DUE: Tuesday April 18, 2023

1. Consider the set $\mathbb{R}^2$ consisting of pairs of real numbers. For $(x, y) \in \mathbb{R}^2$, define scalar multiplication by: $c(x, y) = (cx, cy)$ for any real number c, and define vector addition and multiplication as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\,, \tag{1}$$

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 y_2)\,. \tag{2}$$

(a) Is $\mathbb{R}^2$ a group?

(b) Is $\mathbb{R}^2$ a field?

(c) Is $\mathbb{R}^2$ a linear vector space (over $\mathbb{R}$)?

(d) Is $\mathbb{R}^2$ a linear algebra (over $\mathbb{R}$)?

Suppose that the multiplication law given by eq. (2) is replaced by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2\,,\, x_1 y_2 + x_2 y_1)\,. \tag{3}$$

Do any of the results obtained in parts (a)–(d) above change? Identify a well know mathematical object that is isomorphic to $\mathbb{R}^2$ if eq. (2) is replaced by eq. (3).

2. Consider the possibility that a set G of $n \times n$ matrices forms a group with respect to matrix multiplication.

(a) Prove that if G is a group and if one of the elements of G is a non-singular matrix then all of the elements of G must be non-singular matrices. Conclude that all the elements of G are either non-singular matrices or singular matrices.

(b) Consider the set of 2×2 singular matrices G of the form

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\,, \tag{4}$$

where $x \in \mathbb{R}$ and $x \neq 0$. Prove that G is a group with respect to matrix multiplication. Determine the matrix corresponding to the identity element of G. Determine the inverse of the element specified in eq. (4).

(c) The group defined in part (b) is isomorphic to a well known group. Identify this group.

3. Consider the dihedral group D_4.

(a) Write down the group multiplication table.

(b) Enumerate the subgroups, the normal subgroups and the conjugacy classes.

(c) Identify the factor groups. Is the full group the direct product of some of its subgroups?

4. The *center* of a group G, denoted by $Z(G)$, is defined as the set of elements $z \in G$ that commute with all elements of the group. That is,

$$Z(G) = \{z \in G \,|\, zg = gz \,, \forall\, g \in G\}\,.$$

(a) Show that $Z(G)$ is an abelian subgroup of G.

(b) Show that $Z(G)$ is a normal subgroup of G.

(c) Find the center of D_4 and construct the group $D_4/Z(D_4)$. Determine whether the isomorphism $D_4 \cong [D_4/Z(D_4)] \otimes Z(D_4)$ is valid.

5. An automorphism is defined as an isomorphism of a group G onto itself.

(a) Show that for any $g \in G$, the mapping $T_g(x) = gxg^{-1}$ is an automorphism (called an *inner automorphism*), where $x \in G$.

(b) Show that the set of all inner automorphisms of G, denoted by $\mathcal{I}(G)$, is a group.

(c) Show that $\mathcal{I}(G) \simeq G/Z(G)$, where $Z(G)$ is the center of G.

(d) Show that the set of all automorphisms of G, denoted by $\mathcal{A}(G)$, is a group and that $\mathcal{I}(G)$ is an invariant subgroup. (The factor group $\mathcal{A}(G)/\mathcal{I}(G)$ is called the group of *outer automorphisms* of G.)

6. Consider an arbitrary orthogonal matrix R, which satisfies $RR^{\mathsf{T}} = \mathbb{1}$ (where $\mathbb{1}$ is the identity matrix).

(a) Prove that the possible values of $\det R$ are ± 1. [*HINT:* Consider $\det(RR^{\mathsf{T}})$ and use one of the well-known properties of determinants.]

(b) The group SO(2) consists of all 2×2 orthogonal matrices with unit determinant. Prove that SO(2) is an abelian group.

(c) The group O(2) consists of all 2×2 orthogonal matrices, with no restriction on the sign of its determinant. Is O(2) abelian or non-abelian? (If the latter, exhibit two O(2) matrices that do not commute.)