How to add sine functions of different amplitude and phase

In these notes, I will show you how to add two sinusoidal waves, each of different amplitude and phase, to get a third sinusoidal wave. That is, we wish to show that given

$$E_1 = E_{10} \sin \omega t \,, \tag{1}$$

$$E_2 = E_{20} \sin(\omega t + \delta) \,, \tag{2}$$

the sum $E_\theta \equiv E_1 + E_2$ can be written in the form:

$$\boxed{E_\theta = E_{10} \sin \omega t + E_{20} \sin(\omega t + \delta) = E_{\theta 0} \sin(\omega t + \phi)} \tag{3}$$

where the amplitude $E_{\theta 0}$ and phase ϕ are determined in terms of E_{10}, E_{20} and δ. In these notes, we shall derive that the amplitude $E_{\theta 0}$ is given by

$$\boxed{E_{\theta 0} = \sqrt{E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos\delta}}$$

and the phase ϕ is determined modulo 2π by[1]

$$\boxed{\sin\phi = \frac{E_{20}\sin\delta}{E_{\theta 0}}}\,, \qquad \boxed{\cos\phi = \frac{E_{10} + E_{20}\cos\delta}{E_{\theta 0}}}\,.$$

By definition, the amplitudes E_{10} and E_{20} are positive numbers. If we divide the last two equations, then ϕ can be determined modulo π from:

$$\tan\phi = \frac{E_{20}\sin\delta}{E_{10} + E_{20}\cos\delta}\,.$$

To determine ϕ modulo 2π, we need to supplement the result for $\tan\phi$ with

$$\text{sign}(\sin\phi) = \text{sign}(\sin\delta)\,, \qquad \sin\delta \neq 0\,,$$

where sign$(\sin\phi)$ literally means the sign (i.e. either $+1$ or -1) of the quantity $\sin\phi$. Finally, if $\sin\delta = 0$ then $\cos\delta = \pm 1$, and ϕ can be fixed modulo 2π by

$$\cos\phi = \begin{cases} \;\;1\,, & \cos\delta = 1\,, \\ \;\;1\,, & \cos\delta = -1\,, \quad \text{and} \quad E_{10} > E_{20}\,, \\ -1\,, & \cos\delta = -1\,, \quad \text{and} \quad E_{10} < E_{20}\,. \end{cases}$$

Note that in the case of $\cos\delta = -1$ and $E_{10} = E_{20}$, we have $E_\theta = 0$ in which case $E_{\theta 0} = 0$ and ϕ is no longer meaningful. In fact, this is the only circumstance in which $E_{\theta 0}$ can vanish.

I shall provide two different derivations of the above formulae. Finally, in an appendix, I will provide a mathematically more advanced derivation that makes use of complex numbers. If you are unfamiliar with complex numbers, you can skip the appendix for now and return to this last derivation later after you take Physics 116A or an equivalent course. This last method also provides the real motivation for the method of phasors introduced in Section 2 below.

[1] The phrase, "ϕ is determined modulo 2π," means that ϕ is determined up to an additive integer multiple of 2π. This is all that is needed, since adding a multiple of 2π to the phase angle ϕ does not change the value of $\sin(\omega t + \phi)$.

1. Algebraic method

First, set $t = 0$ in eq. (3) to obtain $E_{20} \sin\delta = E_{\theta 0} \sin\phi$. Solving for $\sin\phi$ yields:

$$\boxed{\sin\phi = \frac{E_{20}\sin\delta}{E_{\theta 0}}}\,.$$

Next, set $\omega t = \pi/2$ in eq.(3). Noting that $\sin(\delta + \pi/2) = \cos\delta$, it follows that $E_{10} + E_{20}\cos\delta = E_{\theta 0}\cos\phi$. Solving for $\cos\phi$ yields:

$$\boxed{\cos\phi = \frac{E_{10} + E_{20}\cos\delta}{E_{\theta 0}}}\,.$$

Finally, using $\cos^2\phi + \sin^2\phi = 1$, and inserting the expressions for $\cos\phi$ and $\sin\phi$ just obtained, one finds:

$$\begin{aligned} E_{\theta 0}^2 &= (E_{10} + E_{20}\cos\delta)^2 + E_{20}^2 \sin^2\delta \\ &= E_{10}^2 + 2E_{10}E_{20}\cos\delta + E_{20}^2(\cos^2\delta + \sin^2\delta) \\ &= E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos\delta\,. \end{aligned}$$

