Extending the symmetries of the generalized *CP*-symmetric 2HDM scalar potential to the Yukawa sector

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There are only six independent types of symmetry-constrained (renormalizable) scalar potentials in the two Higgs doublet model (2HDM). For example, the scalar sector symmetry known as $\mathbb{Z}_2 \otimes \Pi_2$, generated by the simultaneous application of two independent symmetries acting on the scalar fields, and the generalized *CP* symmetry known as GCP2 yield equivalent 2HDM scalar potentials. A similar situation arises for the scalar sector symmetries known as $\mathrm{U}(1) \otimes \Pi_2$ and GCP3, respectively. In this paper, we show that this "degeneracy" remains when the definitions of the corresponding symmetries are extended to the Yukawa sector with three quark generations. The proof involves the exploration of all possible extensions of the corresponding symmetries to the Yukawa sector, consistent with the phenomenological constraints of nonzero quark masses and a nontrivial quark mixing matrix. Moreover, we find that this result is a peculiarity of a Yukawa sector with three quark generations. In particular, with two quark generations, we find that models based on the extension of $\mathbb{Z}_2 \otimes \Pi_2$ to the Yukawa sector are inequivalent with those based on GCP2.

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I. INTRODUCTION

The discovery at LHC in 2012 of a neutral scalar (h_{125}) with $m_h \simeq 125$ GeV [1,2] was a major milestone in particle physics. In particular, the Standard Model (SM) of fundamental particles and their interactions posits the existence of an elementary $\mathrm{SU}(2)_L$ doublet of scalar fields. The spontaneous breaking of the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge symmetry generates masses for the gauge bosons, quarks, and charged leptons, while leaving one physical scalar degree of freedom—the Higgs boson. In the analysis of LHC data collected over the past 13 yrs, the observed properties of h_{125} are consistent with the predictions of the SM within the statistical uncertainties of the measurements [3,4]. Nevertheless, one cannot currently rule out the possibility that additional scalar particles exist with masses of order the electroweak scale. Indeed, in light of the nonminimality of the quark and lepton sectors of the SM, which exhibits three replicas (families/generations) of each fermion type, one might expect a nonminimal scalar sector as well. Therefore, it is of fundamental interest to ascertain the number of elementary scalars in nature.

One of the simplest extensions of the SM scalar sector is the two-Higgs doublet model (2HDM) [5]. In spite of its apparent simplicity, the 2HDM has a very rich and vast phenomenology, having been used, for example, to propose the origin of *CP* violation as a consequence of a spontaneously broken symmetry [6], to explain the baryon–antibaryon asymmetry [7], and to provide a plausible dark-matter candidate [8,9]. The diversity in its phenomenology is due in part to the fact that the most general 2HDM scalar potential initially consists of 14 real parameters, and the corresponding Higgs-quark Yukawa sector initially consists of 72 real parameters (prior to identifying the independent physical parameters of the model). Thus, as in any model of an extended Higgs sector, it is of central importance to impose additional global (discrete or

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continuous) symmetries to increase the predictability of the model by reducing the number of parameters, to avoid (tree-level) flavor-changing neutral couplings mediated by scalars [10,11], and/or to explain some relations among observables [12].

Many theoretical aspects of the 2HDM are now known. The impact of symmetries in the scalar sector has been studied in detail by many authors. Symmetries can be of two types: flavor symmetry transformations, which change a given scalar field into linear combinations of scalar fields (but not their complex conjugates), and generalized *CP* (GCP) symmetry transformations, which transform a given scalar field into linear combinations of complex conjugated scalar fields. It has been shown that in the 2HDM there are only six inequivalent symmetry-constrained (renormalizable) scalar potentials [13,14]. As an example, consider the symmetries

$$\mathbb{Z}_2:\ \Phi_1 \to \Phi_1, \qquad \Phi_2 \to -\Phi_2; \tag{1}$$

$$\Pi_2:\ \Phi_1 \leftrightarrow \Phi_2. \tag{2}$$

The scalar potential invariant under a $\mathbb{Z}_2 \otimes \Pi_2$ $[\mathrm{U}(1) \otimes \Pi_2]$ symmetry is the same scalar potential, although in a different scalar field basis, as the scalar potential invariant under a generalized *CP* symmetry, which is denoted by GCP2 (GCP3) [14–19]. These features can be understood by considering basis invariant quantities [15,20]. More recently, a more sophisticated method that employs novel invariant theory techniques has been shown to yield the same conclusions noted above [21–23].

The extension of flavor symmetries to the Yukawa sector was initiated with an examination of Abelian symmetries in Refs. [24–26]. Extensions of generalized *CP* transformations into the Yukawa sector of the 2HDM were considered in Ref. [27], where it was shown that there is only one model that is consistent with nonzero quark masses and a nondiagonal Cabibbo-Kobayashi-Maskawa (CKM) matrix.[1] However, there are subtleties involved in the study of symmetries. Indeed, when extending symmetries to the Yukawa sector, two different scalar symmetries that yield the same scalar potential can result in different Yukawa couplings [24]. For example, the scalar potential of the $\mathbb{Z}_3$-symmetric 2HDM coincides with the scalar potential of the $\mathbb{Z}_4$-symmetric 2HDM. Nevertheless, when extended to the Yukawa sector, the $\mathbb{Z}_3$-symmetric 2HDM and the $\mathbb{Z}_4$-symmetric 2HDM yield different Yukawa textures.[2]

Hence, the following two questions arise, which we propose to address in this paper:

(i) Is it possible to impose symmetries in the Yukawa sector in such a way that the resulting Yukawa matrices are both compatible with a $\mathbb{Z}_2 \otimes \Pi_2$ $[\mathrm{U}(1) \otimes \Pi_2]$ symmetry of the scalar sector and consistent with experimental observations?

(ii) Are the Yukawa sectors of the $\mathbb{Z}_2 \otimes \Pi_2$ $[\mathrm{U}(1) \otimes \Pi_2]$-symmetric 2HDM and the GCP2 [GCP3]-symmetric 2HDM equivalent, or is there a "removal of the degeneracy"?

