1. Properties of the Matrix Exponential

Let A be a real or complex $n \times n$ matrix. The exponential of A is defined via its Taylor series,

$$e^A = I + \sum_{n=1}^{\infty} \frac{A^n}{n!}\,, \tag{1}$$

where I is the $n \times n$ identity matrix. The radius of convergence of the above series is infinite. Consequently, eq. (1) converges for all matrices A. In these notes, we discuss a number of key results involving the matrix exponential and provide proofs of three important theorems. First, we consider some elementary properties.

Property 1: If $\big[A\,,\,B\big] \equiv AB - BA = 0$, then

$$e^{A+B} = e^A e^B = e^B e^A\,. \tag{2}$$

This result can be proved directly from the definition of the matrix exponential given by eq. (1). The details are left to the ambitious reader.

Remarkably, the converse of property 1 is FALSE. One counterexample is sufficient. Consider the 2×2 complex matrices

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 2\pi i \end{pmatrix}\,, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 2\pi i \end{pmatrix}\,. \tag{3}$$

An elementary calculation yields

$$e^A = e^B = e^{A+B} = I\,, \tag{4}$$

where I is the 2×2 identity matrix. Hence, eq. (2) is satisfied. Nevertheless, it is a simple matter to check that $AB \neq BA$, i.e., $[A\,,\,B] \neq 0$.

Indeed, one can use the above counterexample to construct a second counterexample that employs only real matrices. Here, we make use of the well known isomorphism between the complex numbers and real 2×2 matrices, which is given by the mapping

$$z = a + ib \quad \longmapsto \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\,. \tag{5}$$

It is straightforward to check that this isomorphism respects the multiplication law of two complex numbers. Using eq. (5), we can replace each complex number in eq. (3) with the corresponding real 2×2 matrix,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\pi \\ 0 & 0 & -2\pi & 0 \end{pmatrix}\,, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2\pi \\ 0 & 1 & -2\pi & 0 \end{pmatrix}\,.$$

One can again check that eq. (4) is satisfied, where I is now the 4×4 identity matrix, whereas $AB \neq BA$ as before.

It turns out that a small modification of Property 1 is sufficient to avoid any such counterexamples.

Property 2: If $e^{t(A+B)} = e^{tA}e^{tB} = e^{tB}e^{tA}$, where $t \in (a, b)$ (where $a < b$) lies within some open interval of the real line, then it follows that $[A\,,\,B] = 0$.

Property 3: If S is a non-singular matrix, then for any matrix A,

$$\exp\bigl\{SAS^{-1}\bigr\} = Se^{A}S^{-1}\,. \tag{6}$$

The above result can be derived simply by making use of the Taylor series definition [cf. eq. (1)] for the matrix exponential.

Property 4: For all complex $n \times n$ matrices A,

$$\lim_{m\to\infty}\left(I + \frac{A}{m}\right)^m = e^A\,.$$

Property 4 can be verified by employing the matrix logarithm, which is treated in Sections 4 and 5 of these notes.

Property 5: If $[A(t)\,,\,dA/dt] = 0$, then

$$\frac{d}{dt}e^{A(t)} = e^{A(t)}\frac{dA(t)}{dt} = \frac{dA(t)}{dt}e^{A(t)}\,.$$

This result is self evident since it replicates the well known result for ordinary (commuting) functions. Note that Theorem 2 below generalizes this result in the case of $[A(t)\,,\,dA/dt] \neq 0$.

Property 6: If $\bigl[A\,,\,[A\,,\,B]\bigr] = 0$, then $e^A\,B\,e^{-A} = B + [A\,,\,B]$.

To prove this result, we define

$$B(t) \equiv e^{tA}Be^{-tA}\,,$$

and compute

$$\frac{dB(t)}{dt} = Ae^{tA}Be^{-tA} - e^{tA}Be^{-tA}A = [A\,,\,B(t)]\,,$$

$$\frac{d^2B(t)}{dt^2} = A^2e^{tA}Be^{-tA} - 2Ae^{tA}Be^{-tA}A + e^{tA}Be^{-tA}A^2 = \bigl[A\,,\,[A\,,\,B(t)]\bigr]\,.$$

By assumption, $\bigl[A\,,\,[A\,,\,B]\bigr] = 0$, which must also be true if one replaces $A \to tA$ for any number t. Hence, it follows that $\bigl[A\,,\,[A\,,\,B(t)]\bigr] = 0$, and we can conclude that $d^2B(t)/dt^2 = 0$. It then follows that $B(t)$ is a linear function of t, which can be written as

$$B(t) = B(0) + t\left(\frac{dB(t)}{dt}\right)_{t=0}\,.$$

Noting that $B(0) = B$ and $(dB(t)/dt)_{t=0} = [A\,,\,B]$, we end up with

$$e^{tA}\,B\,e^{-tA} = B + t[A\,,\,B]\,. \tag{7}$$

By setting $t = 1$, we arrive at the desired result. If the double commutator does not vanish, then one obtains a more general result, which is presented in Theorem 1 below.

If $\big[A\,,\,B\big] \neq 0$, the $e^A e^B \neq e^{A+B}$. The general result is called the Baker-Campbell-Hausdorff formula, which will be proved in Theorem 4 below. Here, we shall prove a somewhat simpler version.

Property 7: If $\big[A\,,\,[A\,,\,B]\big] = \big[B\,,\,[A\,,\,B]\big] = 0$, then

$$e^A e^B = \exp\big\{A + B + \tfrac{1}{2}[A\,,\,B]\big\}\,. \tag{8}$$

To prove eq. (8), we define a function,

$$F(t) = e^{tA}e^{tB}\,.$$

We shall now derive a differential equation for $F(t)$. Taking the derivative of $F(t)$ with respect to t yields

$$\frac{dF}{dt} = Ae^{tA}e^{tB} + e^{tA}e^{tB}\;B = AF(t) + e^{tA}Be^{-tA}F(t) = \big\{A + B + t[A\,,\,B]\big\}F(t)\,, \tag{9}$$

after noting that B commutes with e^{Bt} and employing eq. (7). By assumption, both A and B, and hence their sum, commutes with $[A\,,\,B]$. Thus, in light of Property 5 above, it follows that the solution to eq. (9) is

$$F(t) = \exp\big\{t(A + B) + \tfrac{1}{2}t^2[A\,,\,B]\big\}F(0)\,.$$

Setting $t = 0$, we identify $F(0) = I$, where I is the identity matrix. Finally, setting $t = 1$ yields eq. (8).

Property 8: For any matrix A,

$$\det \exp A = \exp\big\{\operatorname{Tr} A\big\}\,. \tag{10}$$

If A is diagonalizable, then one can use Property 3, where S is chosen to diagonalize A. In this case, $D = SAS^{-1} = \operatorname{diag}(\lambda_1\,,\,\lambda_2\,,\,\ldots\,,\,\lambda_n)$, where the λ_i are the eigenvalues of A (allowing for degeneracies among the eigenvalues if present). It then follows that

$$\det e^A = \prod_i e^{\lambda_i} = e^{\lambda_1+\lambda_2+\ldots+\lambda_n} = \exp\big\{\operatorname{Tr} A\big\}\,.$$

However, not all matrices are diagonalizable. One can modify the above derivation by employing the Jordan canonical form. But, here I prefer another technique that is applicable to all matrices whether or not they are diagonalizable. The idea is to define a function

$$f(t) = \det e^{At}\,,$$

and then derive a differential equation for $f(t)$. If $|\delta t/t| \ll 1$, then

$$\det e^{A(t+\delta t)} = \det(e^{At}e^{A\delta t}) = \det e^{At} \det e^{A\delta t} = \det e^{At} \det(I + A\delta t)\,, \tag{11}$$

after expanding out $e^{A\delta t}$ to linear order in δt.

