Math 317 C1 John Sullivan Spring 2003

Groups of small order

We know that, in a group G of order n, the order of any element g divides n. This has important consequences, like the fact that when n is prime, every nonidentity element has order n and thus generates G, so G must be cyclic.

We will see later that when $n = p^2$ is the square of a prime, a group of order n still must be abelian, although there are now two possibilities: $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \times \mathbb{Z}_p$. Similarly, there are exactly two possibilities for a group of order $2p$, namely the cyclic group $\mathbb{Z}_{2p}$ and the dihedral group D_p.

We stop this table with $|G| = 15$ because there are 14 different groups of order 16. All abelian groups are products of cyclic groups, and are easily classified. The nonabelian groups in this table are dihedral, symmetric or alternating groups D_n, S_n or A_n, with two exceptions.

One exception is the *semi-direct product* T of $\mathbb{Z}_3$ by $\mathbb{Z}_4$, with order twelve. To say $G = K \rtimes H$ means that $H \leq G$, $K \lhd G$, $G = KH$, and $K \cap H = \{e\}$. As an example, $D_n = \langle\rho\rangle \rtimes \langle\tau\rangle \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$. This dihedral group is generated by ρ and τ, and its operation is determined by the relations $\rho^n = e = \tau^2$ and $\tau\rho = \rho^{-1}\tau$. Similarly, $T = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ is generated by elements ρ and τ, with the relations $\rho^3 = e = \tau^4$ and again $\tau\rho = \rho^{-1}\tau$. Note that ρ and τ^2 commute, so $\rho\tau^2$ has order six.

The other exception is the quaternion group $Q = \{\pm1, \pm i, \pm j, \pm k\}$ of order eight. It has center $\{\pm1\}$, and each noncentral element g satisfies $g^2 = -1$. Finally, we have $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. Every subgroup is normal, even though Q is nonabelian. (Q is a $\mathbb{Z}_2$ extension of the four-group V, and is a quotient of the semi-direct product $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$.) Both Q and T can be represented as subgroups of $SL_2(\mathbb{C})$, and thus sometimes both are called quaternionic groups.

$\lvert G\rvert$	Abelian G	Nonabelian G
1	$\mathbb{Z}_1 \cong S_1 \cong \langle e\rangle \cong A_2 \cong U(2)$	
2	$\mathbb{Z}_2 \cong S_2 \cong D_1 \cong \{\pm1\} \cong U(3) \cong U(4) \cong U(6)$	
3	$\mathbb{Z}_3 \cong A_3$	
4	$\mathbb{Z}_4 \cong \{\pm1, \pm i\} \cong U(5) \cong U(10)$ $V \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong D_2 \cong U(8) \cong U(12)$	
5	$\mathbb{Z}_5$	
6	$\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \cong U(7) \cong U(9) \cong U(14) \cong U(18)$	$S_3 \cong D_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$
7	$\mathbb{Z}_7$	
8	$\mathbb{Z}_8$ $\mathbb{Z}_4 \times \mathbb{Z}_2 \cong U(15) \cong U(16) \cong U(20) \cong U(30)$ $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong U(24)$	$D_4 \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2 \cong V \rtimes \mathbb{Z}_2$ $Q = \pm\{1, i, j, k\}$
9	$\mathbb{Z}_9$ $\mathbb{Z}_3 \times \mathbb{Z}_3$	
10	$\mathbb{Z}_{10} \cong \mathbb{Z}_5 \times \mathbb{Z}_2 \cong U(11) \cong U(22)$	$D_5 \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_2$
11	$\mathbb{Z}_{11}$	
12	$\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \cong U(13) \cong U(26)$ $\mathbb{Z}_6 \times \mathbb{Z}_2 \cong U(21) \cong U(28) \cong U(36) \cong U(42)$	$D_6 \cong S_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6 \rtimes \mathbb{Z}_2$ $A_4 = V \rtimes \mathbb{Z}_3$ $T = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$
13	$\mathbb{Z}_{13}$	
14	$\mathbb{Z}_{14} \cong \mathbb{Z}_7 \times \mathbb{Z}_2$	$D_7 \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_2$
15	$\mathbb{Z}_{15} \cong \mathbb{Z}_5 \times \mathbb{Z}_3$	