In Sec. II, we introduce our notation, examine the consequences of flavor and of GCP symmetries, and review the six inequivalent symmetries that can be imposed on the 2HDM scalar potential. Our primary goal is to determine whether an extension of the $\mathbb{Z}_2 \otimes \Pi_2$ or $\mathrm{U}(1) \otimes \Pi_2$ symmetry to the Yukawa sector can yield a phenomenologically viable model. In Sec. IV, we show that all Yukawa coupling matrices arising in models with the $\mathbb{Z}_2 \otimes \Pi_2$ symmetry extended to the Yukawa sector with three quark generations must possess at least one massless quark. For the $\mathrm{U}(1) \otimes \Pi_2$ model, we show in Sec. V that only one case exists in which the Yukawa coupling matrices yield nonzero quark masses, mixing angles, and a *CP*-violating phase that can be compatible with experimental observations. Moreover, as shown in Sec. VI, under a change of scalar field basis, the corresponding Yukawa coupling matrices of the $\mathrm{U}(1) \otimes \Pi_2$ model coincide with those of the GCP3 model. Next we consider models with two quark generations that yield nonvanishing, nondegenerate quark masses and a nonzero Cabibbo angle. We show in Sec. VII that the $\mathbb{Z}_2 \otimes \Pi_2$ and GCP2 symmetries can be extended to the Yukawa sector with two quark generations. However, the corresponding Yukawa matrices obtained in $\mathbb{Z}_2 \otimes \Pi_2$ models with two generations are *not* compatible with those found for the Yukawa matrices of the GCP2 model. This result demonstrates that the two questions posed above are indeed relevant. Although the $\mathbb{Z}_2 \otimes \Pi_2$ and GCP2 models cannot be extended to the three-generation Yukawa sector in a way compatible with experiment, they can both be extended to the two-generation Yukawa sector. However, the extension of $\mathbb{Z}_2 \otimes \Pi_2$ and GCP2 to the Yukawa sector *yields inequivalent models*. That is, in this case the degeneracy of the two symmetries of the scalar potential is removed by the extension to the Yukawa sector. We outline our conclusions in Sec. VIII. For completeness, we have included some useful details in four Appendixes.

II. NOTATION

Consider the most general (renormalizable) 2HDM scalar potential V_H, parametrized by

[1] Very recently, this study has been generalized to the GCP-symmetric 3HDM with Yukawa interactions in Ref. [28].

[2] This is analogous to the usual quantum mechanical removal of degeneracies by the addition of a new term in the Hamiltonian.

$$\begin{aligned}V_H &= m_{11}^2\Phi_1^\dagger\Phi_1 + m_{22}^2\Phi_2^\dagger\Phi_2 - [m_{12}^2\Phi_1^\dagger\Phi_2 + \text{H.c.}]\\&\quad+\frac{1}{2}\lambda_1(\Phi_1^\dagger\Phi_1)^2+\frac{1}{2}\lambda_2(\Phi_2^\dagger\Phi_2)^2+\lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2)+\lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\\&\quad+\left[\frac{1}{2}\lambda_5(\Phi_1^\dagger\Phi_2)^2+\lambda_6(\Phi_1^\dagger\Phi_1)(\Phi_1^\dagger\Phi_2)+\lambda_7(\Phi_2^\dagger\Phi_2)(\Phi_1^\dagger\Phi_2)+\text{H.c.}\right],\end{aligned}\tag{3}$$

where H.c. stands for Hermitian conjugate; m_{12}^2, λ_5, λ_6, λ_7 are potentially complex parameters; and all other scalar potential parameters are real.

The scalar potential may also be written in a more compact notation as

$$V_H = Y_{ij}(\Phi_i^\dagger\Phi_j) + Z_{ij,k\ell}(\Phi_i^\dagger\Phi_j)(\Phi_k^\dagger\Phi_\ell),\tag{4}$$

where $i, j, k, \ell \in \{1, 2\}$ with an implicit sum over repeated indices and Hermiticity implies

$$Y_{ij} = Y_{ji}^*, \qquad Z_{ij,k\ell} \equiv Z_{k\ell,ij} = Z_{ji,\ell k}^*.\tag{5}$$

We assume that the parameters of the scalar potential have been chosen such that the minimization of V_H yields charge preserving scalar field vacuum expectation values (VEVs) $\langle\Phi_i\rangle = (0, v_i)^\mathsf{T}$. In the spirit of Refs. [24,27–29], we allow the VEVs v_i to take any complex value, consistent with possible soft-symmetry breaking terms that one might wish to add to the potential for phenomenological reasons. Note that

$$v^2 \equiv |v_1|^2 + |v_2|^2,\tag{6}$$

where $v \equiv (2\sqrt{2}G_F)^{-1/2} \simeq 174$ GeV is fixed by the value of the Fermi constant.

As for the quark Yukawa sector, it involves the n generations of left-handed quark doublets (q_L), right-handed down-type quarks (n_R), and right-handed up-type quarks (p_R). The Yukawa Lagrangian may be written in the interaction-eigenstate basis as (e.g., see Ref. [30])

$$\begin{aligned}-\mathscr{L}_\text{Y} &= \bar{q}_L[(\Gamma_1\Phi_1+\Gamma_2\Phi_2)n_R+(\Delta_1\tilde{\Phi}_1+\Delta_2\tilde{\Phi}_2)p_R]\\&\quad+\text{H.c.},\end{aligned}\tag{7}$$

where $\tilde{\Phi} \equiv i\tau_2\Phi^*$ [with $i\tau_2 \equiv \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$], and q_L, n_R, and p_R are n-component vectors in flavor space.[3] The $n\times n$ matrices Γ_i, Δ_i, contain the complex Yukawa couplings to the right-handed down-type quarks and up-type quarks, respectively. After spontaneous symmetry breaking, the quark mass terms appear as

$$-\mathscr{L}_\text{Y} \supset \bar{n}_L M_n n_R + \bar{p}_L M_p p_R + \text{H.c.},\tag{8}$$

where

$$M_n = \Gamma_1 v_1 + \Gamma_2 v_2,\tag{9}$$

$$M_p = \Delta_1 v_1^* + \Delta_2 v_2^*.\tag{10}$$

In general, the matrices M_n and M_p are not diagonal, corresponding to the fact that the interaction-eigenstate fermion fields n_L, n_R, p_L, and p_R are not mass-eigenstate fields. To obtain the physical fermion fields, we perform the transformations

$$\bar{n}_L = \bar{d}_L V_{dL}^\dagger, \qquad \bar{p}_L = \bar{u}_L V_{uL}^\dagger,\tag{11}$$

$$n_R = V_{dR} d_R, \qquad p_R = V_{uR} u_R,\tag{12}$$

where the unitary matrices V_{dL}, V_{dR}, V_{uL}, and V_{uR} are chosen such that

$$\text{diag}(m_d, m_s, m_b) = D_d = V_{dL}^\dagger M_n V_{dR},\tag{13}$$

$$\text{diag}(m_u, m_c, m_t) = D_u = V_{uL}^\dagger M_p V_{uR},\tag{14}$$

where the diagonal entries of D_d and D_u are real and non-negative (corresponding to the singular value decomposition of the mass matrices M_d and M_u, respectively), and d_L, d_R, u_L, and u_R are mass-eigenstate fermion fields. The basis change from interaction eigenstates to mass eigenstates in the left-handed quark sector yields the couplings of quarks of different generations to the W bosons that are governed by the unitary CKM matrix

$$V_\text{CKM} \equiv V_{uL}^\dagger V_{dL}.\tag{15}$$

We now define the Hermitian matrices

$$H_d = M_n M_n^\dagger, \qquad H_u = M_p M_p^\dagger.\tag{16}$$

Notice that these matrices are bilinear in left-handed spaces (d_L and u_L, respectively); effectively, the right-handed spaces have been traced over. It immediately follows that

$$D_d^2 = V_{dL}^\dagger H_d V_{dL}, \qquad D_u^2 = V_{uL}^\dagger H_u V_{uL}.\tag{17}$$