We now consider

$$\det(I + A\delta t) = \det \begin{pmatrix} 1 + A_{11}\delta t & A_{12}\delta t & \ldots & A_{1n}\delta t \\ A_{21}\delta t & 1 + A_{22}\delta t & \ldots & A_{2n}\delta t \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}\delta t & A_{n2}\delta t & \ldots & 1 + A_{nn}\delta \end{pmatrix}$$

$$= (1 + A_{11}\delta t)(1 + A_{22}\delta t)\cdots(1 + A_{nn}\delta t) + \mathcal{O}\big((\delta t)^2\big)$$

$$= 1 + \delta t(A_{11} + A_{22} + \cdots + A_{nn}) + \mathcal{O}\big((\delta t)^2\big) = 1 + \delta t\ \mathrm{Tr}\, A + \mathcal{O}\big((\delta t)^2\big)\,.$$

Inserting this result back into eq. (11) yields

$$\frac{\det e^{A(t+\delta t)} - \det e^{At}}{\delta t} = \mathrm{Tr}\, A\ \det e^{At} + \mathcal{O}(\delta t)\,.$$

Taking the limit as $\delta t \to 0$ yields the differential equation,

$$\frac{d}{dt} \det e^{At} = \mathrm{Tr}\, A\ \det e^{At}\,. \tag{12}$$

The solution to this equation is

$$\ln\, \det e^{At} = t\ \mathrm{Tr}\, A\,, \tag{13}$$

where the constant of integration has been determined by noting that $(\det e^{At})_{t=0} = \det I = 1$. Exponentiating eq. (13), we end up with

$$\det e^{At} = \exp\big\{t\ \mathrm{Tr}\, A\big\}\,.$$

Finally, setting $t = 1$ yields eq. (10).

Note that this last derivation holds for any matrix A (including matrices that are singular and/or are not diagonalizable).

Remark: For any invertible matrix function $A(t)$, Jacobi's formula is

$$\frac{d}{dt} \det A(t) = \det A(t)\ \mathrm{Tr}\left(A^{-1}(t)\frac{dA(t)}{dt}\right)\,. \tag{14}$$

Note that for $A(t) = e^{At}$, eq. (14) reduces to eq. (12) derived above. Another result related to eq. (14) is

$$\left(\frac{d}{dt}\det(A + tB)\right)_{t=0} = \det A\ \mathrm{Tr}(A^{-1}B)\,.$$

2. Five Important Theorems Involving the Matrix Exponential

The adjoint operator ad_A, which is a linear operator acting on the vector space of $n \times n$ matrices, is defined by

$$\mathrm{ad}_A(B) = [A, B] \equiv AB - BA\,. \tag{15}$$

Note that

$$(\mathrm{ad}_A)^n(B) = \underbrace{[A, \cdots [A, [A, B]] \cdots]}_{n} \tag{16}$$

involves n nested commutators.

<u>Theorem 1:</u>

$$e^A B e^{-A} = \exp(\mathrm{ad}_A)(B) \equiv \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{ad}_A)^n(B) = B + [A, B] + \tfrac{1}{2}\,[A, [A, B]] + \cdots\,. \tag{17}$$

Proof: Define

$$B(t) \equiv e^{tA} B e^{-tA}\,, \tag{18}$$

and compute the Taylor series of $B(t)$ around the point $t = 0$. A simple computation yields $B(0) = B$ and

$$\frac{dB(t)}{dt} = A e^{tA} B e^{-tA} - e^{tA} B e^{-tA} A = [A, B(t)] = \mathrm{ad}_A(B(t))\,. \tag{19}$$

Higher derivatives can also be computed. It is a simple exercise to show that:

$$\frac{d^n B(t)}{dt^n} = (\mathrm{ad}_A)^n(B(t))\,. \tag{20}$$

Theorem 1 then follows by substituting $t = 1$ in the resulting Taylor series expansion of $B(t)$.

We now introduce two auxiliary functions that are defined by their power series:

$$f(z) = \frac{e^z - 1}{z} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}\,, \qquad |z| < \infty\,, \tag{21}$$

$$g(z) = \frac{\ln z}{z - 1} = \sum_{n=0}^{\infty} \frac{(1-z)^n}{n+1}\,, \qquad |1 - z| < 1\,. \tag{22}$$

These functions satisfy:

$$f(\ln z)\, g(z) = 1\,, \qquad \text{for } |1 - z| < 1\,, \tag{23}$$

$$f(z)\, g(e^z) = 1\,, \qquad \text{for} |z| < \infty\,. \tag{24}$$

Theorem 2:

$$e^{A(t)}\frac{d}{dt}e^{-A(t)} = -f(\mathrm{ad}_A)\left(\frac{dA}{dt}\right)\,, \tag{25}$$

where $f(z)$ is defined via its Taylor series in eq. (21). Note that in general, $A(t)$ does not commute with dA/dt. A simple example, $A(t) = A + tB$ where A and B are independent of t and $[A, B] \neq 0$, illustrates this point. In the special case where $[A(t), dA/dt] = 0$, eq. (25) reduces to

$$e^{A(t)}\frac{d}{dt}e^{-A(t)} = -\frac{dA}{dt}\,, \qquad \text{if} \quad \left[A(t), \frac{dA}{dt}\right] = 0\,. \tag{26}$$

Proof: Define

$$B(s,t) \equiv e^{sA(t)}\frac{d}{dt}e^{-sA(t)}\,, \tag{27}$$

and compute the Taylor series of $B(s,t)$ around the point $s = 0$. It is straightforward to verify that $B(0,t) = 0$ and

$$\left.\frac{d^n B(s,t)}{ds^n}\right|_{s=0} = -(\mathrm{ad}_{A(t)})^{n-1}\left(\frac{dA}{dt}\right)\,, \tag{28}$$

for all positive integers n. Assembling the Taylor series for $B(s,t)$ and inserting $s = 1$ then yields Theorem 2. Note that if $[A(t), dA/dt] = 0$, then $(d^n B(s,t)/ds^n)_{s=0} = 0$ for all $n \geq 2$, and we recover the result of eq. (26).

There are two additional forms of Theorem 2, which we now state for completeness.

