Group Theory of Magnetic Monopoles

Yan Yu
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Dirac Monopoles

• A magnetic monopole has the vector potential as:

\[ \vec{A}_N = \frac{g}{4\pi r} \frac{1 - \cos \theta}{\sin \theta} \hat{\phi} \quad \vec{A}_S = \frac{g}{4\pi r} \frac{1 + \cos \theta}{\sin \theta} \hat{\phi} \]

• Then we have

\[ \vec{B} = \nabla \times \vec{A} = \frac{g}{4\pi r^2} \hat{r} \]

• There are singularities at \( \theta = 0 \) and \( \theta = \pi \) correspond to Dirac string. Moving in a circle around string, particle wave function picks up a phase \( \exp\{-i q g\} \). The Dirac string is undetectable and so we require that the phase factor must be equal to 1, which lead to the Dirac quantization condition

\[ qg = 2\pi n \]
General gauge theory formalism

- Consider the faithful representation of $G$, given by $D(g)$ and a scalar field $\phi$ which transforms under the representation
  $$\phi \rightarrow D(g)\phi$$

- $g$ is a function of spacetime: $g=g(x)$, under such transformations:
  $$\partial^\mu \phi \rightarrow D(g)\partial^\mu \phi + \partial^\mu D(g)\phi$$

- To keep it covariant, we introduce gauge fields $W^\mu_\alpha$ and associate with them a matrix in the Lie algebra of $G$:
  $$W^\mu = W^\mu_\alpha T^\alpha \in L(G)$$
• If we specify the gauge transformation as

\[ W^\mu \rightarrow g W^\mu g^{-1} + \frac{i}{e}(\partial^\mu g)g^{-1} \]

• Then the modified covariant derivative is given by

\[ D^\mu \phi = \partial^\mu \phi + i e D(W^\mu)\phi \rightarrow D(g)D^\mu \phi \]

• If we define the antisymmetric gauge field tensor as

\[ G^{\mu\nu} = G^{\mu\nu}_a T^a = \partial^\mu W^\nu - \partial^\nu W^\mu + i e [W^\mu, W^\nu] \]

• We will find that \([D^\mu, D^\nu]\phi = i e D(G^{\mu\nu})\phi\) and consequently

\[ G^{\mu\nu} \rightarrow g G^{\mu\nu} g^{-1} \]
• Since we assume the group $G$ is compact, we can always arrange that

$$\text{Tr}(T^a T^b) = \kappa \delta^{ab}$$

• Then we will find the field tensor is invariant under the action of the gauge group

$$G_a{}^{\mu\nu} G_a{}_{\mu\nu} = \frac{1}{\kappa} \text{Tr}(G^{\mu\nu} G_{\mu\nu}) \rightarrow \frac{1}{\kappa} \text{Tr}(g G^{\mu\nu} g^{-1} g G_{\mu\nu} g^{-1}) = \frac{1}{\kappa} \text{Tr}(G^{\mu\nu} G_{\mu\nu})$$
The structure of the Higgs vacuum

• Consider the Lagrangian density

\[ L = -\frac{1}{4} G_{\mu\nu}^a G_{a\mu\nu} + (D^\mu \phi)^\dagger (D_\mu \phi) - V(\phi) \]

• Where \( V(\phi) \) is invariant under the action of gauge group \( G \)

\[ V(D(g)\phi) = V(\phi) \]

• At any given time we expect the Lagrangian to satisfy the following equations everywhere in space apart from a finite number of compact regions which we call monopoles, this is the **Higgs vacuum**.

\[ V(\phi) = 0, \quad D^\mu \phi = 0 \]
• We use $M_0$ to denote the manifold of Higgs field $\phi$ which minimize the potential function

$$M_0 = \{ \phi : V(\phi) = 0 \}$$

• A non-trivial vacuum manifold consists of more than one point, forms an orbit of $G$. Any two points which can be related by an element of $G$ are said to be on the same orbit.

$$\phi_1 = D(g)\phi_2$$

• Here we assume that $M_0$ consists of a single orbit of the gauge group $G$, that is to say, given

$$\phi_1, \phi_2 \in M_0, \exists g_{12} \in G, \text{such that } \phi_1 = D(g_{12})\phi_2$$
Little group

• For representation D of gauge group G, the little group of a point $\phi \in M_0$ is defined as

$$H_\phi = \{ h \in G : D(h)\phi = \phi \}$$

• Lemma: Representations belonging to the same orbit have their little groups interrelated as

$$H = g^{-1}H'g$$

• Thus for single orbit H is isomorphic for different $\phi$, if we choose any point $\phi_0 \in M_0$ we will have $H = H_{\phi_0}$ and the manifold structure is only determined by G.
• In fact, 

\[ M_0 = G/H \]