[3] In Secs. III–VI, we will take the number of quark generations to be $n = 3$. However, in Sec. VII and in Appendixes C and D, we will discuss some toy models with $n = 2$.

For definiteness, let us take the usual generation number $n = 3$. Then, several consequences follow. First, the quark masses (say, those of the down-type quarks) may be accessed by looking at the three invariants that are obtained from H_d, which can be taken to be either: (i) the three eigenvalues of H_d; (ii) $\mathrm{Tr}(H_d)$, $\det(H_d)$, and the third coefficient of the characteristic equation; or (iii) the traces of H_d, H_d^2, and H_d^3. Second, the elements of the CKM matrix can be accessed by beating H_d against H_u and taking traces. For example, one can find the four independent magnitudes of CKM matrix elements by calculating $\mathrm{Tr}(H_d H_u)$, $\mathrm{Tr}(H_d H_u^2)$, $\mathrm{Tr}(H_d^2 H_u)$, and $\mathrm{Tr}(H_d^2 H_u^2)$ [31,32]. This does not define the sign of the CP violating CKM phase, which can be inferred through the Jarlskog invariant J_{CP} [33–35]. Following Refs. [20,36],

$$J_{\mathrm{CP}} = \mathrm{Im}\,\{\mathrm{Tr}(H_u H_d H_u^2 H_d^2)\} = (m_t^2 - m_c^2)(m_t^2 - m_u^2)(m_c^2 - m_u^2)(m_b^2 - m_s^2)(m_b^2 - m_d^2)(m_s^2 - m_d^2)\, J_{\mathrm{CKM}}, \tag{18}$$

where

$$J_{\mathrm{CKM}} = |\,\mathrm{Im}\,(V_{\alpha a} V_{\beta b} V^*_{\alpha b} V^*_{\beta a})|, \tag{19}$$

for any choice of up-type and down-type quark flavor indices, $\alpha \neq \beta$ and $a \neq b$, respectively. Alternatively, one can note that [33,35,37]:

$$\mathrm{Tr}[H_u, H_d]^3 = 3 \det\,[H_u, H_d] = 6i\ \mathrm{Im}\,\{\mathrm{Tr}(H_u H_d H_u^2 H_d^2)\}. \tag{20}$$

Clearly, through the squared-mass prefactors, J_{CP} vanishes whenever two same-charge quark masses are degenerate. Moreover, J_{CP} vanishes (through J_{CKM}) if the CP-violating phase in the CKM matrix vanishes. Finally, J_{CP} vanishes (again through J_{CKM}) whenever the CKM matrix is block diagonal.

A. Basis transformations and flavor symmetries

Physical observables are independent of the choice of the basis for the scalar fields and fermion fields employed in the Higgs Lagrangian. Since the Lagrangian parameters are basis-dependent quantities, some care is needed in identifying the physical parameters of the theory. Moreover, the presence of a symmetry can impart physical significance to the parameters of a particular basis choice.

The Higgs Lagrangian specified in Sec. II was written in terms of fields Φ_i, q_L, n_R, and p_R. The most general *basis transformation* that preserves the form of the gauge covariant kinetic energy terms yields new scalar and fermion fields,

$$\Phi_i \to \Phi'_i = U_{ij}\Phi_j, \qquad q_L \to q'_L = U_L q_L, \tag{21}$$

$$n_R \to n'_R = U_{n_R} n_R, \qquad p_R \to p'_R = U_{p_R} p_R, \tag{22}$$

where U is an arbitrary 2×2 unitary matrix and U_L, U_{n_R}, U_{p_R} are arbitrary $n \times n$ unitary matrices (where n is the number of quark generations). With respect to the transformed basis of scalar and quark fields, the scalar potential parameters and VEVs likewise transform as

$$Y_{ij} \to Y'_{ij} = U_{ik} Y_{k\ell} U^*_{j\ell}, \tag{23}$$

$$Z_{ij,k\ell} \to Z'_{ij,k\ell} = U_{im} U_{ko} Z_{mn,op} U^*_{jn} U^*_{\ell p}, \tag{24}$$

$$v_i \to v'_i = U_{ij} v_j, \tag{25}$$

whereas the Yukawa matrices transform as

$$\Gamma_i \to \Gamma'_i = U_L \Gamma_j U^\dagger_{n_R} (U^\dagger)_{ji}, \tag{26}$$

$$\Delta_i \to \Delta'_i = U_L \Delta_j U^\dagger_{p_R} (U^{\mathsf{T}})_{ji}. \tag{27}$$

Since physical observables do not depend on the choice of basis, only basis invariant combinations of the Higgs Lagrangian parameters are physical [15,20] (which can be experimentally measured).

In contrast to basis transformations, consider the implications of a *flavor symmetry* transformation of fields. The flavor symmetry transformation groups are subgroups of the corresponding groups of basis transformations of scalar and fermion fields that leave the Higgs Lagrangian invariant. We shall denote the corresponding Higgs and fermion flavor symmetry transformations by

$$\Phi_i \to \Phi^S_i = S_{ij}\Phi_j, \qquad q_L \to q^S_L = S_L q_L, \tag{28}$$

$$n_R \to n^S_R = S_{n_R} n_R, \qquad p_R \to p^S_R = S_{p_R} p_R, \tag{29}$$

where S is a 2×2 unitary matrix and S_L, S_{n_R}, S_{p_R} are unitary $n \times n$ matrices such that the corresponding Higgs Lagrangian parameters are left invariant. The flavor symmetry corresponding to the transformation of scalar fields is sometimes called a Higgs family (HF) symmetry. If the Higgs Lagrangian is invariant under a HF symmetry transformation, then the scalar potential parameters defined in Eq. (4) satisfy

$$Y_{ij} = Y^S_{ij} = S_{ik} Y_{k\ell} S^*_{j\ell}, \tag{30}$$

$$Z_{ij,k\ell} = Z^S_{ij,k\ell} = S_{im} S_{kp} Z_{mn,pr} S^*_{jn} S^*_{\ell r}. \tag{31}$$

