We recall that:

1. If α is a root vector, so is $-\alpha$. (6.39)
2. If α and β are root vectors, then $2(\alpha,\beta)/(\alpha,\alpha)$ is an integer. (6.40)
3. If α and β are root vectors, then so is

$$\beta-2\alpha\frac{(\alpha,\beta)}{(\alpha,\alpha)} \tag{6.41}$$

The angle φ between two roots α and β is given by

$$\cos\varphi=\frac{(\alpha,\beta)}{\sqrt{(\alpha,\alpha)(\beta,\beta)}} \tag{6.42}$$

or

$$\cos^2\varphi=\frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}=0,\ \tfrac{1}{4},\ \tfrac{1}{2},\ \tfrac{3}{4},\ \text{or } 1 \tag{6.43}$$

using Theorems 6.2 and 6.4.

Because of Eq. 6.39 we need only consider positive angles, and thus we are restricted to the angles

$$\varphi=0°,30°,45°,60°,90° \tag{6.44}$$

which in turn restricts the ratios of the scalar products as follows:

1. $\varphi=0°$. This case only arises for $\alpha=\beta$ and is thus trivial.
2. $\varphi=30°$. Then $(\alpha,\beta)/(\alpha,\alpha)=\frac{1}{2}$ or $\frac{3}{2}$, and hence $(\alpha,\beta)/(\beta,\beta)=\frac{3}{2}$ or $\frac{1}{2}$, respectively, and therefore $(\beta,\beta)/(\alpha,\alpha)=\frac{1}{3}$ or 3.
3. $\varphi=45°$. Then $(\alpha,\beta)/(\alpha,\alpha)=\frac{1}{2}$ or 1, and hence $(\alpha,\beta)/(\beta,\beta)=1$ or $\frac{1}{2}$, respectively, and therefore $(\beta,\beta)/(\alpha,\alpha)=\frac{1}{2}$ or 2.
4. $\varphi=60°$. Then $(\alpha,\beta)/(\alpha,\alpha)=\frac{1}{2}$, and hence $(\alpha,\beta)/(\beta,\beta)=\frac{1}{2}$, and therefore $(\alpha,\alpha)=(\beta,\beta)$.
5. $\varphi=90°$. Then $(\alpha,\beta)=0$, and hence $(\alpha,\alpha)/(\beta,\beta)$ is indeterminate.

The ratio $k_{\alpha\beta}$ of the *lengths* of the root vectors α and β are given by

$$k_{\alpha\beta}=\sqrt{\frac{(\alpha,\alpha)}{(\beta,\beta)}} \tag{6.45}$$

Thus we have

$\varphi=30°,\quad k=3$

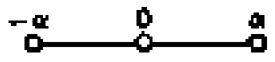
$\varphi=45°,\quad k=2$

$\varphi=60°,\quad k=1$

$\varphi=90°,\quad k$ undetermined

We now have sufficient information to construct root-vector diagrams for all the simple Lie algebras.

For $l=1$ we have from Eqs. 6.24 and 6.39 that there are just two nonzero roots $\pm\alpha$, and hence the only diagram ($\varphi=0°$) is



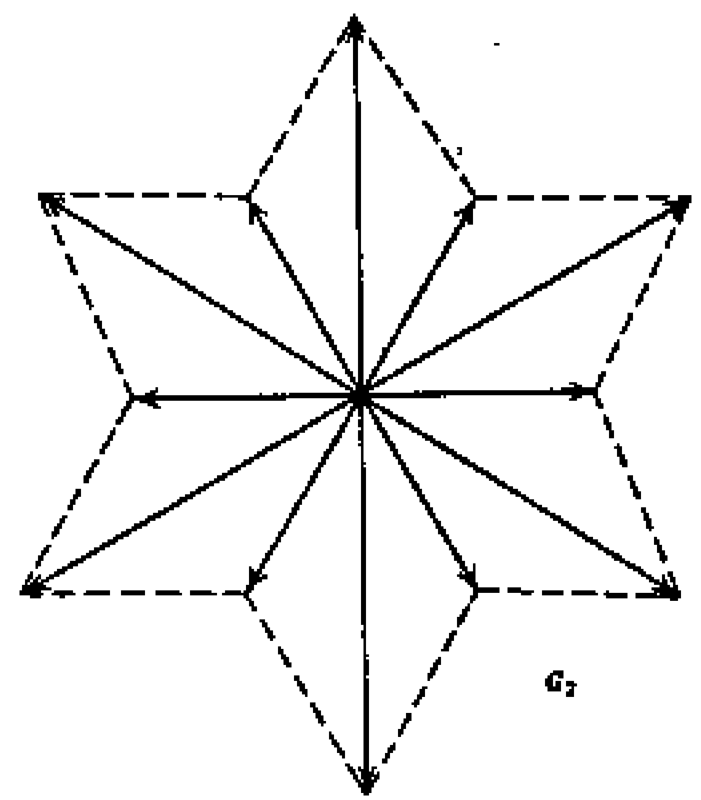
There is only one Lie algebra of rank 1 associated with this diagram, namely $su(2)$, which is isomorphic with $so(3)$. This Lie algebra is normally designated as A_1.