Theorem 2(a):

$$\frac{d}{dt}e^{A(t)} = e^{A(t)}\,\tilde{f}(\mathrm{ad}_A)\left(\frac{dA}{dt}\right)\,, \tag{29}$$

where $\tilde{f}(z)$ is defined via its Taylor series,

$$\tilde{f}(z) = \frac{1 - e^{-z}}{z} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\,z^n\,, \qquad |z| < \infty. \tag{30}$$

Eq. (29) is an immediate consequence of eq. (25) since,

$$e^{-A(t)}\frac{d}{dt}e^{A(t)} = \frac{dA}{dt} - \frac{1}{2!}\left[A, \frac{dA}{dt}\right] + \frac{1}{3!}\left[A, \left[A, \frac{dA}{dt}\right]\right] - \cdots = \tilde{f}(\mathrm{ad}_A)\left(\frac{dA}{dt}\right)\,.$$

Theorem 2(b):

$$\left(\frac{d}{dt}e^{A+tB}\right)_{t=0} = e^A\,\tilde{f}(\mathrm{ad}_A)(B)\,, \tag{31}$$

where $f(z)$ is defined via its Taylor series in eq. (30). Eq. (31) defines that Gâteau derivative of e^A (also called the directional derivative of e^A along the direction of B).[1]

[1]In the present application, the Gâteau derivative exists, is a linear function of B, and is continuous in A, in which case it coincides with the Fréchet derivative.

Corollary:

$$\exp(A+\epsilon B) = e^A\big[1+\epsilon\,\tilde{f}(\mathrm{ad}_A)(B)+\mathcal{O}(\epsilon^2)\big]\,. \tag{32}$$

Proof: Starting from Theorem 2(b), let us denote the right hand side of eq. (31) by

$$F(A,B) \equiv e^A\, f(\mathrm{ad}_A)(B)\,. \tag{33}$$

Then, using the definition of the derivative, it follows that

$$\left(\frac{d}{dt}e^{A+tB}\right)_{t=0} = \left(\lim_{\epsilon\to 0}\frac{e^{A+(t+\epsilon)B}-e^{A+tB}}{\epsilon}\right)_{t=0} = \lim_{\epsilon\to 0}\frac{e^{A+\epsilon B}-e^{A}}{\epsilon} = F(A,B)\,. \tag{34}$$

In particular, eq. (34) implies that,

$$e^{A+\epsilon B} = e^A + \epsilon F(A,B) + \mathcal{O}(\epsilon^2)\,. \tag{35}$$

Employing the definition of $F(A,B)$ yields eq. (32).

The relation between Theorems 2(a) and 2(b) can be seen more clearly as follows. The proof of Theorem 2(b) shows that it follows directly from Theorem 2(a). One can also show that Theorem 2(a) is a consequence of Theorem 2(b) as follows. Working consistently to first order in ϵ and employing eq. (35) in the final step,

$$\frac{d}{dt}e^{A(t)} = \lim_{\epsilon\to 0}\frac{e^{A(t+\epsilon)}-e^{A(t)}}{\epsilon} = \lim_{\epsilon\to 0}\frac{e^{A(t)+\epsilon A'(t)}-e^{A(t)}}{\epsilon} = F\big(A(t),A'(t)\big)\,, \tag{36}$$

where $A'(t)\equiv dA/dt$. Finally, multiplying eq. (36) by $e^{-A(t)}$ yields eq. (29). That is, eqs. (29) and (31) are equivalent forms of the same theorem.

Theorem 3:

$$\frac{d}{dt}e^{-A(t)} = -\int_0^1 e^{-sA}\,\frac{dA}{dt}\,e^{-(1-s)A}\,ds\,. \tag{37}$$

This integral representation is an alternative version of Theorem 2.

Proof: Consider

$$\begin{aligned}\frac{d}{ds}\left(e^{-sA}\,e^{-(1-s)B}\right) &= -Ae^{-sA}\,e^{-(1-s)B}+e^{-sA}\,e^{-(1-s)B}B\\ &= e^{-sA}(B-A)e^{-(1-s)B}\,.\end{aligned} \tag{38}$$

Integrate eq. (38) from $s=0$ to $s=1$.

$$\int_0^1 \frac{d}{ds}\left(e^{-sA}\,e^{-(1-s)B}\right) = e^{-sA}\,e^{-(1-s)B}\bigg|_0^1 = e^{-A}-e^{-B}\,. \tag{39}$$

Using eq. (38), it follows that:

$$e^{-A}-e^{-B} = \int_0^1 ds\, e^{-sA}(B-A)e^{-(1-s)B}\,. \tag{40}$$

In eq. (40), we can replace $B \longrightarrow A + hB$, where h is an infinitesimal quantity:

$$e^{-A} - e^{-(A+hB)} = h \int_0^1 ds\, e^{-sA} B e^{-(1-s)(A+hB)} \,. \tag{41}$$

Taking the limit as $h \to 0$,

$$\lim_{h\to 0} \frac{1}{h} \left[e^{-(A+hB)} - e^{-A} \right] = -\int_0^1 ds\, e^{-sA} B e^{-(1-s)A} \,. \tag{42}$$

Finally, we note that the definition of the derivative can be used to write:

$$\frac{d}{dt} e^{-A(t)} = \lim_{h\to 0} \frac{e^{-A(t+h)} - e^{-A(t)}}{h} \,. \tag{43}$$

Using

$$A(t+h) = A(t) + h\frac{dA}{dt} + \mathcal{O}(h^2) \,, \tag{44}$$

it follows that:

$$\frac{d}{dt} e^{-A(t)} = \lim_{h\to 0} \frac{\exp\left[-\left(A(t) + h\frac{dA}{dt} \right) \right] - \exp[-A(t)]}{h} \,. \tag{45}$$

Thus, we can use the result of eq. (42) with $B = dA/dt$ to obtain

$$\frac{d}{dt} e^{-A(t)} = -\int_0^1 e^{-sA}\, \frac{dA}{dt}\, e^{-(1-s)A}\, ds \,, \tag{46}$$

which is the result quoted in Theorem 3.

As in the case of Theorem 2, there are two additional forms of Theorem 3, which we now state for completeness.

Theorem 3(a):

$$\frac{d}{dt} e^{A(t)} = \int_0^1 e^{(1-s)A}\, \frac{dA}{dt}\, e^{sA}\, ds \,. \tag{47}$$

This follows immediately from eq. (37) by taking $A \to -A$ and $s \to 1 - s$. In light of eqs. (34) and (36), it follows that,

Theorem 3(b):

$$\left(\frac{d}{dt} e^{A+tB} \right)_{t=0} = \int_0^1 e^{(1-s)A}\, B\, e^{sA}\, ds \,. \tag{48}$$

Second proof of Theorem 2: One can now derive Theorem 2 directly from Theorem 3. Multiply eq. (37) by $e^{A(t)}$ to obtain:

$$e^{A(t)} \frac{d}{dt} e^{-A(t)} = -\int_0^1 e^{(1-s)A}\, \frac{dA}{dt}\, e^{-(1-s)A}\, ds \,. \tag{49}$$

Using Theorem 1 [see eq. (17)],

$$
\begin{aligned}
e^{A(t)}\frac{d}{dt}e^{-A(t)} &= -\int_0^1 \exp\left[\mathrm{ad}_{(1-s)A}\right]\left(\frac{dA}{dt}\right) ds \\
&= -\int_0^1 e^{(1-s)\mathrm{ad}_A}\left(\frac{dA}{dt}\right) ds\,. \qquad (50)
\end{aligned}
$$

Changing variables $s \longrightarrow 1-s$, it follows that:

$$
e^{A(t)}\frac{d}{dt}e^{-A(t)} = -\int_0^1 e^{s\,\mathrm{ad}_A}\left(\frac{dA}{dt}\right) ds\,. \qquad (51)
$$

The integral over s is trivial, and one finds:

$$
e^{A(t)}\frac{d}{dt}e^{-A(t)} = \frac{1-e^{\,\mathrm{ad}_A}}{\mathrm{ad}_A}\left(\frac{dA}{dt}\right) = -f(\mathrm{ad}_A)\left(\frac{dA}{dt}\right)\,, \qquad (52)
$$

which coincides with Theorem 2.