Likewise, if the Yukawa Lagrangian is invariant under the HF and quark flavor symmetry transformations, then the Yukawa matrices defined in Eq. (7) satisfy

$$\Gamma_i = S_L \Gamma_j S^\dagger_{n_R} (S^\dagger)_{ji}, \tag{32}$$

$$\Delta_i = S_L \Delta_j S^\dagger_{p_R} (S^T)_{ji}. \tag{33}$$

The HF symmetry is unbroken if $v_i = v_i^S$, and it is spontaneously broken if $v_i \neq v_i^S$, where

$$v_i^S \equiv S_{ij} v_j. \tag{34}$$

Note that the symmetry transformation matrices introduced in Eqs. (28) and (29) are defined with respect to a particular basis choice for the scalar and fermions fields. One can perform a basis transformation specified in Eqs. (21) and (22) to express the corresponding symmetry transformation matrices with respect to the new basis of scalar and fermion fields [24],

$$S' = USU^\dagger, \qquad S'_L = U_L S_L U_L^\dagger, \tag{35}$$

$$S'_{n_R} = U_{n_R} S_{n_R} U^\dagger_{n_R}, \qquad S'_{p_R} = U_{p_R} S_{p_R} U^\dagger_{p_R}. \tag{36}$$

When expressed in terms of the new basis fields, one can check that the corresponding basis-transformed parameters are invariant with respect to the symmetry transformations exhibited in Eqs. (35) and (36). In studies of all possible inequivalent symmetries, it is often useful to employ a basis corresponding to the choice of unitary matrices U, U_L, U_{n_R}, and U_{p_R} such that S', S'_L, S'_{n_R}, and S'_{p_R} are diagonal matrices.

B. GCP symmetries

The flavor symmetries discussed in the previous section are not the only type of symmetries that leave the gauge covariant kinetic terms invariant. In addition, one can also consider GCP symmetries that transform the fields into linear combinations of the corresponding CP conjugate fields. These symmetries act on the scalar and fermion fields as

$$\Phi_i(t,\vec{x}) \to \Phi_i^{\rm GCP}(t,\vec{x}) = X_{ij}\Phi_j^*(t,-\vec{x}), \tag{37}$$

$$Q_L(t,\vec{x}) \to Q_L^{\rm GCP}(t,\vec{x}) = X_L \gamma^0 C \bar{Q}_L^{\mathsf{T}}(t,-\vec{x}), \tag{38}$$

$$n_R(t,\vec{x}) \to n_R^{\rm GCP}(t,\vec{x}) = X_{n_R} \gamma^0 C \bar{n}_R^{\mathsf{T}}(t,-\vec{x}), \tag{39}$$

$$p_R(t,\vec{x}) \to p_R^{\rm GCP}(t,\vec{x}) = X_{p_R} \gamma^0 C \bar{p}_R^{\mathsf{T}}(t,-\vec{x}), \tag{40}$$

where X, X_L, X_{n_R}, and X_{p_R} are generic unitary matrices in the respective flavor spaces, γ^0 is a Dirac matrix, and C is the charge conjugation matrix. Henceforth, we will suppress the reference to the spacetime coordinates.

Invariance of the Higgs Lagrangian under the GCP transformations exhibited in Eqs. (37)–(40) implies that

$$Y^*_{ij} = X^*_{ki} Y_{k\ell} X_{\ell j} = (X^\dagger Y X)_{ij}, \tag{41}$$

$$Z^*_{ij,k\ell} = X^*_{mi} X^*_{pk} Z_{mn,pr} X_{nj} X_{r\ell}, \tag{42}$$

$$\Gamma^*_j = X_L^\dagger X_{ij} \Gamma_i X_{n_R}, \tag{43}$$

$$\Delta^*_j = X_L^\dagger X^*_{ij} \Delta_i X_{p_R}. \tag{44}$$

One can again perform a basis transformation specified in Eqs. (21) and (22) to express the corresponding GCP symmetry transformation matrices with respect to the new basis of scalar and fermion fields,

$$X' = UXU^{\mathsf{T}}, \qquad X'_L = U_L X_L U_L^{\mathsf{T}}, \tag{45}$$

$$X'_{n_R} = U_{nR} X_{n_R} U^{\mathsf{T}}_{nR}, \qquad X'_{p_R} = U_{pR} X_{p_R} U^{\mathsf{T}}_{pR}. \tag{46}$$

In the special cases where the unitary X matrices are also symmetric, then one can prove that a unitary matrix V exists such that $X = V^{\mathsf{T}} V$ (e.g., see Appendix B of Ref. [38]). Then, the basis choice of $U = V$ yields $X' = \mathbb{1}$ (the identity matrix), with a similar result for X'_L, X'_{n_R}, and X'_{p_R}. In these special cases, the GCP transformations reduce to ordinary CP transformations. More generally, the unitary X matrices are not symmetric, in which case one cannot employ a basis (corresponding to a choice of unitary matrices U, U_L, U_{n_R}, and U_{p_R}) in which the transformed X matrices are diagonal. However, it is still possible to reduce the general form of the X matrices by using a theorem proved in Ref. [39], which states that for any unitary matrix X, there exists a unitary matrix U such that the transformed X matrices above can be reduced to the forms

$$X' = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix}, \qquad X'_\sigma = \begin{pmatrix} c_{\theta_\sigma} & s_{\theta_\sigma} & 0 \\ -s_{\theta_\sigma} & c_{\theta_\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad \sigma \in \{L, n_R, p_R\}, \tag{47}$$

where $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$, and all angles lie in the closed interval $[0, \pi/2]$. This result is very useful in classifying the GCP symmetries and in studying their effects in the Yukawa sector.

Alternatively, one can start from Eq. (47) and make a further basis transformation using Eqs. (21) and (22) with

$$U = \frac{e^{3i\pi/4}}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

$$U_L = U_{n_R} = U_{p_R} = \frac{e^{3i\pi/4}}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & \sqrt{2}e^{-3i\pi/4} \end{pmatrix}, \quad (48)$$

to obtain the corresponding GCP symmetry transformation matrices

$$X'' = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}, \qquad X''_\sigma = \begin{pmatrix} 0 & e^{-i\theta_\sigma} & 0 \\ e^{i\theta_\sigma} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\text{with} \quad \sigma \in \{L, n_R, p_R\}. \quad (49)$$

Note that the basis choices introduced above can be made independently on each of the four spaces (i.e., the scalar, q_L, n_R, and p_R spaces).

