DUE: TUESDAY, JUNE 13, 2023

FINAL PROJECTS ALERT: The presentations of the final projects will take place on Thursday June 1 from 11:45 am–1:15 pm and on Wednesday June 14 from 4–7 pm in ISB 231. You are strongly encouraged to attend both sessions. The slides from your presentations will be posted to the class website.

1. (a) Show that the Lie algebra of $\mathrm{U}(n)$ can be written as a direct sum, $\mathfrak{u}(n) \cong \mathfrak{su}(n) \oplus \mathfrak{u}(1)$.

(b) As for the corresponding Lie groups, show that $\mathrm{U}(n) \cong \mathrm{SU}(n) \otimes \mathrm{U}(1)/\mathbb{Z}_n$.

HINT: Consider the homomorphism of $(A, e^{i\theta}) \longmapsto e^{i\theta} A$, where $A \in \mathrm{SU}(n)$ and $e^{i\theta} \in \mathrm{U}(1)$. What is the kernel of this homomorphism?

2. This problem concerns the Lie group $\mathrm{SO}(4)$ and its Lie algebra $\mathfrak{so}(4)$.

(a) Work out the Lie algebra $\mathfrak{so}(4)$ and verify that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

HINT: Show that there is a choice of basis for $\mathfrak{so}(4)$ consisting of 4×4 antisymmetric matrices that contain precisely two non-zero entries: 1 and -1. Evaluate the commutation relations of these $\mathfrak{so}(4)$ generators. Then, by choosing a new basis consisting of sums and differences of pairs of the original $\mathfrak{so}(4)$ generators, show that the resulting commutation relations are isomorphic to the commutation relations of the Lie algebra $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

(b) What is the universal covering group of $\mathrm{SO}(4)$? What is the center of $\mathrm{SO}(4)$? Identify the adjoint group $\mathrm{Ad}\big(\mathrm{SO}(4)\big)$.

(c) Calculate the Killing form of $\mathfrak{so}(4)$ and verify that this Lie algebra is semisimple and compact.

(d) Consider the complexification of the corresponding Lie algebras of part (a). Show that $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$. Do the conclusions of part (c) still hold? If not, explain how these conclusions are modified?

(e) Using the methods used in part (a), show that $\mathfrak{so}(3, 1) \cong \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$, where $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$ is the realification of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. By complexification of this result, show that one recovers the result of part (d).

REMARK: Note the Lie algebra isomorphisms, $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ and $\mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$. The latter is obtained from the former via complexification.

3. A Lie algebra $\mathfrak{g}$ is defined by the commutation relations of the generators,

$$[e_a, e_b] = f_{ab}^{c} e_c\,.$$

Consider the finite-dimensional matrix representations of the e_a. We shall denote the corresponding generators in the adjoint representation by F_a and in an arbitrary irreducible representation R by R_a. The dimension of the adjoint representation, d, is equal to the dimension of the Lie algebra $\mathfrak{g}$, while the dimension of the representation R will be denoted by d_R.

(a) Show that the Cartan-Killing metric g_{ab} can be written as $g_{ab} = \mathrm{Tr}(F_a F_b)$.

(b) If $\mathfrak{g}$ is a simple real compact Lie algebra, prove that for any irreducible representation R,

$$\mathrm{Tr}(R_a R_b) = c_R\, g_{ab}\,,$$

where c_R is called the *index* of the irreducible representation R.

HINT: Choose a basis where g_{ab} is proportional to δ_{ab}. Then the f_{ab}^{c} are antisymmetric in all three indices. Show that $\mathrm{Tr}\,[R_a, R_b]R_c = \mathrm{Tr}\,R_a[R_b, R_c]$ and argue that this implies that $\mathrm{Tr}\,R_a R_b$, viewed as the ab element of a $d \times d$ matrix, commutes with all Lie algebra elements in the adjoint representation. Finally, invoke Schur's lemma.[1]

(c) The quadratic Casimir operator is defined as $C_2 \equiv g^{ab} e_a e_b$ where g^{ab} is the inverse of g_{ab}. Recall that C_2 commutes with all elements of the Lie algebra. Hence, by Schur's lemma, C_2 must be a multiple of the identity operator. Let us write $C_2 = C_2(R)\mathbf{I}$ where $\mathbf{I}$ is the $d_R \times d_R$ identity matrix and $C_2(R)$ is the eigenvalue of the Casimir operator in the irreducible representation R. Show that $C_2(R)$ is related to the index c_R by

$$C_2(R) = \frac{d c_R}{d_R}\,,$$

where d is the dimension of the Lie algebra $\mathfrak{g}$. Check the above formula in the case that R is the adjoint representation.