Likewise, starting from eq. (48) and making use of eq. (17), it follows that,

$$
\begin{aligned}
\left(\frac{d}{dt}\right)_{t=0} e^{A+tB} &= e^A\int_0^1 e^{-sA}\,B\,e^{sA}\,ds = e^A\int_0^1 ds\,\exp\big[\mathrm{ad}_{-sA}\big](B) \\
&= e^A\int_0^1 ds\,\exp\big[-s\,\mathrm{ad}_A\big](B) = e^A\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\mathrm{ad}_A)^n(B)\int_0^1 s^n\,ds \\
&= e^A\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}(\mathrm{ad}_A)^n(B) = e^A\tilde{f}(\mathrm{ad}_A)(B)\,, \qquad (53)
\end{aligned}
$$

which coincides with Theorem 2(b).

Theorem 4: The Baker-Campbell-Hausdorff (BCH) formula

$$
\ln\left(e^A e^B\right) = B + \int_0^1 g\left[\exp(t\,\mathrm{ad}_A)\exp(\mathrm{ad}_B)\right](A)\,dt\,, \qquad (54)
$$

where $g(z)$ is defined via its Taylor series in eq. (22). Since $g(z)$ is only defined for $|1-z|<1$, it follows that the BCH formula for $\ln\left(e^A e^B\right)$ converges provided that $\|e^A e^B - I\| < 1$, where I is the identity matrix and $\|\cdots\|$ is a suitably defined matrix norm. Expanding the BCH formula, using the Taylor series definition of $g(z)$, yields:

$$
e^A e^B = \exp\left(A + B + \tfrac{1}{2}[A,B] + \tfrac{1}{12}\,[A,[A,B]] + \tfrac{1}{12}\,[B,[B,A]] + \ldots\right)\,, \qquad (55)
$$

assuming that the resulting series is convergent. An example where the BCH series does not converge occurs for the following elements of SL(2,$\mathbb{R}$):

$$
M = \begin{pmatrix} -e^{-\lambda} & 0 \\ 0 & -e^{\lambda}\end{pmatrix} = \exp\left[\lambda\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\,\exp\left[\pi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]\,, \qquad (56)
$$

where λ is any nonzero real number. It is easy to prove[2] that no matrix C exists such that $M = \exp C$. Nevertheless, the BCH formula is guaranteed to converge in a neighborhood of the identity of any Lie group.

Two corollaries of the BCH formula are noteworthy.

Corollary 1: The Trotter Product formula

$$\lim_{k\to\infty} \left(e^{A/k}e^{B/k}\right)^k = e^{A+B}\,. \tag{57}$$

Corollary 2: The Commutator formula

$$\lim_{k\to\infty} \left(e^{A/k}e^{B/k}e^{-A/k}e^{-B/k}\right)^{k^2} = \exp\bigl([A,B]\bigr)\,. \tag{58}$$

The proofs of eqs. (57) and (58) are left as an exercise for the reader.

Proof of the BCH formula: Define

$$C(t) = \ln(e^{tA}e^{B})\,. \tag{59}$$

or equivalently,

$$e^{C(t)} = e^{tA}e^{B}\,. \tag{60}$$

Using Theorem 1, it follows that for any complex $n \times n$ matrix H,

$$\begin{aligned}\exp\left[\mathrm{ad}_{C(t)}\right](H) &= e^{C(t)}He^{-C(t)} = e^{tA}e^{B}He^{-tA}e^{-B}\\ &= e^{tA}\left[\exp(\mathrm{ad}_B)(H)\right]e^{-tA}\\ &= \exp(\mathrm{ad}_{tA})\exp(\mathrm{ad}_B)(H)\,.\end{aligned} \tag{61}$$

Hence, the following operator equation is valid:

$$\exp\left[\mathrm{ad}_{C(t)}\right] = \exp(t\,\mathrm{ad}_A)\exp(\mathrm{ad}_B)\,, \tag{62}$$

after noting that $\exp(\mathrm{ad}_{tA}) = \exp(t\,\mathrm{ad}_A)$. Next, we use Theorem 2 to write:

$$e^{C(t)}\frac{d}{dt}e^{-C(t)} = -f(\mathrm{ad}_{C(t)})\left(\frac{dC}{dt}\right)\,. \tag{63}$$

[2]The characteristic equation for any 2×2 matrix A is given by: $\lambda^2 - (\mathrm{Tr}\;A)\lambda + \det\;A = 0$. Hence, the eigenvalues of any 2×2 traceless matrix $A \in \mathfrak{sl}(2,\mathbb{R})$ [that is, A is an element of the Lie algebra of SL(2,$\mathbb{R}$)] are given by $\lambda_\pm = \pm(-\det\;\mathrm{A})^{1/2}$. Then,

$$\mathrm{Tr}\;e^A = \exp(\lambda_+) + \exp(\lambda_-) = \begin{cases} 2\cosh|\det\;A|^{1/2}\,, & \text{if } \det\;A \le 0\,,\\ 2\cos|\det\;A|^{1/2}\,, & \text{if } \det\;A > 0\,.\end{cases}$$

Thus, if $\det A \le 0$, then $\mathrm{Tr}\;e^A \ge 2$, and if $\det A > 0$, then $-2 \le \mathrm{Tr}\;e^A < 2$. It follows that for any $A \in \mathfrak{sl}(2,\mathbb{R})$, $\mathrm{Tr}\;e^A \ge -2$. For the matrix M defined in eq. (56), $\mathrm{Tr}\;M = -2\cosh\lambda < -2$ for any nonzero real λ. Hence, no matrix C exists such that $M = \exp C$.

However, we can compute the left-hand side of eq. (63) directly:

$$e^{C(t)}\frac{d}{dt}e^{-C(t)} = e^{tA}e^{B}\frac{d}{dt}e^{-B}e^{-tA} = e^{tA}\frac{d}{dt}e^{-tA} = -A\,, \tag{64}$$

since B is independent of t, and tA commutes with $\frac{d}{dt}(tA)$. Combining the results of eqs. (63) and (64),

$$A = f(\mathrm{ad}_{C(t)})\left(\frac{dC}{dt}\right)\,. \tag{65}$$

Multiplying both sides of eq. (65) by $g(\exp \mathrm{ad}_{C(t)})$ and using eq. (24) yields:

$$\frac{dC}{dt} = g(\exp \mathrm{ad}_{C(t)})(A)\,. \tag{66}$$

Employing the operator equation, eq. (62), one may rewrite eq. (66) as:

$$\frac{dC}{dt} = g(\exp(t\,\mathrm{ad}_A)\exp(\mathrm{ad}_B))(A)\,, \tag{67}$$

which is a differential equation for $C(t)$. Integrating from $t=0$ to $t=T$, one easily solves for C. The end result is

$$C(T) = B + \int_0^T g(\exp(t\,\mathrm{ad}_A)\exp(\mathrm{ad}_B))(A)\,dt\,, \tag{68}$$

where the constant of integration, B, has been obtained by setting $T=0$. Finally, setting $T=1$ in eq. (68) yields the BCH formula.

