Introduction to SU(5) and SO(10) Grand Unification

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Overview

1. Motivation for Grand Unification
2. SM introduction (first generation)
3. SU(5) Unification
4. SO(10) Unification (briefly)
Motivation

- Can the large number of parameters inserted into the standard model be reconciled?
- Is it possible to unify SM forces, account for seemingly arbitrary coupling constants?
- Likewise can the fermions be united?
- Why are the hypercharge values of multiplets the way they are?
- Similarly, why are the electric charges what they are?
- SM incompatible with massive neutrinos
Isospin

- Analogous to spin-\(\frac{1}{2}\), described by SU(2)
- Strong isospin states up/down equivalent to spin-\(\frac{1}{2}\) projections: \(\pm \frac{1}{2}\)
- Quite descriptive due to the proximity of the masses of the u,d quarks
- Strong interaction acts on multiplets of the total isospin, e.g.
  \[ 2 \otimes 2 = 3 \oplus 1: \pi^0, \pm \oplus \eta \]
Hypercharge

- Utility in assigning these particle families a particular number: \( Y \)
- Gell-Mann-Nishijima formula: \( Q = I_3 + \frac{Y}{2} \)
Weak isospin

- Extend idea to leptons, create second doublet: \[
\begin{pmatrix}
\nu_L \\
\ell^-_L
\end{pmatrix}
\]
- Weak isospin mediated by \( W^\pm, Z^0 \)
- Observed asymmetry in chirality, only left-handed particles transform non-trivially under \( SU(2) \)
- Extend hypercharge as well, Gell-Mann-Nishijima still holds if the weak isospin doublet has \( Y = -1 \)
- Right-handed particles have no weak isospin projections
Gauge bosons, symmetry breaking

- **SM gauge group**: $SU(3) \otimes SU(2) \otimes U(1)$
- **SU(3) gauge bosons**: 8 massless gluons
- **Unified electroweak gauge bosons** (massless): $W^0, \pm, B$, with $\gamma, Z^0$ being linear combinations of $W^0$ and $B$
- **SU(2) \otimes U(1) symmetry broken**, electromagnetic and weak force behave differently, $W^\pm, Z^0$ bosons become massive
16 total fermions in $SU(3) \otimes SU(2) \otimes U(1)$

- $\nu_R = (1 \otimes 1)_0$
- $e_R = (1 \otimes 1)_{-2}$
- Left-handed lepton doublet $(1 \otimes 2)_{-1}$
- Right-handed down quark triplet $d_R = (3 \otimes 1)_{-\frac{2}{3}}$
- Right-handed up quark triplet $u_R = (3 \otimes 1)_{\frac{4}{3}}$
- Left-handed quark sextuplet $Q_L = (3 \otimes 2)_{\frac{1}{3}}$

16 fermions and 16 anti-fermions: $16 \oplus \bar{16} = 32$
Why SU(5)?

- Sensible to start with a rank 4 group
- SM has 4 commuting generators, $T_{3,8}$ from SU(3), $\sigma_3$ from SU(2) and the U(1) generator
- Is it possible to embed the SM into SU(5)?
- By construction, the diagonal generators can be made equivalent to those of the SM
Fitting $SU(3) \otimes SU(2) \otimes U(1)$ in $SU(5)$

- Represent matrices in $SU(5)$ as block diagonal, with block elements that are $SU(3)$, $SU(2)$ matrices multiplied by an appropriate $U(1)$ factor.

\[
A = \begin{pmatrix} \alpha^{-2} M & 0 \\ 0 & \alpha^3 N \end{pmatrix}, \alpha \in U(1), M \in SU(3), N \in SU(2)
\]

- Kernel of $\phi : (\alpha, M, N) \mapsto A$ is $(\alpha, \alpha^2, \alpha^{-3})$.

- Satisfying determinant condition requires that $\alpha$ be a sixth root of unity: $\mathbb{Z}_6$. 
To be able to embed the SM group ($G_{SM}$) into SU(5), the kernel must act trivially.

With some algebra, this yields constraints on the hypercharge, e.g. for left-handed quarks, $Y = 2k + \frac{1}{3}$, $k \in \mathbb{Z}$.

Structure introduced to the hypercharges via Gell-Mann-Nishijima, electric charge determined.
Where the first generation particles fit

- Examining the structure of that matrix $A$, its action would be on a vector made up of a doublet and a triplet
- The fundamental $5 = (1 \otimes 2)_1 \oplus (3 \otimes 1)_{-2/3}$
- Utilise $\bar{5} = (1 \otimes 2)_{-1} \oplus (\bar{3} \otimes 1)_{2/3}$, identify with doublet of leptons: $(\nu, e)^T$ and triplet of $\bar{d}$
- The antisymmetric part of the product of two $5$s: $5 \otimes_A 5 = 10$ can describe the remainder
- $10 = (1 \otimes 1)_2 \oplus (3 \otimes 2)_{1/3} \oplus (\bar{3} \otimes 1)_{-4/3}$
- Identify $\bar{e}$, quark sextuplet, $\bar{u}$ triplet
Gauge bosons