C. The inequivalent symmetries of the 2HDM potential

Consider the most general renormalizable 2HDM scalar potential V_H given in Eq. (3). To reduce the number of scalar potential parameters, one can impose symmetries on the Higgs Lagrangian that leave the gauge covariant scalar and fermion kinetic terms unchanged. These symmetries can be classified according to two different types: flavor symmetries exhibited in Eqs. (28) and (29), or GCP symmetries exhibited in Eqs. (37)–(40). Moreover, contrary to what one might expect, imposing two different symmetries on the scalar fields does not necessarily give rise to two different scalar potentials. As shown in Refs. [14,40,41], there are only six inequivalent symmetries that can be imposed on the scalar potential which leave invariant the gauge covariant kinetic energy terms of the scalar fields. Of these six inequivalent symmetries, three are HF symmetries and three are GCP symmetries. The three HF symmetries are $\mathbb{Z}_2$, the Peccei-Quinn symmetry U(1) [42], and the maximal Higgs flavor symmetry $U(2)/U(1)_Y$. With respect to a certain basis, the HF symmetries act on the scalar fields as[4]

$$\mathbb{Z}_2\colon\ \Phi_1 \to \Phi_1, \qquad \Phi_2 \to -\Phi_2, \quad (50)$$

$$U(1)\colon \Phi_1 \to \Phi_1, \quad \Phi_2 \to e^{i\theta}\Phi_2, \quad 0 < \theta < 2\pi, \quad (51)$$

$$U(2)/U(1)_Y\colon \Phi_a \to \Phi_a^S = S_{ab}\Phi_b, \ \text{with}\ S \in U(2)/U(1)_Y. \quad (52)$$

[4]In light of electroweak gauge invariance, the scalar potential is invariant under a hypercharge $\mathrm{U}(1)_Y$ transformation $\Phi_i \to e^{i\theta}\Phi_i$ (for $i = 1, 2$) for *arbitrary* choice of scalar potential parameters. Thus, we remove this symmetry from the definition of the maximal U(2) flavor symmetry in Eq. (52).

The three GCP symmetries are the standard *CP* transformation (sometimes called GCP1), GCP2, and GCP3. Their action on the scalar fields is given by

$$\text{Standard}\, CP\colon\ \Phi_1 \to \Phi_1^*, \qquad \Phi_2 \to \Phi_2^*, \quad (53)$$

$$\text{GCP2}\colon\ \Phi_1 \to \Phi_2^*, \qquad \Phi_2 \to -\Phi_1^*, \quad (54)$$

$$\text{GCP3}\colon \begin{cases} \Phi_1 \to c_\theta\Phi_1^* + s_\theta\Phi_2^* \\ \Phi_2 \to c_\theta\Phi_2^* - s_\theta\Phi_1^* \end{cases}, \qquad 0 < \theta < \frac{1}{2}\pi. \quad (55)$$

In the case of GCP3, any choice of $0 < \theta < \frac{1}{2}\pi$ imposes the same conditions on the scalar potential parameters.

In principle, any subgroup of U(2) provides a possible HF symmetry that can be imposed on the 2HDM scalar potential. But, any such symmetry must be equivalent to $\mathbb{Z}_2$, U(1), or $\mathrm{U}(2)/\mathrm{U}(1)_Y$. For example, suppose one imposes a discrete symmetry $\mathbb{Z}_n$ (with integer $n \geq 3$) on the scalar potential. Then, one obtains a scalar potential that is in fact (accidentally) invariant under the full continuous U(1) symmetry.

In this work, we will consider yet another HF symmetry, Π_2, defined as

$$\Pi_2\colon\ \Phi_1 \leftrightarrow \Phi_2. \quad (56)$$

However, the Π_2 symmetry is equivalent to the $\mathbb{Z}_2$ symmetry specified in Eq. (50) after performing a change of scalar field basis [15]. In particular, with U given by

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (57)$$

Eq. (35) implies that the Π_2 symmetry transformation matrix changes to

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (58)$$

which we recognize as the $\mathbb{Z}_2$ symmetry transformation matrix.

If we now impose the symmetries discussed above in the scalar field basis $\{\Phi_1, \Phi_2\}$ where they are written as Eqs. (50)–(56), we obtain the constraints on the parameters of the scalar potential listed in Table I.

So far, we have only considered the case where we impose symmetries on the scalar sector Lagrangian with one symmetry generator, dubbed simple symmetries in Ref. [43]. However, one could also require the scalar potential to be invariant under multiple symmetries in the (Φ_1, Φ_2) basis. For example, consider a potential invariant under $\mathbb{Z}_2$ or U(1). One can, in addition, impose *in the same basis* that the scalar potential is also symmetric

TABLE I. Classification of 2HDM scalar potential symmetries [13,14] defined in Eqs. (50)–(55) and their impact on the parameters of the scalar potential with respect to the basis $\{\Phi_1, \Phi_2\}$ defined in Eq. (3). Empty entries correspond to a lack of constraints on the corresponding parameters. Note that Π_2 [defined in Eq. (56)], $\mathbb{Z}_2 \otimes \Pi_2$, and $\mathrm{U}(1) \otimes \Pi_2$ are not independent from other symmetry conditions, since a change of scalar field basis can be performed in each case to yield a new basis in which the $\mathbb{Z}_2$, GCP2, and GCP3 symmetries, respectively, are manifestly realized.

Symmetry	m_{22}^2	m_{12}^2	λ_2	λ_4	Reλ_5	Im λ_5	λ_6	λ_7
$\mathbb{Z}_2$		0					0	0
U(1)		0			0	0	0	0
$\mathrm{U}(2)/\mathrm{U}(1)_Y$	m_{11}^2	0	λ_1	$\lambda_1 - \lambda_3$	0	0	0	0
CP real			0	Real	Real			
GCP2	m_{11}^2	0	λ_1					$-\lambda_6$
GCP3	m_{11}^2	0	λ_1		$\lambda_1 - \lambda_3 - \lambda_4$	0	0	0
Π_2	m_{11}^2	Real	λ_1			0		λ_6^*
$\mathbb{Z}_2 \otimes \Pi_2$	m_{11}^2	0	λ_1			0	0	0
$\mathrm{U}(1) \otimes \Pi_2$	m_{11}^2	0	λ_1		0	0	0	0

under the action of another simple symmetry such as Π_2. In such cases, we say that the potential is invariant under $\mathbb{Z}_2 \otimes \Pi_2$ or $\mathrm{U}(1) \otimes \Pi_2$, respectively.[5] Furthermore, it follows from the analysis in Ref. [14], that both of these must be equivalent to one of the six symmetries presented above. In particular, concerning the impact on the scalar potential, $\mathbb{Z}_2 \otimes \Pi_2$ was shown to be equivalent to GCP2 and $\mathrm{U}(1) \otimes \Pi_2$ was shown to be equivalent to GCP3. To be more precise, there is a change of basis that can take the potential invariant under $\mathbb{Z}_2 \otimes \Pi_2$ to the expression of the potential invariant under GCP2, and similarly for $\mathrm{U}(1) \otimes \Pi_2$ and GCP3. The specific basis choices that relate these symmetry relations can be found in Ref. [19].