It is instructive to use eq. (54) to obtain the terms exhibited in eq. (55). In light of the series definition of $g(z)$ given in eq. (22), we need to compute

$$\begin{aligned} I - \exp(t\,\mathrm{ad}_A)\exp(\mathrm{ad}_B) &= I - (I + t\,\mathrm{ad}_A + \tfrac{1}{2}t^2\,\mathrm{ad}_A^2)(I + \mathrm{ad}_B + \tfrac{1}{2}\mathrm{ad}_B^2) \\ &= -\mathrm{ad}_B - t\,\mathrm{ad}_A - t\,\mathrm{ad}_A\,\mathrm{ad}_B - \tfrac{1}{2}\mathrm{ad}_B^2 - \tfrac{1}{2}t^2\,\mathrm{ad}_A^2\,, \end{aligned} \tag{69}$$

and

$$\big[I - \exp(t\,\mathrm{ad}_A)\exp(\mathrm{ad}_B)\big]^2 = \mathrm{ad}_B^2 + t\,\mathrm{ad}_A\mathrm{ad}_B + t\,\mathrm{ad}_B\mathrm{ad}_A + t^2\,\mathrm{ad}_A^2\,, \tag{70}$$

after dropping cubic terms and higher. Hence, using eq. (22),

$$g(\exp(t\,\mathrm{ad}_A)\exp(\mathrm{ad}_B)) = I - \tfrac{1}{2}\mathrm{ad}_B - \tfrac{1}{2}t\,\mathrm{ad}_A - \tfrac{1}{6}t\,\mathrm{ad}_A\,\mathrm{ad}_B + \tfrac{1}{3}t\,\mathrm{ad}_B\,\mathrm{ad}_A + \tfrac{1}{12}\mathrm{ad}_B^2 + \tfrac{1}{12}t^2\,\mathrm{ad}_A^2\,. \tag{71}$$

Noting that $\mathrm{ad}_A(A) = [A,A] = 0$, it follows that to cubic order,

$$\begin{aligned} B + \int_0^1 g(\exp(t\,\mathrm{ad}_A)\exp(\mathrm{ad}_B))(A)\,dt &= B + A - \tfrac{1}{2}[B,A] - \tfrac{1}{12}\big[A,[B,A]\big] + \tfrac{1}{12}\big[B,[B,A]\big] \\ &= A + B + \tfrac{1}{2}[A,B] + \tfrac{1}{12}\big[A,[A,B]\big] + \tfrac{1}{12}\big[B,[B,A]\big]\,, \end{aligned} \tag{72}$$

which confirms the result of eq. (55).

Theorem 5: The Zassenhaus formula

The Zassenhaus formula for matrix exponentials is sometimes referred to as the dual of the Baker-Campbell Hausdorff formula. It provides an expression for $\exp(A+B)$ as an infinite produce of matrix exponentials. It is convenient to insert a parameter t into the argument of the exponential. Then, the Zassenhaus formula is given by

$$\exp\bigl\{t(A+B)\bigr\} = e^{tA}e^{tB}\exp\bigl\{-\tfrac{1}{2}t^2\bigl[A,B\bigr]\bigr\}\exp\bigl\{\tfrac{1}{6}t^3\bigl(2\bigl[B,[A,B]\bigr]+\bigl[A,[A,B]\bigr]\bigr)\bigr\}\cdots\,, \tag{73}$$

where the exponents of higher order in t involve nested commutators.[3]

More explicitly,

$$e^{t(A+B)} = e^{tA}e^{tB}e^{t^2C_2}e^{t^3C_3}\cdots\,, \tag{74}$$

where the C_n are defined recursively as

$$C_2 = \frac{1}{2}\left[\frac{\partial^2}{\partial t^2}\left(e^{-tB}e^{-tA}e^{t(A+B)}\right)\right]_{t=0} = -\tfrac{1}{2}[A,B]\,, \tag{75}$$

$$C_3 = \frac{1}{3!}\left[\frac{\partial^3}{\partial t^3}\left(e^{-t^2C_2}e^{-tB}e^{-tA}e^{t(A+B)}\right)\right]_{t=0} = -\tfrac{1}{3}[A+2B,C_2]\,, \tag{76}$$

and in general

$$C_n = \frac{1}{n!}\left[\frac{\partial^n}{\partial t^n}\left(e^{-t^{n-1}C_{n-1}}\cdots e^{-t^2C_2}e^{-tB}e^{-tA}e^{t(A+B)}\right)\right]_{t=0}\,. \tag{77}$$

A proof of the Zassenhaus formula can be found in M. Suzuki, Commun. Math. Phys. **57**, 193 (1977).

We can now rederive eq. (32), which we repeat here for the reader's convenience.

Corollary:

$$\exp(A+\epsilon B) = e^A\bigl[1+\epsilon\,\tilde{f}(\mathrm{ad}_A)(B)+\mathcal{O}(\epsilon^2)\bigr]\,. \tag{78}$$

Proof: In eq. (73), replace $A \to A/t$ and $t \to \epsilon$. Then it follows immediately that

$$t^2C_2 = -\tfrac{1}{2}\,\epsilon\,[A,B]+\mathcal{O}(\epsilon^2)\,, \tag{79}$$

$$t^3C_3 = -\tfrac{1}{3}\,\epsilon\,[A,C_2]+\mathcal{O}(\epsilon^2) = \frac{1}{3!}\,\epsilon\,\bigl[A,[A,B]\bigr]+\mathcal{O}(\epsilon^2) = \frac{1}{3!}\,\epsilon\,\bigl(\mathrm{ad}_A\bigr)^2(B)+\mathcal{O}(\epsilon^2)\,, \tag{80}$$

and in general

$$t^{n+1}C_{n+1} = -\frac{1}{n+1}\,\epsilon\,[A,C_n]+\mathcal{O}(\epsilon^2) = \frac{(-1)^n}{(n+1)!}\,\epsilon\,\bigl(\mathrm{ad}_A\bigr)^n(B)+\mathcal{O}(\epsilon^2)\,. \tag{81}$$

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[3] An algorithm for deriving the expansions exhibited in eqs. (55) and (73) can be found in R.M. Wilcox, J. Math. Phys. **8**, 962 (1967).

Hence, eq. (73) yields,

$$\begin{aligned}\exp(A+\epsilon B) &= e^A e^{\epsilon B}\prod_{n=1}^{\infty}\exp\left\{\frac{(-1)^n}{(n+1)!}\,\epsilon\,(\mathrm{ad}_A)^n(B)+\mathcal{O}(\epsilon^2)\right\}\\ &= e^A\left[1+\epsilon B+\mathcal{O}(\epsilon^2)\right]\prod_{n=1}^{\infty}\left\{1+\frac{(-1)^n}{(n+1)!}\,\epsilon\,(\mathrm{ad}_A)^n(B)+\mathcal{O}(\epsilon^2)\right\}\\ &= e^A\left[1+\epsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}(\mathrm{ad}_A)^n(B)+\mathcal{O}(\epsilon^2)\right]\\ &= e^A\left[1+\epsilon\,\tilde{f}(\mathrm{ad}_A)(B)+\mathcal{O}(\epsilon^2)\right],\end{aligned}\tag{82}$$

after employing eq. (30). The proof is complete.