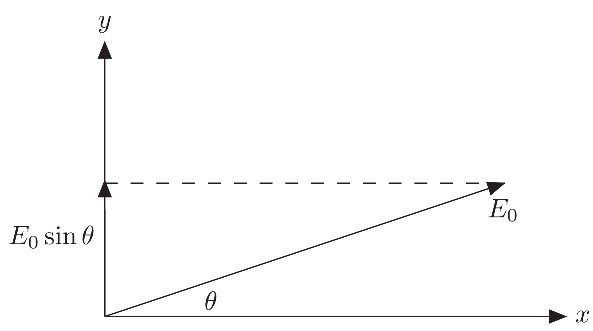
By definition, $E_{\theta 0}$ is a non-negative number.[2] Thus, we take the positive square root to obtain

$$\boxed{E_{\theta 0} = \sqrt{E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos\delta}}\,.$$

This completes our derivation.

2. Geometric method—the method of phasors

Consider a fictitious vector in a two-dimensional space, whose length is E_0, which makes an angle θ with respect to the x-axis as shown below:

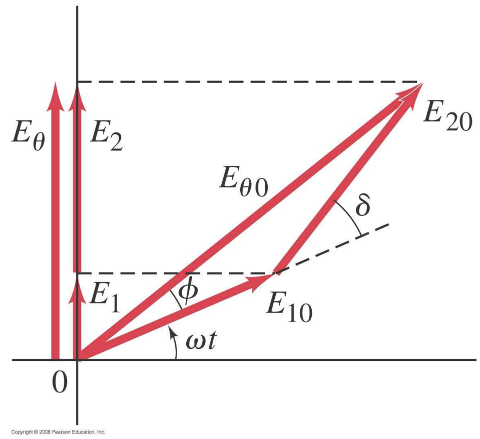


If we project this vector onto the y-axis, then its projected length is $E_0 \sin\theta$ as shown above. This vector is called a *phasor* and represents a quantity with an amplitude E_0 and an angle θ.

The utility of such a representation is that we can perform the sum of eq. (3) by considering the phasors corresponding to each sine term in the sum, and then adding the phasors vectorially! The projection of the vector sum of the two phasors onto the y-axis is just the sum of the two sine functions that we wish to compute. This vector sum can be carried out geometrically, and provides a second method for evaluating $E_{\theta 0}$ and ϕ.

[2] Warning! This is a matter of convention, which Giancoli chooses not to follow (without warning you).

To see how this works, consider the computation of eq. (3) by the method of phasors. We represent E_1 and E_2 [cf. eqs. (1) and (2)] as shown in the figure below.



Then, the phasor representation of E_θ is just the vector sum shown above. We identify E_{10}, E_{20} and $E_{\theta 0}$, as the lengths of the phasors representing E_1, E_2 and E_θ, respectively. To evaluate $E_{\theta 0}$ and ϕ, we focus on the triangle in the figure above. First, using the law of cosines,

$$E_{\theta 0}^2 = E_{10}^2 + E_{20}^2 - 2E_{10}E_{20}\cos(\pi - \delta)\,,$$

since $\pi - \delta$ is the angle between the phasors representing E_1 and E_2. Using $\cos(\pi - \delta) = -\cos\delta$, we end up with

$$E_{\theta 0} = \sqrt{E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos\delta}\,,$$

as before. Next, using the law of sines,

$$\frac{\sin\phi}{E_{20}} = \frac{\sin(\pi - \delta)}{E_{\theta 0}}\,.$$

Using $\sin(\pi - \delta) = \sin\delta$, we can solve for $\sin\phi$. We find that