From a phenomenological point of view, one must be careful in imposing a U(1), $\mathrm{U}(2)/\mathrm{U}(1)_Y$, or GCP3 symmetry on the scalar potential. In particular, if any of these symmetries are spontaneously broken, then the scalar spectrum will contain an unwanted massless scalar [43]. In such cases, one will need to softly break the corresponding symmetry with dimension-two squared-mass terms to give a phenomenologically acceptable mass to the would-be Goldstone boson.

III. EXTENSIONS OF SYMMETRIES TO THE YUKAWA SECTOR

So far, we have only considered and classified the effect of symmetries in the scalar sector. Now, we will be interested in determining how these symmetries can be extended to the Yukawa sector.

We will start by considering flavor symmetries. Recall from Sec. II A that requiring invariance under this type of symmetries implies that the Yukawa matrices must satisfy Eqs. (32) and (33). On the other hand, if we consider GCP symmetries, we require that the Lagrangian is invariant under the transformations in Eqs. (37)–(40), which in turn means that the Yukawa matrices must satisfy Eqs. (41)–(44). What we then mean by extending the symmetry S (X) of the scalar sector to the Yukawa sector is to find a set $\{S_L, S_{nR}, S_{pR}\}$ ($\{X_L, X_{n_R}, X_{p_R}\}$) such that the Yukawa couplings are compatible with experimental observations; i.e., (i) non-vanishing quark masses and (ii) a non zero Jarlskog invariant J_{CP}. Recall from Eqs. (18)–(20) that the latter means that there exists a nonzero CP-violating CKM phase *and* the CKM matrix is not block diagonal. The first condition is equivalent to requiring

$$\det H_d \neq 0, \qquad \det H_u \neq 0. \tag{59}$$

The second condition can be directly extracted from

$$\det\{[H_u, H_d]\} \neq 0. \tag{60}$$

In 2010, Ferreira and Silva [27] performed a complete study of the extensions of GCP symmetries to the Yukawa sector and have found two interesting results. First, they proved that GCP2 cannot be extended to the Yukawa sector in a way compatible with the two criteria discussed above. Second, they proved that there was only one possible extension of the GCP3 symmetry to the fermions, with the angles in the reduced form of Eq. (47) given by $\theta = \theta_L = \theta_{n_R} = \pi/3$. In this model, the corresponding down-type quark Yukawa matrices were given by

$$\Gamma_1 = \begin{pmatrix} ia_{11} & ia_{12} & a_{13} \\ ia_{12} & -ia_{11} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} ia_{12} & -ia_{11} & -a_{23} \\ -ia_{11} & -ia_{12} & a_{13} \\ -a_{32} & a_{31} & 0 \end{pmatrix}, \tag{61}$$

where the a_{ij} are real parameters.

This result prompts us to ask the following question: Can the symmetries $\mathbb{Z}_2 \otimes \Pi_2$ and $\mathrm{U}(1) \otimes \Pi_2$ be extended to the quark sector in a way compatible with the two conditions discussed above? And, if so, how do they relate to the extensions of GCP2 and GCP3? This is what we propose to address in the following sections.

IV. $\mathbb{Z}_2 \otimes \Pi_2$ FOR THREE GENERATIONS

We begin our analysis by considering how the symmetry $\mathbb{Z}_2 \otimes \Pi_2$ should act on the quark fields such that the Yukawa Lagrangian is invariant with respect to $\mathbb{Z}_2 \otimes \Pi_2$ transformations of the scalars and quarks. In their study of

[5] Because we are requiring the symmetries to be imposed on the same basis, there is no unitary change of basis one can make that simultaneously diagonalizes both the generator of $\mathbb{Z}_2$ [or U(1)] and that of Π_2.

the extension of Abelian symmetries to the Yukawa sector, the authors of Ref. [24] found all possible extensions of the $\mathbb{Z}_2$ symmetry to the fermions. For the convenience of the reader, the list given in Ref. [24] of all possible extensions of the $\mathbb{Z}_2$ symmetry to the down-type quark Yukawa couplings is provided in Table V of Appendix A. The various models listed there correspond to choosing the $\mathbb{Z}_2$ symmetry matrices $S^{(\mathbb{Z}_2)} = \mathrm{diag}\{1, -1\}$ while surveying over possible choices for the $\mathbb{Z}_2$ symmetry matrices $S_L^{(\mathbb{Z}_2)}$ and $S_{n_R}^{(\mathbb{Z}_2)}$ such that the $\mathbb{Z}_2$ symmetry equations given in Eqs. (30) and (32) are satisfied. For example, the Models 67, 71, and 73 of Ref. [24][6] can be obtained by choosing[7]

$$\text{Case I}: \; S^{(\mathbb{Z}_2)} = \mathrm{diag}\{1, -1\}, \qquad S_L^{(\mathbb{Z}_2)} = \mathbb{1}, \qquad S_{n_R}^{(\mathbb{Z}_2)} = \mathrm{diag}\{1, 1, -1\}, \tag{62}$$

$$\text{Case II}: \; S^{(\mathbb{Z}_2)} = \mathrm{diag}\{1, -1\}, \qquad S_L^{(\mathbb{Z}_2)} = \mathrm{diag}\{1, 1, -1\}, \qquad S_{n_R}^{(\mathbb{Z}_2)} = \mathbb{1}, \tag{63}$$

$$\text{Case III}: \; S^{(\mathbb{Z}_2)} = \mathrm{diag}\{1, -1\}, \qquad S_L^{(\mathbb{Z}_2)} = S_{n_R}^{(\mathbb{Z}_2)} = \mathrm{diag}\{1, 1, -1\}, \tag{64}$$

where $\mathbb{1}$ is the 3×3 identity matrix. These choices yield the following forms for the down-type quark Yukawa coupling matrices:

$$\text{Case I}: \; \Gamma_1 = \begin{pmatrix} \mathrm{x} & \mathrm{x} & 0 \\ \mathrm{x} & \mathrm{x} & 0 \\ \mathrm{x} & \mathrm{x} & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 & \mathrm{x} \\ 0 & 0 & \mathrm{x} \\ 0 & 0 & \mathrm{x} \end{pmatrix}, \tag{65}$$

$$\text{Case II}: \; \Gamma_1 = \begin{pmatrix} \mathrm{x} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \end{pmatrix}, \tag{66}$$

$$\text{Case III}: \; \Gamma_1 = \begin{pmatrix} \mathrm{x} & \mathrm{x} & 0 \\ \mathrm{x} & \mathrm{x} & 0 \\ 0 & 0 & \mathrm{x} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 & \mathrm{x} \\ 0 & 0 & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & 0 \end{pmatrix}, \tag{67}$$

where x stands for an arbitrary complex number. These three cases will be of particular interest in the analysis that follows.