- \(5^2 - 1 = 24\) generators

\[
T^a = \left( \begin{array}{cc} \frac{\lambda^a}{2} & 0 \\ 0 & 0 \end{array} \right), \quad T^i = \left( \begin{array}{cc} 0 & 0 \\ 0 & \frac{\sigma^i}{2} \end{array} \right)
\]

- \(\lambda^a\) and \(\sigma^i\) are the Gell-Mann and Pauli matrices respectively.
- for \(a = 3, 8\) and \(i = 3\), there are 3 diagonal matrices. We expect one more
- \(\tilde{Y} = \frac{1}{\sqrt{60}} \text{diag}(-2, -2, -2, 3, 3)\)
- \(24 = (8 \otimes 1)_0 \oplus (1 \otimes 3)_0 \oplus (1 \otimes 1)_0 \oplus (\bar{3} \otimes 2)^{\frac{5}{6}} \oplus (3 \otimes 2)^{-\frac{5}{6}}\)
- Twelve off-diagonal matrices, linear combinations of which give 12 additional bosons
- These X,Y bosons can mediate processes that don’t conserve baryon or lepton number
Proton Decay

Figure: Retrieved from wikipedia. public domain.
Proton Decay

Figure: Retrieved from wikipedia. public domain.
Proton Decay

\[ \Gamma(qqq \rightarrow q\bar{q}l = \pi^0 e^+) \approx \frac{m_p^5}{M_{GUT}^4} \]

- Experimental searches for proton decay constrain \( M_{GUT} \gtrsim 10^{16} \text{ GeV} \)
For the Higgs, again use fundamental 5 containing a triplet and a doublet.

Symmetry breaking $SU(5) \rightarrow SU(3) \otimes SU(2) \otimes U(1)$

Doublet energy goes to the weak energy scale (100 GeV)

Energy of triplet would remain at GUT energy scale, $10^{14}$ GeV
difference: fine tuning problem

SO(10) encounters same problem
Weinberg angle

- \[ \sin^2(\theta_W) = \frac{g'^2}{g^2+g'^2} \text{ where } g', g \text{ are the couplings for hypercharge and weak isospin respectively} \]
- Denotes the rotation of the basis vectors \( \gamma, Z^0 \) in the \( W^0, B \) plane after electroweak symmetry breaking.
- Can express it in terms of single SU(5) coupling \( g \), replacing the \( g \) in the above equation.
- Inspect the action of \( \tilde{\gamma} \) on the states in the \( \bar{5} \) and we can read off \( \tilde{\gamma} = \frac{\sqrt{60}}{6} \gamma \)
- Comparing \( \gamma \) to \( \tilde{\gamma} \) yields \( (g/g')^2 = \frac{5}{3} \)
- Predicts \( \sin^2(\theta^{GUT}_W) = \frac{3}{8} \)
SU(5) summary

- A single gauge coupling
- Structure of hypercharge, electric charge
- Energy scale large due to proton stability
- Weinberg angle prediction is motivating but simple scheme untenable
- Not all SM particles embedded into single irrep
- Fine tuning problem present
Pati-Salam model

- Idea: promote right handed singlets to doublets
- Weak isospin now acts like $SU(2) \otimes SU(2)$, with each $SU(2)$ acting on a doublet of different handedness
- Additionally, can think of lepton number as a fourth colour. $SU(3) \rightarrow SU(4)$
- Full model: $SU(4) \otimes SU(2) \otimes SU(2)$
What can be gained by adding another commuting generator?

- B-L conservation is a naturally motivated choice: gives rise to Pati-Salam

- Can represent SM fermions inside a single irrep: \( 16 = 10 \oplus \bar{5} \oplus 1 \)

- All particles inside a single irrep
Can break symmetry in various ways, some examples:

- $SO(10) \rightarrow SU(3) \otimes SU(2) \otimes U(1)$
- $SO(10) \rightarrow SU(4) \otimes SU(2) \otimes SU(2)$
- $SO(10) \rightarrow SU(3) \otimes SU(2) \otimes SU(2) \otimes U(1)$
- $SO(10) \rightarrow SU(5) \otimes U(1)$
We saw that the 5 of SU(5) was an SU(3) colour triplet and an SU(2) isospin doublet.

With $16 \oplus \bar{16} = 32$ SM fermions (and anti-fermions), can represent every particle in a 5 element vector, whose elements are binary flags for some kind of charge: $2^5 = 32$.

Can think of a binary flag as some SU(2) projection.

$\psi = (r, g, b, up, down)^T$
\[2^5 \text{ representation}\]

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<th>(Y)</th>
<th>(r)</th>
<th>(g)</th>
<th>(b)</th>
<th>(u)</th>
<th>(d)</th>
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Various ways to unite leptons and quarks
Hypercharge seen to be less arbitrary, electric charge follows
Still a large number of parameters to be inserted
SU(5): $M_{GUT} \gtrsim 10^{16}$ GeV
SU(5) and SO(10) encounter fine tuning issues
Extending these theories with SUSY possible, though thoroughly outside the scope here


