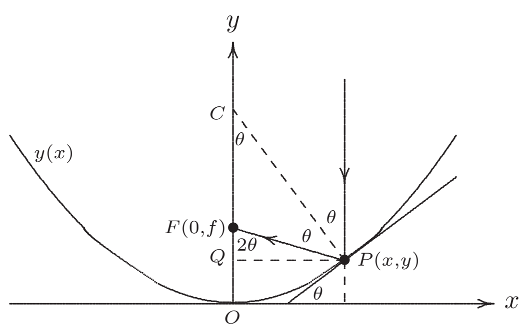
Focusing properties of spherical and parabolic mirrors

1. General considerations

Consider a curved mirror surface that is constructed as follows. Start with a curve, denoted by $y(x)$ in the x–y plane, that is symmetrical under a reflection through the y axis; i.e. $y(-x) = y(x)$. The y-axis is thus the symmetry-axis of the two-dimensional curve $y(x)$. The three-dimensional curved mirror surface is then obtained by rotating the curve about the y-axis, thereby producing a "surface of revolution" corresponding to the surface of the mirror. The projection of this surface onto the x–y plane yields the original curve $y(x)$.

Due to the symmetry of the three-dimensional surface, it is sufficient to examine the light rays propagating in the x–y plane. Consider two parallel light rays that strike a curved mirror surface. The first ray is initially propagating in a direction parallel to the y-axis. It then strikes the mirror with an angle of incidence θ with respect to the normal to the curve $y(x)$ at the point P, labeled by coordinates (x, y). Using the law of reflection, the angle of reflection of the resulting reflected ray is equal to the angle of incidence, θ. The second ray heads down the y-axis, strikes the mirror at O and then is reflected back up the y-axis. Both reflected rays intersect at the focal point F, labeled by coordinates $(0, f)$, as shown in the figure below.



The tangent line to the curve at P is explicitly shown above. Simple geometrical considerations imply that the angle the tangent line makes with the x-axis is also given by θ. Thus,

$$\tan\theta = \frac{dy}{dx}\,.$$

where the derivative of the curve $y(x)$ is evaluated at the point P.

In addition, the angle FCP is also equal to θ (as the two initial light rays are parallel), from which we conclude that the angle QFP is equal to 2θ, as indicated in the above figure. Thus, the distance of the line segment FQ is given by $x/\tan 2\theta$. Hence, the focal length f is given by:

$$f = y + \frac{x}{\tan 2\theta}\,.$$

Using the identity

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\,,$$

it follows that:

$$\boxed{f = y + \frac{x(1-\tan^2\theta)}{2\tan\theta}}\,. \tag{1}$$

2. The spherical mirror

For a spherical mirror, the curve shown above is part of a circle of radius r. Moreover, C is the center of the circle, since the line segment CP is perpendicular to the tangent line at point P. Hence the length of CP is equal to r. Note that the triangle CFP is isosceles, hence the length of the sides FP and FC are equal. Denote this length by a. Then by the law of cosines,

$$\begin{aligned} r^2 &= 2a^2[1-\cos(\pi-2\theta)] \\ &= 2a^2(1+\cos 2\theta) \\ &= 4a^2\cos^2\theta\,, \end{aligned} \tag{2}$$

after using $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$. Hence, $a = r/(2\cos\theta)$. Finally, noting that $f + a = r$, we end up with