With the list all possible extensions of the $\mathbb{Z}_2$ symmetry to the Yukawa sector in hand, the problem of finding the extensions of $\mathbb{Z}_2 \otimes \Pi_2$ to the Yukawa sector is equivalent to imposing the Π_2 symmetry equations on the Yukawa coupling matrices listed in Appendix A,

$$\Gamma_1 = S_L \Gamma_2 S_{nR}^\dagger, \qquad \Gamma_2 = S_L \Gamma_1 S_{nR}^\dagger, \tag{68}$$

$$\Delta_1 = S_L \Delta_2 S_{pR}^\dagger, \qquad \Delta_2 = S_L \Delta_1 S_{pR}^\dagger, \tag{69}$$

after employing Eqs. (32) and (33) with $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where S_L, S_{nR}, and S_{pR} are arbitrary 3×3 unitary matrices.[8] We can now immediately exclude Models 66 and 69 of Appendix A, since for these two models one obtains $\Gamma_1 = \Gamma_2 = 0$ after imposing Eq. (68), which would imply vanishing quark masses. One can further reduce the number of inequivalent models by noting that the Π_2 symmetry that interchanges Φ_1 and Φ_2 retains the same form under the change of basis

$$\begin{pmatrix} \Phi_1' \\ \Phi_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \tag{70}$$

Hence, when imposing Π_2, it is equivalent to work in the original $\{\Phi_1, \Phi_2\}$-basis or in the transformed $\{\Phi_1', \Phi_2'\}$-basis. In particular, any two models that are related by the basis change specified in Eq. (70) will be affected by the Π_2 symmetry in the same way. Consequently, the elements of the model pairs $\{67, 68\}$, $\{71, 79\}$, and $\{73, 75\}$ can be viewed as equivalent models. We are therefore left with three models, $\{67, 71, 73\}$, that cannot be related by any family permutations, with corresponding down-type quark Yukawa matrices given by Eqs. (65)–(67) prior to imposing the Π_2 symmetry. We will treat the models corresponding to these three cases independently in the following.

[6]The model numbers refer to the corresponding equation numbers appearing in Ref. [24], which are replicated in Appendix A.

[7]There is some phase freedom in defining the symmetry matrices as the matrices employed in Eqs. (62)–(64) are not the unique choices that yield the Yukawa matrices exhibited in Eqs. (65)–(67). For example, in Eq. (62), one would also obtain Eq. (65) by choosing $S = \mathrm{diag}\{1, e^{i\theta}\}$, $S_L = e^{i\eta}\mathbb{1}$, and $S_{n_R} = e^{i\eta}\mathrm{diag}\{1, 1, e^{-i\theta}\}$ for $0 < \theta < 2\pi$ and $0 \le \eta < 2\pi$. Strictly speaking, the corresponding symmetry group is $\mathbb{Z}_2$ only in the case of $\theta = \frac{1}{2}\pi$. Nevertheless, the resulting constraint on the down-type quark Yukawa matrices yields Eq. (65) for *all* allowed values of θ and η.

[8]Note that when applying the Π_2 symmetry conditions, the matrices S_L, S_{nR}, and S_{pR} are taken to be arbitrary because the freedom of choosing a basis for the quarks is fixed, up to permutations, once the $\mathbb{Z}_2$ symmetry matrix $S^{(\mathbb{Z}_2)}$ is chosen to be diagonal. See Sec. (II.C) of Ref. [24] for further details.

Starting from the basis where Γ_1 and Γ_2 take the forms shown in Eqs. (65)–(67), we now impose the Π_2 symmetry, which adds the additional constraints exhibited in Eq. (68). The impact of the additional constraints can be determined by employing the following analysis. Consider the quantity

$$H(c_1,c_2) \equiv (c_1\Gamma_1 + c_2\Gamma_2)(c_1\Gamma_1 + c_2\Gamma_2)^\dagger = |c_1|^2\Gamma_1\Gamma_1^\dagger + c_1c_2^*\Gamma_1\Gamma_2^\dagger + c_1^*c_2\Gamma_2\Gamma_1^\dagger + |c_2|^2\Gamma_2\Gamma_2^\dagger, \tag{71}$$

where c is an arbitrary complex number. In light of Eq. (68),

$$\Gamma_1\Gamma_1^\dagger = S_L(\Gamma_2\Gamma_2^\dagger)S_L^\dagger, \qquad \Gamma_1\Gamma_2^\dagger = S_L(\Gamma_2\Gamma_1^\dagger)S_L^\dagger, \tag{72}$$

$$\Gamma_2\Gamma_1^\dagger = S_L(\Gamma_1\Gamma_2^\dagger)S_L^\dagger, \qquad \Gamma_2\Gamma_2^\dagger = S_L(\Gamma_1\Gamma_1^\dagger)S_L^\dagger. \tag{73}$$

Inserting these results back into Eq. (71), we see that under the Π_2 symmetry $H(c_1,c_2)$ is transformed into

$$H'(c_1,c_2) \equiv S_L H(c_1,c_2) S_L^\dagger = |c_2|^2\Gamma_1\Gamma_1^\dagger + c_1^*c_2\Gamma_1\Gamma_2^\dagger + c_1c_2^*\Gamma_2\Gamma_1^\dagger + |c_1|^2\Gamma_2\Gamma_2^\dagger = H(c_2,c_1). \tag{74}$$

Note that $\det H'(c_1,c_2) = \det H(c_1,c_2)$ since S_L is a unitary matrix.