3. Properties of the Matrix Logarithm

The matrix logarithm should be an inverse function to the matrix exponential. However, in light of the fact that the complex logarithm is a multi-valued function, the concept of the matrix logarithm is not as straightforward as was the case of the matrix exponential. Let A be a complex $n \times n$ matrix with no real negative or zero eigenvalues. Then, there is a unique logarithm, denoted by $\ln A$, all of whose eigenvalues lie in the strip, $-\pi < \mathrm{Im}\, z < \pi$ of the complex z-plane. We refer to $\ln A$ as the principal logarithm of A, which is defined on the cut complex plane, where the cut runs from the origin along the negative real axis. If A is a real matrix (subject to the conditions just stated), then its principal logarithm is real.[4]

For an $n \times n$ complex matrix A, we can define $\ln A$ via its Taylor series expansion, under the assumption that the series converges. The matrix logarithms is then defined as,

$$\ln A = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{(A-I)^m}{m}\,,\tag{83}$$

whenever the series converges, where I is the $n \times n$ identity matrix. The series converges whenever $\|A - I\| < 1$, where $\|\cdots\|$ indicates a suitable matrix norm.[5] If the matrix A satisfies $(A-I)^m = 0$ for all integers $m > N$ (where N is some fixed positive integer), then $A - I$ is called nilpotent and A is called unipotent. If A is unipotent, then the series given by eq. (83) terminates, and $\ln A$ is well defined independently of the value of $\|A - I\|$. For later use, we also note that if $\|A - I\| < 1$, then $I - A$ is non-singular, and $(I - A)^{-1}$ can be expressed as an infinite geometric series,

$$(I-A)^{-1} = \sum_{m=0}^{\infty} A^m\,.\tag{84}$$

[4]For further details, see Sections 1.5–1.7 of Nicholas J. Higham, *Functions of Matrices* (Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008).

[5]One possible choice is the Hilbert-Schmidt norm, which is defined as $\|X\| = \left[\mathrm{Tr}(X^\dagger X)\right]^{1/2}$, where the positive square root is chosen.

One can also define the matrix logarithm by employing the Gregory series,[6]

$$\ln A = -2\sum_{m=1}^{\infty}\frac{1}{2m+1}\big[(I-A)(I+A)^{-1}\big]^{2m+1}\,, \tag{85}$$

which converges under the assumption that all eigenvalues of A possess a positive real part. In particular, eq. (85) converges for any Hermitian positive definite matrix A. Hence, the region of convergence of the series in eq. (85) is considerably larger than the corresponding region of convergence of eq. (83).

Before discussing a number of key results involving the matrix logarithm, we first consider some elementary properties.[7]

Property 1: For all A with $\|A-I\|<1$, $\exp(\ln A)=A$.

Property 2: For all A with $\|A\|<\ln 2$, $\ln(e^A)=A$.

Note that although $\|A\|<\ln 2$ implies that $\|e^A-I\|<1$, the converse is not necessarily true. This means that it is possible that $\ln(e^A)\neq A$ despite the fact that the series that defines $\ln(e^A)$ via eq. (83) converges. For example, if $A=2\pi iI$, then $e^A=e^{2\pi i}I=I$ and $e^A-I=0$, whereas $\|A\|=2\pi>\ln 2$. In this case, $\ln(e^A)=0\neq A$.

A slightly stronger version of property 2 states that for any $n\times n$ complex matrix, $\ln(e^A)=A$ if and only if $|\operatorname{Im}\lambda_i|<\pi$ for every eigenvalue λ_i of A.

One can extend the definition of the matrix logarithm given in eq. (83) by adopting the following integral definition.[8] If A is a complex $n\times n$ matrix with no real negative or zero eigenvalues,[9] then

$$\ln A = (A-I)\int_0^1\big[s(A-I)+I\big]^{-1}ds\,. \tag{86}$$

It is straightforward to check that if $\|A-I\|<1$, then one can expand the integrand of eq. (86) in a Taylor series in s [cf. eq. (84)]. Integrating over s term by term then yields eq. (83). Of course, eq. (86) applies to a much broader class of matrices, A.

Property 3: Employing the extended definition of the matrix logarithm given in eq. (86), if A is a complex $n\times n$ matrix with no real negative or zero eigenvalues, then $\exp(\ln A)=A$.

To prove Property 3, we define a matrix valued function f of a complex variable z,[10]

$$f(z)=z(A-I)\int_0^1\big[sz(A-I)+I\big]^{-1}ds\,.$$

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[6]See, e.g., Section 11.3 of Nicholas J. Higham, *Functions of Matrices*, op. cit.

[7]Proofs of some of these results can be found in Chapter 2.3 of Brian Hall, *Lie Groups, Lie Algebras, and Representations* (Second Edition), (Springer International Publishing, Cham, Switzerland, 2015). See also Nicholas J. Higham, *Functions of Matrices*, previously cited in footnote 4.

[8]See Chapter 11 of Nicholas J. Higham, *Functions of Matrices*, previously referenced in footnote 4.

[9]The absence of zero eigenvalues implies that A is an invertible matrix.

[10]Here, we follow Jacques Faraut, *Analysis on Lie Groups* (Cambridge University Press, Cambridge, UK, 2008), problem 9 on pp. 31–32.

It is straightforward to show that $f(z)$ is analytic in a complex neighborhood of the real interval between $z = 0$ and $z = 1$. In a neighborhood of the origin, one can verify by expanding in z and dropping terms of $\mathcal{O}(z^2)$ that

$$\exp f(z) = I + z(A - I)\,. \tag{87}$$

Using the analyticity of $f(z)$, we can insert $z = 1$ in eq. (87) to conclude that

$$\exp(\ln A) = \exp f(1) = A\,.$$

Property 4: If A is a complex $n \times n$ matrix with no real negative or zero eigenvalues and $|p| \leq 1$, then $\ln(A^p) = p \ln A$. In particular, $\ln(A^{-1}) = -\ln A$ and $\ln(A^{1/2}) = \frac{1}{2}\ln A$.

Property 5: If $A(t)$ is a complex $n \times n$ matrix with no real negative or zero eigenvalues that depends on a parameter t, and A commutes with dA/dt, then

$$\frac{d}{dt}\ln A(t) = A^{-1}\frac{dA}{dt} = \frac{dA}{dt}A^{-1}\,.$$

Property 6: If A is a complex $n \times n$ matrix with no real negative or zero eigenvalues and S is a non-singular matrix, then

$$\ln(SAS^{-1}) = S(\ln A)S^{-1}\,. \tag{88}$$

Property 7: Suppose that X and Y are complex $n\times n$ complex matrices such that $XY = YX$. Moreover, if $|\arg\lambda_j + \arg\mu_j| < \pi$, for every eigenvalue λ_j of X and the corresponding eigenvalue μ_j of Y, then $\ln(XY) = \ln X + \ln Y$.

Note that if X and Y do not commute, then the corresponding formula for $\ln(XY)$ is quite complicated. Indeed, if the matrices X and Y are sufficiently close to I, so that $\exp(\ln X) = X$ and $\ln(e^X) = X$ (and similarly for Y), then we can apply eq. (55) with $A = \ln X$ and $B = \ln Y$ to obtain,

$$\ln(XY) = \ln X + \ln Y + \tfrac{1}{2}\big[\ln X, \ln Y\big] + \cdots\,.$$

4. Important Theorems involving the Matrix Logarithm

Before considering the theorems of interest, we prove the following lemma.

