end{matrix} \right\rangle$$

$E_{11} = b^\dagger b$

$b = \frac{1}{\sqrt{2}}(x + ip) \rightarrow H = b^\dagger b + \frac{1}{2} = \hat{N} + \frac{1}{2}$

$E = N + \frac{1}{2}$

Boson Realisations - u(2) containing u(1)

$b_1^\dagger b_1 , \quad b_2^\dagger b_2 , \quad b_1^\dagger b_2 , \quad b_2^\dagger b_1$

$$\left| \begin{array}{cc} u(2) \supset & u(1) \\ \downarrow & \downarrow \\ N & n_1 \end{array} \right\rangle$$

$$\left(\begin{array}{cc} N & 0 \\ & n_1 \end{array} \right)$$

$n_1 + n_2 = N$

$b_1^\dagger b_1 + b_2^\dagger b_2 = \hat{N}$

$H = \hat{N} + 1$

$E = N + 1$

Boson Realisations - u(2) containing so(2)

$$\hat{F}_- = b_2^\dagger b_1$$

$$\hat{F}_+ = b_1^\dagger b_2$$

$$\hat{F}_z = \frac{1}{2}(b_2^\dagger b_2 - b_1^\dagger b_1)$$

$$= \frac{1}{2}(\hat{N}_2 - \hat{N}_1)$$

su(2) : $[F_+, F_-] = 2F_z$

$[F_z, F_\pm] = \pm F_\pm$

so(2)

$\hookrightarrow M$

Boson Realisations - u(2) containing so(2)

trade $\frac{1}{2}(N_2 - N_1) = \frac{1}{2}N - N_1$

$$\left| \begin{matrix} u(2) & \supset & so(2) \\ \downarrow & & \downarrow \\ N & & M \end{matrix} \right\rangle \qquad F \equiv \frac{N}{2} \qquad F_z \equiv \frac{M}{2}$$

$$\left| \begin{matrix} su(2) & \supset & so(2) \\ \downarrow & & \downarrow \\ F & & F_z \end{matrix} \right\rangle \qquad F_z = -F, -F+1 \ldots F$$

Fermion realisations

- New operators as expected,

$$\left\{a_i,\, a^\dagger_{i'}\right\} = \delta_{ii'}; \quad \{a_i,\, a_{i'}\} = \left\{a^\dagger_i, a^\dagger_{i'}\right\} = 0$$

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$$\left.\begin{array}{c}\square\\ \square\\ \vdots\\ \square\end{array}\right\} N_F \qquad \{N_F\} \equiv \left[\overbrace{\underbrace{1, 1, \ldots, 1}_{N_F}, 0, .., 0}^{n}\right] .$$

Fermion realisations - u(2)

$\square$	$\begin{array}{c}\square\\\square\end{array}$	
$[1]$	$[1,1]$	
$N_F=1$	$N_F=2$	$u(2)$
$J=\frac{1}{2}$	$J=0$	$su(2)$

Fermion realisations - u(2)

[1] [1, 1]

$N_F = 1$ $N_F = 2$ $u(2)$

$J = \frac{1}{2}$ $J = 0$ $su(2)$

Classification of antisymmetric states of $u(2)$

$j = 1/2$	$u(2)$	$su(2)$
	[0]	0
	[1]	1/2
	[1, 1]	0

references

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- Sakurai, J. J., & Commins, E. D. (1995). Modern quantum mechanics, revised edition.