HINT: The matrix elements of the R_a are $(R_a)_{ij}$, where $i, j = 1, \ldots, d_R$. If you keep the matrix element indices explicit, then the derivation of the above result is straightforward.

(d) Compute the index of an arbitrary irreducible representation of $\mathfrak{su}(2)$.

(e) Compute the index of the fundamental (defining) representation of $\mathfrak{su}(3)$. Generalize this result to $\mathfrak{su}(n)$.

4. Various subalgebras of $\mathfrak{su}(3)$ may be identified with specific subsets of the $\mathfrak{su}(3)$ generators.

(a) Show that the Gell-Mann matrices λ_1, λ_2, and λ_3 generate an $\mathfrak{su}(2)$ subalgebra.

[1] Note that by complexifying the simple real compact Lie algebra, one can easily show that the above result also holds for any simple complex Lie algebra.

(b) Show that the Gell-Mann matrices λ_2, λ_5, and λ_7 generate an $\mathfrak{so}(3)$ subalgebra. (Why do you think I called this an $\mathfrak{so}(3)$ subalgebra rather than an $\mathfrak{su}(2)$ subalgebra?)

(c) Decompose (if necessary) the three-dimensional irreducible representation of $\mathfrak{su}(3)$ into representations that are irreducible under the subalgebras of parts (a) and (b).

5. Consider the simple Lie algebra $\mathfrak{g}$ generated by the ten 4×4 matrices: $\sigma_a\otimes\mathbf{I}$, $\sigma_a\otimes\tau_1$, $\sigma_a\otimes\tau_3$ and $\mathbf{I}\otimes\tau_2$, where $(\mathbf{I},\sigma_a)$ and $(\mathbf{I},\tau_a)$ are the 2×2 identity and Pauli matrices in orthogonal spaces. For example, since $\tau_3=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, we obtain in block matrix form:

$$\sigma_a\otimes\tau_3=\left(\begin{array}{c|c}\sigma_a & \mathbf{0}\\ \hline \mathbf{0} & -\sigma_a\end{array}\right)\,,\qquad (a=1,2,3)\,,$$

where $\mathbf{0}$ is the 2×2 zero matrix. The remaining seven matrices can be likewise obtained. Take $H_1=\sigma_3\otimes\mathbf{I}$ and $H_2=\sigma_3\otimes\tau_3$ as the generators of the Cartan subalgebra. Note that if A, B, C, and D are 2×2 matrices, then $(A\otimes B)(C\otimes D)=AC\otimes BD$.

(a) Find the roots of $\mathfrak{g}$. Normalize the roots such that the shortest root vector has length 1. What is the rank of $\mathfrak{g}$?

(b) Determine the simple roots and evaluate the corresponding Cartan matrix. Deduce the Dynkin diagram for this Lie algebra and identify it by name.

(c) The fundamental weights $\boldsymbol{m}_i$ are defined in terms of the simple roots $\boldsymbol{\alpha}_j\in\Pi$ such that

$$\frac{2(\boldsymbol{m}_i,\boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_j,\boldsymbol{\alpha}_j)}=\delta_{ij}\,,\qquad\text{for } i,j=1,2,\ldots,r\,,$$

where $r\equiv\operatorname{rank}\mathfrak{g}$. Using the results of part (b), determine all the fundamental weights of $\mathfrak{g}$.

HINT: Expand the $\boldsymbol{m}_i$ as a linear combination of the simple roots and solve for the coefficients.

(d) [*EXTRA CREDIT*] Each of the r fundamental weights is the highest weight for an irreducible representation of $\mathfrak{g}$. Collectively, these are called the fundamental (or basic) representations of $\mathfrak{g}$. For each fundamental representation of $\mathfrak{g}$, compute the complete set of weights and draw the corresponding weight diagrams.[2] What are the corresponding dimensions of the fundamental representations of $\mathfrak{g}$.

HINT: In this example, all weights of the fundamental representations of $\mathfrak{g}$ appear with multiplicity equal to one. The complete set of weights for the irreducible representations of $\mathfrak{sp}(2,\mathbb{C})\cong\mathfrak{so}(5,\mathbb{C})$ corresponding to the highest weights $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, respectively, can be obtained by the method of block weight diagrams given in Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Dover Publications, Inc., Mineola, NY, 2006).[3] Note that the Cartan matrix employed by Cahn is the transpose of Cartan matrix defined in class.

[2] The weight diagrams should be plotted on a two dimensional plane, where the axes correspond to the diagonalized generators normalized such that the shortest root vector has length 1.

[3] A link to an electronic copy of this book can be found on the Physics 251 course webpage.