$$\boxed{f = r\left(1 - \frac{1}{2\cos\theta}\right)}\,. \tag{3}$$

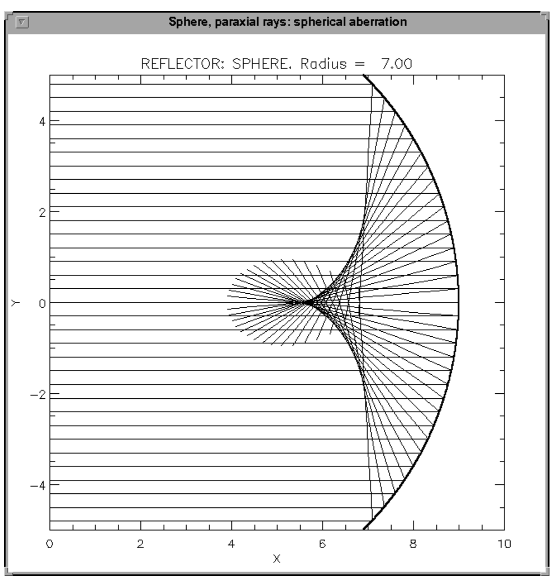
This last equation shows that there is no unique focal point, since f depends on the angle θ. However, if $\theta \ll 1$, then

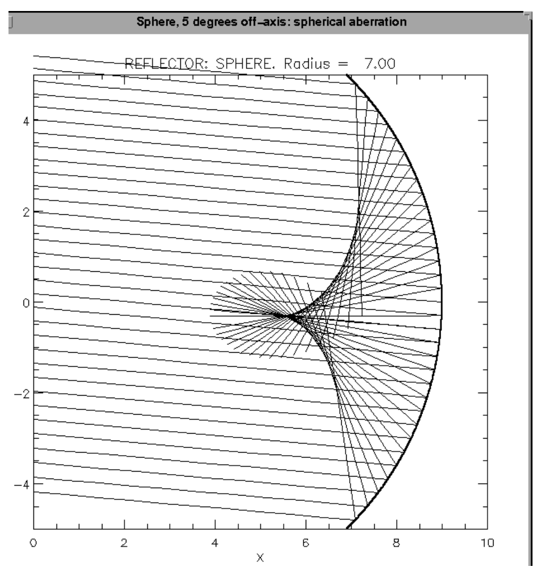
$$\cos\theta \simeq 1 - \tfrac{1}{2}\theta^2 \simeq 1\,, \qquad \text{for } \theta \ll 1\,,$$

in which case, we can approximate

$$\boxed{f \simeq \tfrac{1}{2}r\,, \qquad \text{for } \theta \ll 1}\,.$$

That is, for small angles (or equivalently for a mirror whose length is much smaller than the radius r), the location of the focal point F is independent of the angle of incidence, which means that all parallel rays that strike the spherical mirror (at small angle) pass through the focal point F. The non-uniqueness of the focal point when larger angles of incidence are considered is illustrated by two figures, which provide a graphic illustration of the focusing properties of a spherical mirror [taken from `http://www.astro.virginia.edu/class/oconnell/astr511/lec16-teloptics-f03.html`].





One can also obtain the above results for the focal point of a spherical mirror directly from eq. (1). The equation for the circle of radius r, whose center is located at C with coordinates $(0, r)$ is given by

$$x^2 + (y - r)^2 = r^2 \,,$$

which simplifies to

$$y^2 - 2ry + x^2 = 0\,.$$

This is a quadratic equation that can be solved for y. Of the two solutions, the one of interest here corresponds to $|y| \leq r$:

$$y = r - \sqrt{r^2 - x^2}\,. \tag{4}$$

It follows that

$$\tan\theta = \frac{dy}{dx} = \frac{x}{\sqrt{r^2 - x^2}}\,. \tag{5}$$

Note that eq. (5) could have been derived directly by considering the right triangle CQP. In particular, $CP = r$ and $QP = x$, so by Pythagoras' theorem, $CQ = \sqrt{r^2 - x^2}$. Then eq. (5) immediately follows. Inserting the results of eqs. (4) and (5) in eq. (1), and simplifying the resulting algebraic expression (exercise: verify this assertion), one ends up with:

$$f = r\left(1 - \frac{r}{2\sqrt{r^2 - x^2}}\right)\,.$$

Since $\sqrt{r^2 - x^2} = r\cos\theta$ [again making use of the triangle CQP], we again reproduce eq. (3).