$$\sin\phi = \frac{E_{20}\sin\delta}{E_{\theta 0}}\,,$$

which again agrees with our previous result. This equation only fixes ϕ modulo π. In order to fix ϕ modulo 2π, we employ the law of sines again (noting that the angle in the triangle between the phasors representing E_2 and E_θ is given by $\delta - \phi$):

$$\frac{\sin\phi}{E_{20}} = \frac{\sin(\delta - \phi)}{E_{10}}\,.$$

This equation can be rearranged in the following form:

$$\begin{aligned}
\frac{E_{10}}{E_{20}} &= \frac{\sin(\delta - \phi)}{E_{10}} \\
&= \frac{\sin\delta\cos\phi - \cos\delta\sin\phi}{\sin\phi} \\
&= \frac{\sin\delta}{\tan\phi} - \cos\delta\,.
\end{aligned}$$

One can solve this equation easily for $\tan\phi$ to obtain

$$\tan\phi = \frac{E_{20}\sin\delta}{E_{10}+E_{20}\cos\delta}\,,$$

in agreement with our previous result. Finally, we can use our results above for $\sin\phi$ and $\tan\phi$ to compute $\cos\phi$ as follows

$$\cos\phi = \frac{\sin\phi}{\tan\phi} = \left(\frac{E_{20}\sin\delta}{E_{\theta 0}}\right)\left(\frac{E_{10}+E_{20}\cos\delta}{E_{20}\sin\delta}\right) = \frac{E_{10}+E_{20}\cos\delta}{E_{\theta 0}}\,,$$

which completes the derivation.

3. The limit of equal amplitudes

As a check, consider the case of equal amplitudes, $E_{10} = E_{20} \equiv E_0$. Then, using the above results,

$$E_{\theta 0} = \sqrt{2E_0(1+\cos\delta)}\,.$$

Recalling the trigonometric identity, $\cos^2(\delta/2) = \frac{1}{2}(1+\cos\delta)$, we end up with:

$$E_{\theta 0} = 2E_0|\cos(\delta/2)|\,.$$

Note the absolute value sign, since by definition the amplitude $E_{\theta 0}$ is defined to be non-negative, which means we must take the positive square root: $\sqrt{\cos^2(\delta/2)} = |\cos(\delta/2)|$. If $\cos\delta = -1$, then $E_{\theta 0} = 0$ and the angle ϕ is undefined. Otherwise, we may use the results derived above for $\sin\phi$ and $\cos\phi$ to obtain.

$$\sin\phi = \frac{\sin\delta}{2|\cos(\delta/2)|} = \frac{2\sin(\delta/2)\cos(\delta/2)}{2|\cos(\delta/2)|} = \begin{cases} \sin(\delta/2)\,, & \text{if } \cos(\delta/2) > 0\,, \\ -\sin(\delta/2)\,, & \text{if } \cos(\delta/2) < 0\,, \end{cases}$$

$$\cos\phi = \frac{1+\cos\delta}{2|\cos(\delta/2)|} = \frac{\cos^2(\delta/2)}{|\cos(\delta/2)|} = |\cos(\delta/2)|\,,$$

after using the trigonometric identity $\sin\delta = 2\sin(\delta/2)\cos(\delta/2)$. Note that

$$\cos(\delta/2) = \tfrac{1}{2}\,(1+\cos\delta) \neq 0\,, \qquad \text{for} \quad \cos\delta \neq -1\,,$$

which must hold if $E_{\theta 0} \neq 0$. The results above for $\sin\phi$ and $\cos\phi$ imply that:

$$\phi = \begin{cases} \delta/2\,, & \text{if} \quad \cos(\delta/2) > 0\,, \\ \pi + \delta/2\,, & \text{if} \quad \cos(\delta/2) < 0\,. \end{cases}$$

