Solution to the Equation of Motion for Forced Oscillations

The equation of motion for forced oscillations is given by Eq. (14-21) of Gioncoli:

$$m\frac{dx^2}{dt^2}+b\frac{dx}{dt}+kx=F_0\cos\omega t$$

We shall show that $x=A_0\sin\left(\omega t+\phi_0\right)$ is a solution of by direct substitution.

$$x=A_0\sin\left(\omega t+\phi_0\right)\ ;\ \frac{dx}{dt}=\omega A_0\cos\left(\omega t+\phi_0\right)\ ;\ \frac{d^2x}{dt^2}=-\omega^2A_0\sin\left(\omega t+\phi_0\right)$$

$$m\frac{dx^2}{dt^2}+b\frac{dx}{dt}+kx=F_0\cos\omega t\quad\rightarrow$$

$$m\left[-\omega^2A_0\sin\left(\omega t+\phi_0\right)\right]+b\left[\omega A_0\cos\left(\omega t+\phi_0\right)\right]+k\left[A_0\sin\left(\omega t+\phi_0\right)\right]=F_0\cos\omega t$$

Expanding the trigonometric functions [cf. page A-4 of Appendix A of Giancoli],

$$\left(kA_0-m\omega^2A_0\right)\left[\sin\omega t\cos\phi_0+\cos\omega t\sin\phi_0\right]+b\omega A_0\left[\cos\omega t\cos\phi_0-\sin\omega t\sin\phi_0\right]=F_0\cos\omega t$$

We now group the various terms by their time dependence.

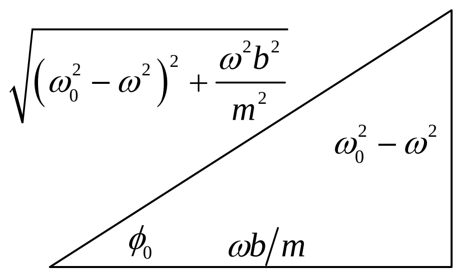
$$\left[\left(kA_0-m\omega^2A_0\right)\cos\phi_0-b\omega A_0\sin\phi_0\right]\sin\omega t+\left[\left(kA_0-m\omega^2A_0\right)\sin\phi_0+b\omega A_0\cos\phi_0\right]\cos\omega t$$
$$=F_0\cos\omega t$$

The above equation must be valid for all *time*, which means that the coefficients of the functions of t must be the same on both sides of the equation. Since there is no $\sin\omega t$ on the right side of the equation, the coefficient of $\sin\omega t$ must be 0.

$$\left(kA_0-m\omega^2A_0\right)\cos\phi_0-b\omega A_0\sin\phi_0=0\quad\rightarrow$$

$$\frac{\sin\phi_0}{\cos\phi_0}=\frac{kA_0-m\omega^2A_0}{b\omega A_0}=\frac{k-m\omega^2}{b\omega}=\frac{m\omega_0^2-m\omega^2}{b\omega}=\frac{\omega_0^2-\omega^2}{\omega b/m}=\tan\phi_0\quad\rightarrow\quad\boxed{\phi_0=\tan^{-1}\frac{\omega_0^2-\omega^2}{\omega b/m}}$$

Thus we see that Eq. 14-24 of Giancoli is necessary for $x = A_0 \sin(\omega t + \phi_0)$ to be the solution. This can be illustrated with the diagram shown below.



Finally, we equate the coefficients of $\cos\omega t$.

$$\left(kA_0 - m\omega^2 A_0\right)\sin\phi_0 + b\omega A_0\cos\phi_0 = F_0 \quad \rightarrow$$

$$A_0\left[\left(k-m\omega^2\right)\frac{\left(\omega_0^2-\omega^2\right)}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\frac{\omega^2 b^2}{m^2}}}+b\omega\frac{\frac{\omega b}{m}}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\frac{\omega^2 b^2}{m^2}}}\right]=F_0 \quad \rightarrow$$

$$A_0 m\left[\frac{\left(\omega_0^2-\omega^2\right)^2}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\frac{\omega^2 b^2}{m^2}}}+\frac{\frac{\omega^2 b^2}{m^2}}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\frac{\omega^2 b^2}{m^2}}}\right]=F_0 \quad \rightarrow$$

$$\boxed{A_0 = \frac{F_0}{m\left[\sqrt{\left(\omega_0^2-\omega^2\right)^2+\frac{\omega^2 b^2}{m^2}}\right]}}$$

Thus we see that Eq. 14-23 of Giancoli is also necessary for $x = A_0 \sin(\omega t + \phi_0)$ to be the solution.