3. The parabolic mirror

Consider a parabola that is described by the equation

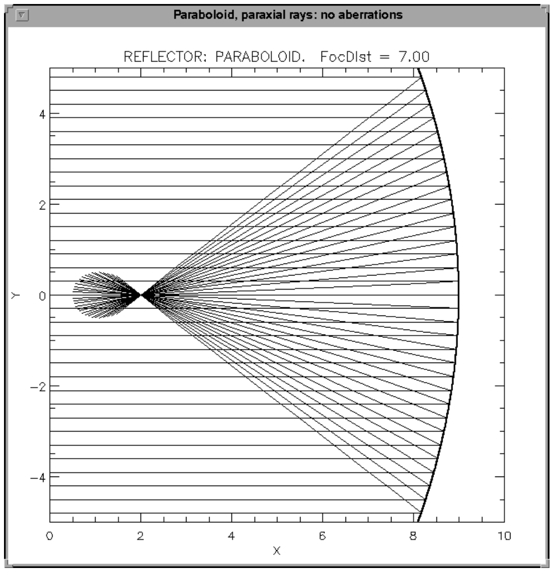
$$y = Ax^2 \,,$$

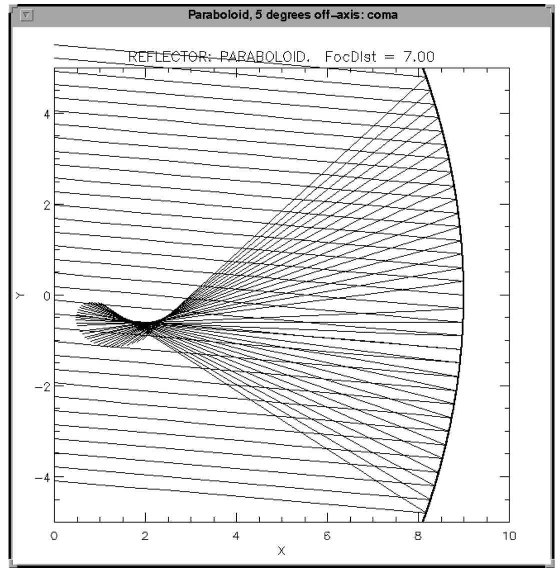
for some positive constant A. Then $dy/dx = 2Ax$. Inserting these results into eq. (1) gives the following result for the focal length:

$$f = Ax^2 + \frac{x(1 - 4A^2x^2)}{4Ax} = \frac{1}{4A}\,. \tag{6}$$

Thus, indeed the focal length f is *independent* of x. That is, *all* light rays that are initially parallel to the y-axis (i.e. the symmetry axis of the parabola) pass through the focal point F after reflecting off the mirror. No small angle approximation is necessary in this case.

The requirement that the initial light rays should be parallel to the symmetry axis of the parabola is critical. One can show that if the initial light rays are parallel to each other but are not parallel to the symmetry axis of the parabola (sometimes called off-axis parallel rays), then the reflected rays are not focused to a unique focal point. These results are illustrated by two figures below [taken from `http://www.astro.virginia.edu/class/oconnell/astr511/lec16-teloptics-f03.html`], which provide a graphic illustration of the focusing properties of a parabolic mirror with respect to on-axis and off-axis parallel light rays. Note that the reflected on-axis rays all converge to a single focal point as indicated by eq. (6). In contrast, the reflected off-axis rays do not quite converge to a single focal point.





4. Uniqueness of exact focusing of on-axis parallel light rays by a parabolic mirror

Referring back to the figure on the first page of these notes, we shall determine the curve $y(x)$ that has the property that all incoming on-axis parallel rays (i.e. rays that are parallel to the y-axis) pass through the focal point F after reflection off the mirror. That is, for any on-axis incoming light ray that strikes the mirror at *any* point P on the curve, $y(x)$, the reflected ray must pass through the point F. This means that the value of f is independent of the coordinate value x of the point P. Thus, starting from eq. (1), we impose the condition that f is independent of x. This is equivalent to the requirement that $df/dx = 0$. Recalling that $\tan\theta = dy/dx$, we can rewrite eq. (1) as:

$$f = y + \frac{x}{2\,dy/dx}\left[1 - \left(\frac{dy}{dx}\right)^2\right].$$

Taking the derivative of this equation and setting $df/dx = 0$, we get a fairly complicated looking expression:

$$\frac{dy}{dx} + \frac{1}{2(dy/dx)^2}\left\{\frac{dy}{dx}\left[1 - \left(\frac{dy}{dx}\right)^2 - 2x\frac{dy}{dx}\frac{d^2y}{dx^2}\right] - x\left[1 - \left(\frac{dy}{dx}\right)^2\right]\frac{d^2y}{dx^2}\right\} = 0\,.$$

Multiplying both sides of the equation by $-2(dy/dx)^2$ and simplifying the resulting expression yields:

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^3 = 0\,.$$

Remarkably, this result factors nicely as follows:

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]\left(x\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) = 0\,.$$

Since $1 + (dy/dx)^2 \geq 1$, we can divide out by the first factor, and end up with a very simple differential equation:

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0\,. \tag{7}$$

The most general solution to this differential equation is given by

$$y = Ax^2 + C\,, \tag{8}$$

where A and C are arbitrary constants. You can verify this by substituting this solution back into eq. (7). A derivation of this result is given in the appendix below. Indeed, $y(x)$ is a parabola, whose symmetry axis coincides with the y-axis. Thus, the parabola is the unique curve such that all incoming on-axis parallel rays pass through the focal point F after reflection off the mirror.

Appendix: Solving the differential equation, eq. (7)

In this appendix, we indicate how to solve the differential equation given in eq. (7), which (after multiplication by x) can be rewritten as:

$$x^2\frac{d^2y}{dx^2} = x\frac{dy}{dx}\,. \tag{9}$$

First, we note the relation:

$$d\ln x = \frac{dx}{x}\,,$$

which implies that

$$x\frac{dy}{dx} = \frac{dy}{d\ln x}\,. \tag{10}$$

Then,

$$\frac{d^2y}{d(\ln x)^2} \equiv \frac{d}{d\ln x}\left(\frac{dy}{d\ln x}\right) = x\frac{d}{dx}\left(x\frac{dy}{dx}\right) = x\frac{dy}{dx} + x^2\,\frac{d^2y}{dx^2}\,, \tag{11}$$

where the last step makes use of the usual rule for the derivative of a product: $d(uv) = udv + vdu$. Using eqs. (10) and (11),

$$x^2\,\frac{d^2y}{dx^2} = \frac{d^2y}{d(\ln x)^2} - \frac{dy}{d\ln x}\,, \tag{12}$$

Thus, defining a new variable $t = \ln x$ (or equivalently, $x = e^t$), eqs. (10) and (12) imply that:

$$x\frac{dy}{dx} = \frac{dy}{dt}\,, \qquad x^2\,\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}\,, \qquad \text{for } x = e^t\,. \tag{13}$$

Using eq. (13), the differential equation, eq. (9), takes the following simple form:

$$\frac{d^2y}{dt^2} = 2\frac{dy}{dt}\,. \tag{14}$$

To solve eq. (14), we define $u = dy/dt$. Then, $du/dt = d^2y/dt^2$, so that eq. (14) reduces to:

$$\frac{du}{dt} = 2u\,. \tag{15}$$

The solution to this last equation is simple. Writing eq. (15) as $du/u = 2dt$, one simply integrates both sides of the equation to get $\ln u = 2t + \ln K$, where $\ln K$ is the integration constant. Hence, $\ln(u/K) = 2t$, which when exponentiated yields $u(t) = Ke^{2t}$. Since $u = dy/dt$, we can obtain $y(t)$ by integration. Hence, $y(t) = Ae^{2t} + C$, where $A \equiv \frac{1}{2}K$ and C is a second constant of integration. Finally, using $x = e^t$ we end up with:

$$y(x) = Ax^2 + C\,,$$

which is the most general solution announced in eq. (8).