6.9 LIE ALGEBRAS OF RANK 2

We now consider Lie algebras of rank $l=2$. The root-vector diagrams span a two-dimensional weight space.

A. $\varphi=30°$

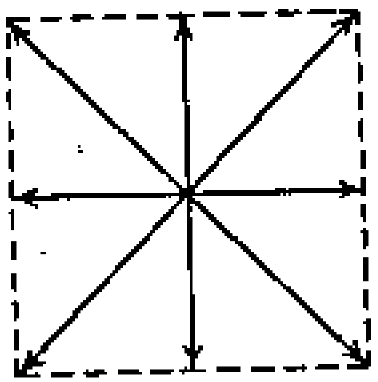
Suppose α is a root vector, and coordinates of its terminus being (1,0). Then there will be another root vector β of length $\sqrt{3}$ at an angle of 30° to α, with its terminus at $(\frac{3}{2}, \sqrt{3}/2)$. It follows from Eq. 6.39 that $-\alpha$ and $-\beta$ will also be root vectors. Taking $(\alpha,\beta)/(\beta,\beta)=\frac{1}{2}$, we have from Eq. 6.41 that $\beta-\alpha$ is also a root, and of course so is $\alpha-\beta$. These two roots have terminii at $(\frac{1}{2}, \sqrt{3}/2)$ and $(-\frac{1}{2}, -\sqrt{3}/2)$, respectively. Continuing in this way we finally obtain the highly symmetrical "Star of David" root-vector diagram containing 12 nonzero roots—and of course two null roots (0,0), since $l=2$. The Lie algebra associated with these root vectors was designated by Cartan as G_2.



G_2

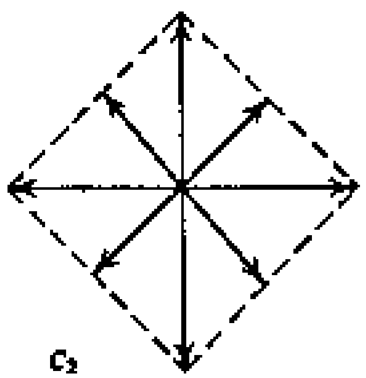
B. $\varphi = 45°$

Proceeding as before, we readily arrive at the figure



B_2

corresponding to Cartan's Lie algebra B_2. There are 10 root vectors (including the two null root vectors), which may be associated with the root-vector scheme of the $so(5)$ Lie algebra. A second diagram arises when the short and long simple roots are interchanged.

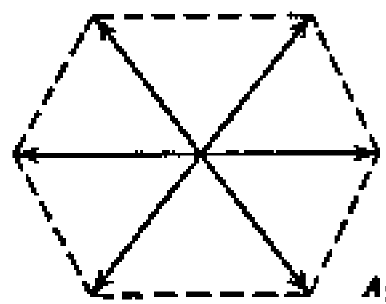


C_2

The second diagram is identified with Cartan's C_2 algebra, which is isomorphic to B_2 and differs only by a rotation of the root figure through 45°. The algebra associated with C_2 is that of the infinitesimal operators of the symplectic group $Sp(4)$ in four dimensions and is written as $sp(4)$.

C. $\varphi = 60°$

The resulting root-vector diagram is the hexagon

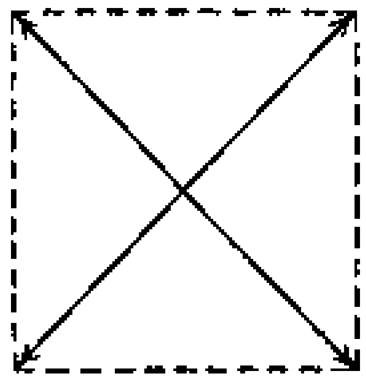


A_2

corresponding to Cartan's A_2 Lie algebra. There are eight root vectors (including the two null root vectors), which may be associated with the roots of the $su(3)$ Lie algebra.

D. $\varphi = 90°$

The resulting root-vector diagram is



D_2

The vector diagram of D_2 corresponds to a Lie algebra having six operators. It may be developed into two sets of mutually orthogonal roots, and thus represents the algebra of $so(4)$, which is isomorphic to the direct sum of two $so(3)$ algebras.

reference: "Classical Groups for Physicists", by Brian G. Wybourne
(with my corrections included)