Lemma: If B is a non-singular matrix that depends on a parameter t, then

$$\frac{d}{dt}B^{-1}(t) = -B^{-1}\frac{dB}{dt}B^{-1}\,. \tag{89}$$

Proof: eq. (89) is easily derived by taking the derivative of the equation $B^{-1}B = I$. It follows that

$$0 = \frac{d}{dt}(B^{-1}B) = \left(\frac{d}{dt}B^{-1}\right)B + B^{-1}\frac{dB}{dt}\,. \tag{90}$$

Multiplying on the right of eq. (90) by B^{-1} yields

$$\frac{d}{dt}B^{-1} + B^{-1}\frac{dB}{dt}B^{-1} = 0\,,$$

which immediately yields eq. (89).

A second form of eq. (89) employs the Gâteau derivative. In light of eqs. (34) and (36) it follows that,

$$\left(\frac{d}{dt}(A+tB)^{-1}\right)_{t=0} = -A^{-1}BA^{-1}\,.$$

Theorem 6:

$$\frac{d}{dt}\ln A(t) = \int_0^1 ds\,\big[sA + (1-s)I\big]^{-1}\frac{dA}{dt}\big[sA+(1-s)I\big]^{-1}\,. \tag{91}$$

Below, we provide two different proofs of Theorem 6.

Proof 1: Employing the integral representation of $\ln A$ given in eq. (86), it follows that

$$\frac{d}{dt}\ln A = \frac{dA}{dt}\int_0^1\big[s(A-I)+I\big]^{-1}ds + (A-I)\int_0^1\frac{d}{dt}\big[s(A-I)+I\big]^{-1}ds\,. \tag{92}$$

We now make use of eq. (89) to evaluate the integrand of the second integral on the right hand side of eq. (92), which yields

$$\frac{d}{dt}\ln A = \frac{dA}{dt}\int_0^1\big[s(A-I)+I\big]^{-1}ds - (A-I)\int_0^1\big[s(A-I)+I\big]^{-1}s\frac{dA}{dt}\big[s(A-I)+I\big]^{-1} \tag{93}$$

We can rewrite eq. (93) as follows,

$$\begin{aligned}\frac{d}{dt}\ln A = &\int_0^1\big[s(A-I)+I\big]\big[s(A-I)+I\big]^{-1}\frac{dA}{dt}\big[s(A-I)+I\big]^{-1}ds\\ &-\int_0^1 s(A-I)[s(A-I)+I\big]^{-1}\frac{dA}{dt}[s(A-I)+I\big]^{-1}\,,\end{aligned} \tag{94}$$

which simplifies to

$$\frac{d}{dt}\ln A = \int_0^1\big[s(A-I)+I\big]^{-1}\frac{dA}{dt}\big[s(A-I)+I\big]^{-1}ds\,.$$

Thus, we have established eq. (91).

Proof 2: Start with the following formula,

$$\ln(A+B) - \ln A = \int_0^\infty du \left\{ \bigl(A+uI\bigr)^{-1} - \bigl(A+B+uI\bigr)^{-1} \right\}, \tag{95}$$

Using the definition of the derivative,

$$\frac{d}{dt}\ln A(t) = \lim_{h\to 0} \frac{\ln(A(t+h) - \ln A(t)}{h} = \lim_{h\to 0} \frac{\ln\bigl[A(t) + hdA/dt + \mathcal{O}(h^2)\bigr] - \ln A(t)}{h}\,.$$

Denoting $B = hdA/dt$ and making use of eq. (95),

$$\frac{d}{dt}\ln A(t) = \lim_{h\to 0}\frac{1}{h}\int_0^\infty du \left\{ \bigl(A+uI\bigr)^{-1} - \bigl(A+hdA/dt+uI\bigr)^{-1} \right\}, \tag{96}$$

For infinitesimal h, we have

$$\begin{aligned}
\bigl(A+hdA/dt+uI\bigr)^{-1} &= \bigl[(A+uI)(I+h(A+uI)^{-1}dA/dt)\bigr]^{-1} \\
&= (I+h(A+uI)^{-1}dA/dt)^{-1}(A+uI)^{-1} \\
&= (I-h(A+uI)^{-1}dA/dt)(A+uI)^{-1} + \mathcal{O}(h^2) \\
&= (A+uI)^{-1} - h(A+uI)^{-1}dA/dt(A+uI)^{-1} + \mathcal{O}(h^2)\,.
\end{aligned} \tag{97}$$

Inserting this result into eq. (96) yields

$$\frac{d}{dt}\ln A(t) = \int_0^\infty du\,(A+uI)^{-1}\frac{dA}{dt}(A+uI)^{-1}\,. \tag{98}$$

Finally, if we change variables using $u = (1-s)/s$, it follows that

$$\frac{d}{dt}\ln A(t) = \int_0^1 ds\,\bigl[sA+(1-s)I\bigr]^{-1}\frac{dA}{dt}\bigl[sA+(1-s)I\bigr]^{-1}\,. \tag{99}$$

which is the result quoted in eq. (91).

A second form of Theorem 6 employs the Gâteau (or equivalently the Fréchet) derivative.

Theorem 6(a):

$$\left(\frac{d}{dt}\ln(A+tB)\right)_{t=0} = \int_0^1 ds\,\bigl[sA+(1-s)I\bigr]^{-1}B\bigl[sA+(1-s)I\bigr]^{-1}\,. \tag{100}$$

after making use of eqs. (34) and (36) .

Theorem 7: [11]

$$A(t)\frac{d}{dt}\ln A(t) = \sum_{n=0}^{\infty}\frac{1}{n+1}(A^{-1}\mathrm{ad}_A)^n\left(\frac{dA}{dt}\right) = \frac{dA}{dt}+\tfrac{1}{2}A^{-1}\left[A,\frac{dA}{dt}\right]+\tfrac{1}{3}A^{-2}\left[A,\left[A,\frac{dA}{dt}\right]\right]+\cdots \tag{101}$$

Proof: A matrix inverse has the following integral representation,

$$B^{-1} = \int_0^\infty e^{-sB}\,ds\,, \tag{102}$$

if the eigenvalues of B lie in the region, $\mathrm{Re}\,z > 0$, of the complex z-plane. If we perform a formal differentiation of eq. (102) with respect to B, it follows that

$$B^{-n-1} = \frac{1}{n!}\int x^m e^{-sB}\,ds\,. \tag{103}$$

Thus, starting with eq. (98), we shall employ eq. (102) to write,

$$(A+uI)^{-1} = \int_0^\infty e^{-v(A+uI)}\,dv\,.$$

Inserting this result into eq. (98) yields

$$\begin{aligned}\frac{d}{dt}\ln A(t) &= \int_0^\infty du\int_0^\infty dv\int_0^\infty dw\, e^{-v(A+uI)}\frac{dA}{dt}e^{-w(A+uI)}\\ &= \int_0^\infty dv\int_0^\infty dw\, e^{-(v+w)A}\,e^{wA}\frac{dA}{dt}e^{-wA}\int_0^\infty e^{-(v+w)u}\,du\\ &= \int_0^\infty dw\int_0^\infty \frac{dv}{v+w}\,e^{-(v+w)A}\,e^{wA}\frac{dA}{dt}e^{-wA}\,.\end{aligned} \tag{104}$$