In Section 34-4 of Giancoli, a convention is chosen in which $E_{\theta 0}$ can be of either sign, so in order to compare Giancoli's results with ours, one must be careful if $\cos(\delta/2) < 0$. Nevertheless, independently of this convention, we can write:

$$\begin{aligned} E_\theta = E_{\theta 0}\sin(\omega t+\phi) &= 2E_0|\cos(\delta/2)|\sin(\omega t+\phi) \\ &= 2E_0|\cos(\delta/2)|(\sin\omega t\cos\phi + \cos\omega t\sin\phi) \\ &= 2E_0|\cos(\delta/2)|\left(\sin\omega t\frac{\cos^2(\delta/2)}{|\cos(\delta/2)|} + \cos\omega t\frac{\sin(\delta/2)\cos(\delta/2)}{|\cos(\delta/2)|}\right) \\ &= 2E_0\cos(\delta/2)\,(\sin\omega t\cos(\delta/2) + \cos\omega t\sin(\delta/2)) \\ &= 2E_0\cos(\delta/2)\sin(\omega t + \delta/2)\,, \end{aligned}$$

which agrees with Eq. 34-5c of Giancoli (on page 907).

Appendix: Adding two sine functions of different amplitude and phase using complex numbers

To perform the sum:

$$E_\theta = E_{10}\sin\omega t + E_{20}\sin(\omega t+\delta) = E_{\theta 0}\sin(\omega t+\phi)\,, \tag{4}$$

we note the famous Euler formula:

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

In particular, $\sin\theta$ is the imaginary part of $e^{i\theta}$. Thus, if we consider the equation:

$$E_{10}e^{i\omega t} + E_{20}e^{i(\omega t+\delta)} = E_{\theta 0}e^{i(\omega t+\phi)}\,, \tag{5}$$

then the imaginary part of this equation coincides with eq. (4). By the way, I can view a complex number $x+iy$ as a vector in a two-dimensional space (called the complex plane) that points from the origin to the point (x,y). This vector is precisely the phasor that we employed in Section 2 of these notes. In particular, in this language, eq. (5) describes the sum of two complex numbers, which is depicted by the sum of the phasors in the figure shown in Section 2. The projected lengths of the phasors on the y-axis simply correspond to the imaginary parts of the corresponding complex numbers.

Thus, to solve for $E_{\theta 0}$ and ϕ, we can simply start from eq. (5). If I multiply this equation by $e^{-i\omega t}$, I obtain:

$$\boxed{E_{10} + E_{20}e^{i\delta} = E_{\theta 0}e^{i\phi}} \tag{6}$$

where E_{10}, E_{20} and $E_{\theta 0}$ are non-negative (and real) by definition. To compute $E_{\theta 0}$, I simply take the complex absolute value of both sides of this equation. For any complex number $z=x+iy$, the complex absolute value is given by $|z|=\sqrt{x^2+y^2}$. Hence,

$$\begin{aligned} E_{\theta 0}^2 = |E_{10}+E_{20}e^{i\delta}|^2 &= |E_{10}+E_{20}\cos\delta + iE_{20}\sin\delta|^2 \\ &= (E_{10}+E_{20}\cos\delta)^2 + E_{20}^2\sin^2\delta \\ &= E_{10}^2 + 2E_{10}E_{20}\cos\delta + E_{20}^2\,, \end{aligned}$$

after using $\sin^2\delta+\cos^2\delta=1$. Taking the positive square roots yields

$$E_{\theta 0} = \sqrt{E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos\delta}\,.$$

To determine ϕ, I simply take the real and imaginary parts of eq. (6):

$$E_{\theta 0}\cos\phi = E_{10}+E_{20}\cos\delta\,, \qquad\qquad E_{\theta 0}\sin\phi = E_{20}\sin\delta\,,$$

Solving for $\cos\phi$ and $\sin\phi$, respectively, we immediately find:

$$\cos\phi = \frac{E_{10}+E_{20}\cos\delta}{E_{\theta 0}}\,,$$

$$\sin\phi = \frac{E_{20}\sin\delta}{E_{\theta 0}}\,.$$

Thus, we have successfully reproduced the main results obtained previously in these notes. I think you must agree that this last approach is by far the simplest from a computational point of view. Once you learn how use and manipulate complex numbers, many tasks in mathematical physics become much simpler!