• Prove: Associate a point \( \phi \) with \( g \in G \) via \( \phi = D(g)\phi_0 \), the elements \( g_1, g_2 \) will be associated with the same \( \phi \) if and only if \( g_1 \) and \( g_2 \) belong to the same right coset of \( H \) in \( G \), That is, \( D(g_1^{-1}g_2)\phi_0 = \phi_0 \), or equivalently, \( g_1^{-1}g_2 \in H \)

• Thus we may identify \( M_0 \) with the right coset space \( G/H \) which means once \( H \) has been determined the other details associated with the Higgs field may be ignored.
Consider a compact monopole region $M$, surrounded by a large region $S$, in which the equations defining the Higgs vacuum hold to a good approximation.

If $r \in S$ then $\phi(r) \in M_0$.

This implies that if we consider a closed surface $\Sigma$, enclosing $M$ once, then $\phi : \Sigma \rightarrow M_0$.
• As time evolves, if the map varies continuously with time we called such change a homotopy, and \( \phi(r, t_1), \phi(r, t_2) \) are said to define homotopic maps.

• Generally, two continuous maps \( f_1 \) and \( f_2 \) between topological spaces \( X \) and \( Y \) are homotopic, if there exists \( F \) sending
\[
(x, t) \rightarrow F(x, t) \in Y \quad (t \in [0,1] \text{ and } x \in X),
\]

such that
\[
F(x,0) = f_1(x) \text{ and } F(x,1) = f_2(x)
\]

• Thus \( F \) maps \( X \times [0,1] \rightarrow Y \) and constitutes a continuous deformation of the map \( f_1 \) into the map \( f_2 \), we denote such classes of maps from n-sphere to \( Y \) by \( \Pi_n(Y) \).
• Consider the magnetic flux through some closed surface, surrounding a region where $D^\mu \phi = 0$ fails.

\[ g_\Sigma = \int_{\Sigma} B \cdot dS \]

• Given the form of the gauge field outside the monopoles region as

\[ W^\mu = \frac{1}{a^2 e} \phi \land \partial^\mu \phi + \frac{1}{a} \phi A^\mu \]

• In fact, $g_\Sigma$ is time independent, gauge invariant, and also independent of $\phi$ on the surface, a small variation in the Higgs field produces no change in the flux. Thus the flux depends only on the homotopy classes of the maps

\[ \lim_{r \to \infty} \phi(\vec{r}) : S^{d-1} \to M_0 \]

• We are interested in $\Pi_{d-1}(M_0)$
• If $G$ is simply connected as well as $M$, then we have

$$\Pi_0(G) = 0 \quad \Pi_1(G) = 0$$

• A theorem in homotopy theory tells us that:

$$\Pi_1(G/H) \simeq \Pi_0(H) \text{ and } \Pi_2(G/H) \simeq \Pi_1(H)$$

• The first isomorphism tells us that assuming $M$ is connected is equivalent to assuming that $H$ is connected.

• The second isomorphism provides a description of the magnetic charges in terms of the first homotopy group of $H$ since $\phi : \Sigma \to M_0$ defines an element of $\Pi_2(M_0)$ under the assumption that $\Pi_1(M_0) = 0$
’t Hooft-Polyakov monopoles

- The gauge group is \( G = SO(3) \)
- Then we have the little group
  \[
  H = SO(2) \cong U(1)
  \]
- And the manifold of Higgs vacuum
  \[
  M_0 = SO(3)/U(1) \cong S^2
  \]
- Thus in 3+1 dimensions,
  \[
  \Pi_{d-1}(G/H) = \Pi_2(S^2) = \mathbb{Z}
  \]
- The equivalence classes are characterized by the number of times \( N \), that \( \phi(r) \) covers the sphere \( M_0 \) as \( r \) covers a two-dimensional sphere once. The number \( N \) determine the homotopy class.
**Charge quantization**

- SO(3) is not simply connected, but we can replace it by SU(2) to obtain a simply connected group. Then the homotopically distinct closed paths in H=U(1) are

\[
h(s) = \exp(iq \int_{\Sigma} B \cdot dS) = e^{iqg}
\]

- It is obtained when we solve the equation \( D^\mu \phi = 0 \) with the parameterized surface \( \Sigma \) as the unit square with its perimeter identified to a single point \( r_0 \in \Sigma \)

\[
\Sigma = \{r(s, t) : s \in [0,1], t \in [0,1]\}
\]

- The closure requires that \( h(0) = h(1) = 1 \) and leads to the Dirac quantization condition \( qg = 2\pi N, N \in \mathbb{Z} \)
Reference