Let us change integration variables by replacing v with $x = v+w$, and then interchange the order of integration,

$$\begin{aligned}\frac{d}{dt}\ln A(t) &= \int_0^\infty dw\int_w^\infty \frac{dx}{x}e^{-xA}\,e^{wA}\frac{dA}{dt}e^{-wA}\\ &= \int_0^\infty \frac{dx}{x}\,e^{-xA}\int_0^x dw\, e^{wA}\frac{dA}{dt}e^{-wA}\,.\end{aligned} \tag{105}$$

We can now employ the result of Theorem 1 [cf. eq. (17)] to obtain

$$e^{wA}\frac{dA}{dt}e^{-wA} = \sum_{n=0}^{\infty}\frac{w^n}{n!}(\mathrm{ad}_A)^n\left(\frac{dA}{dt}\right)\,.$$

[11] I have not seen the next theorem anywhere in the literature, although it is difficult to believe that such an expression has never been derived elsewhere.

Inserting this result into eq. (105), we obtain,

$$\begin{aligned}
\frac{d}{dt}\ln A(t) &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^\infty \frac{dx}{x}\,e^{-xA}(\mathrm{ad}_A)^n\left(\frac{dA}{dt}\right)\int_0^x w^n\,dw \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\,\frac{1}{n+1}\left\{\int_0^\infty x^n e^{-xA}\,dx\right\}(\mathrm{ad}_A)^n\left(\frac{dA}{dt}\right). \qquad (106)
\end{aligned}$$

Finally, using eq. (103), we end up with

$$\frac{d}{dt}\ln A(t) = \sum_{n=0}^{\infty}\frac{1}{n+1}A^{-n-1}(\mathrm{ad}_A)^n\left(\frac{dA}{dt}\right). \qquad (107)$$

If we expand out the series, we find

$$A(t)\frac{d}{dt}\ln A(t) = \frac{dA}{dt} + \tfrac{1}{2}A^{-1}\left[A, \frac{dA}{dt}\right] + \tfrac{1}{3}A^{-2}\left[A, \left[A, \frac{dA}{dt}\right]\right] + \cdots \qquad (108)$$

Note that if $[A, dA/dt] = 0$, then eq. (108) yields:

$$\begin{aligned}
\frac{d}{dt}\ln A(t) &= A^{-1}\frac{dA}{dt} \\
&= A^{-1}\frac{dA}{dt}AA^{-1} = A^{-1}A\frac{dA}{dt}A^{-1} \\
&= \frac{dA}{dt}A^{-1}\,, \qquad (109)
\end{aligned}$$

which coincides with Property 5 given in the previous section.

One can rewrite eq. (107) in a more compact form by defining the function,

$$h(x) = -x^{-1}\ln(1-x) = \sum_{n=0}^{\infty}\frac{x^n}{n+1}\,. \qquad (110)$$

It then follows that[12]

$$A(t)\frac{d}{dt}\ln A(t) = h\big(A^{-1}\mathrm{ad}_A\big)\left(\frac{dA}{dt}\right). \qquad (111)$$

A second form of Theorem 7 employs the Gâteau derivative. In light of eqs. (34) and (36), one can derive the following alternative theorem.

Theorem 7(a):

$$\left(\frac{d}{dt}\ln(A+Bt)\right)_{t=0} = A^{-1}\,h\big(A^{-1}\mathrm{ad}_A\big)(B)\,, \qquad (112)$$

where the function h is defined by its Taylor series given in eq. (110).

[12] In obtaining eq. (111), we made use of the fact that A^{-1} commutes with the operator ad_A. In more detail, $A^{-1}\mathrm{ad}_A(B) - \mathrm{ad}_A(A^{-1}B) = A^{-1}(AB - BA) - (AA^{-1}B - A^{-1}BA) = 0$.

More explicitly,

$$\left(\frac{d}{dt}\ln(A+Bt)\right)_{t=0} = A^{-1}B + \tfrac{1}{2}A^{-2}\,[A,B] + \tfrac{1}{3}A^{-3}\,[A,[A,B]] + \cdots$$

Note that if $[A,B]=0$, then $\big[d\ln(A+Bt)/dt\big]_{t=0} = A^{-1}B = BA^{-1}$, which is also a consequence of Property 5 given in the previous section in the special case of $A(t)\equiv A+Bt$ for t-independent commuting matrices A and B.

Notes and References:

The proofs of Theorems 1, 2 and 4 can be found in section 5.1 of *Symmetry Groups and Their Applications*, by Willard Miller Jr. (Academic Press, New York, 1972). The proof of Theorem 3 is based on results given in section 6.5 of *Positive Definite Matrices*, by Rajendra Bhatia (Princeton University Press, Princeton, NJ, 2007). Bhatia notes that eq. (37) has been attributed variously to Duhamel, Dyson, Feynman and Schwinger. See also R.M. Wilcox, J. Math. Phys. **8**, 962 (1967). Theorem 3 is also quoted in eq. (5.75) of *Weak Interactions and Modern Particle Theory*, by Howard Georgi (Dover Publications, Mineola, NY, 2009) [although the proof of this result is relegated to an exercise]. A derivation of Theorem 5 can be found, e.g., in F. Casas, A. Muruab, and M. Nadinic, Comput. Phys. Commun. **183**, 2386 (2012).

The proof of Theorem 2 using the results of Theorem 3 is based on my own analysis, although I would not be surprised to find this proof elsewhere in the literature. Finally, a nice discussion of the SL(2,$\mathbb{R}$) matrix that cannot be written as a single exponential can be found in section 3.4 of *Matrix Groups: An Introduction to Lie Group Theory*, by Andrew Baker (Springer-Verlag, London, UK, 2002), and in section 10.5(b) of *Group Theory in Physics*, Volume 2, by J.F. Cornwell (Academic Press, London, UK, 1984).

The distinction between the Gâteau derivative and the Fréchet derivative [cf. footnote 1] is noted in Nicholas J. Higham, *Functions of Matrices* (Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008).

The integral definition of the matrix logarithm given in eq. (86) was derived by A. Wouk, J. Math. Anal. and Appl. **11**, 131 (1965). An explicit derivation is also provided on pp. 136–137 of Willi-Hans Stieb, *Problems and Solutions in Introductory and Advanced Matrix Calculus* (World Scientific Publishing Company, Singapore, 2006). The first proof of Theorem 6 was derived by my own analysis, although I expect that others must have produced a similar derivation. The second proof of Theorem 6 is inspired by Stephen L. Adler, *Taylor Expansion and Derivative Formulas for Matrix Logarithms*, which can be found at the following link: `https://www.ias.edu/sites/default/files/sns/files/1-matrixlog_tex(1).pdf`. Note that Theorem 6(a) was obtained previously in eq. (3.13) of L. Dieci, B. Morini and A. Papini, Siam J. Matrix Anal. Appl. **17**, 570 (1996).

In contrast, I have not seen Theorems 7 and 7(a) anywhere in the literature, although it is difficult to believe that such an expression has never been derived elsewhere.