In both Case I [Eq. (65)] and Case II [Eq. (66)], an explicit computation of the determinants of the right-hand sides of Eqs. (71) and (74) yields

$$\det H(c_1,c_2) = |c_1|^4|c_2|^2|\det(\Gamma_1+\Gamma_2)|^2, \tag{75}$$

$$\det H'(c_1,c_2) = |c_1|^2|c_2|^4|\det(\Gamma_1+\Gamma_2)|^2. \tag{76}$$

Consequently,

$$\det H(c_1,c_2) - \det H'(c_1,c_2) = |c_1|^2|c_2|^2(|c_1|^2 - |c_2|^2)|\det(\Gamma_1+\Gamma_2)|^2 = 0. \tag{77}$$

Noting that a multinomial (in $|c_1|$ and $|c_2|$) is zero if and only if all of its coefficients are zero, it follows that $\det(\Gamma_1+\Gamma_2) = 0$. That is, the impact of the Π_2 symmetry is to impose the condition $\det(\Gamma_1+\Gamma_2) = 0$ on the matrices specified in Eqs. (65) and (66). Inserting this result back into Eq. (75), we obtain $\det H(c_1,c_2) = 0$. Finally, using Eqs. (9) and (16), it follows that

$$\det H_d = \det[(\Gamma_1 v_1 + \Gamma_2 v_2)(\Gamma_1 v_1 + \Gamma_2 v_2)^\dagger] = \det H(v_1,v_2) = 0, \tag{78}$$

which implies that at least one of the down-type quarks is massless.

In Case III [Eq. (67)], an explicit calculation yields

$$\det H(c_1,c_2) = |c_1|^2|c_1^2\det\Gamma_1 + c_2^2\det(\tilde{\Gamma}_1+\Gamma_2)|^2, \tag{79}$$

$$\det H'(c_1,c_2) = |c_2|^2|c_2^2\det\Gamma_1 + c_1^2\det(\tilde{\Gamma}_1+\Gamma_2)|^2, \tag{80}$$

where $\tilde{\Gamma}_1$ is obtained from Γ_1 by setting $(\Gamma_1)_{33} = 0$. That is,

$$\tilde{\Gamma}_1 = \begin{pmatrix} \mathrm{x} & \mathrm{x} & 0 \\ \mathrm{x} & \mathrm{x} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{81}$$

Consequently,

$$\det H(c_1,c_2) - \det H'(c_1,c_2) = (|c_1|^6 - |c_2|^6)|\det\Gamma_1|^2 - |c_1|^2|c_2|^2(|v_1|^2 - |v_2|^2)|\det(\tilde{\Gamma}_1+\Gamma_2)|^2 + [c_1c_2(c_1^{*3}c_2 - c_2^{*3}c_1)\det\Gamma_1^*\det(\tilde{\Gamma}_1+\Gamma_2) + \mathrm{c.c.}] = 0, \tag{82}$$

where c.c. stands for complex conjugate of the preceding term. Since Eq. (82) is a multinomial in the variables c_1, c_1^*, c_2, and c_2^*, it is equal to zero if and only if all of its coefficients are zero. It follows that[9]

$$\det\Gamma_1 = \det(\tilde{\Gamma}_1+\Gamma_2) = 0. \tag{83}$$

In light of Eqs. (78), (79), and (83) we again obtain $\det H_d = 0$. As in Case I and Case II, it follows that at least one of the down-type quarks is massless.

It is noteworthy that the analysis presented above does not require one to determine the exact form of the Yukawa matrices after the application of both the $\mathbb{Z}_2$ and Π_2 symmetries. We were able to exclude all cases based exclusively on basis invariant considerations. Since all the possible extensions of $\mathbb{Z}_2 \otimes \Pi_2$ to the Yukawa sector lead to at least one massless down-type quark, we conclude that *it is impossible to extend $\mathbb{Z}_2 \otimes \Pi_2$ to the fermions in a way compatible with experimental results*. Thus, in the 2HDM with three fermion generations, there is no difference between imposing GCP2 (as shown in Ref. [27]) and

[9]Note that $\det\Gamma_1 = 0$ implies that either $(\Gamma_1)_{33} = 0$ or $(\Gamma_1)_{11}(\Gamma_1)_{22} = (\Gamma_1)_{12}(\Gamma_1)_{21}$. In either case, $\det(\Gamma_1+\Gamma_2)$ is independent of the value of $(\Gamma_1)_{33} = 0$ due to the form of Γ_2 specified in Eq. (67). It then follows that $\det(\Gamma_1+\Gamma_2) = \det(\tilde{\Gamma}_1+\Gamma_2) = 0$. In particular, the impact of the Π_2 symmetry is to impose the conditions $\det\Gamma_1 = \det(\Gamma_1+\Gamma_2) = 0$ on the matrices specified in Eq. (67).

$\mathbb{Z}_2 \otimes \Pi_2$ on the Yukawa Lagrangian. In both cases, the corresponding models are phenomenologically unacceptable due to the presence of at least one massless down-type quark.

One can now examine the consequences of Eq. (69), which governs the up-type quark Yukawa couplings. The first step would be to determine the possible structures of Δ_1 and Δ_2 after imposing the $\mathbb{Z}_2$ symmetry constraints. Using Eq. (33), we would make use of the choices of $S^{(\mathbb{Z}_2)}$ and $S_L^{(\mathbb{Z}_2)}$ employed in Eqs. (62)–(64), while surveying all possible choices for $S_{pR}^{(\mathbb{Z}_2)}$. Of course, since we have already concluded that none of the models governed by Cases I–III are phenomenologically viable, there is no need to study further the possible extensions of the $\mathbb{Z}_2$ symmetry to the up-type quark sector.

V. EXTENDING $\mathrm{U}(1) \otimes \Pi_2$ FOR THREE GENERATIONS

In this section, we consider how the symmetry $\mathrm{U}(1) \otimes \Pi_2$ should act on the quark fields such that the Yukawa Lagrangian is invariant with respect to $\mathrm{U}(1) \otimes \Pi_2$ transformations of the scalars and quarks. Following a similar strategy to the one employed in Sec. IV, we begin with a list of all possible extensions of U(1) to the fermions obtained in Ref. [24], which we have organized in Tables VI–IX.

Examining the possible models, one can check that most of them are actually equivalent to or subcases of the extensions of $\mathbb{Z}_2$ analyzed before (up to permutations of the quark doublets and/or scalar doublets). This should not come as a surprise, as U(1) contains a $\mathbb{Z}_2$. However, one can also consider a discrete $\mathbb{Z}_3$ symmetry, which when applied to the scalar sector results in a U(1)-symmetric scalar potential. Extensions of $\mathbb{Z}_3$ to the fermions do not necessarily yield a Yukawa Lagrangian that is invariant under the $\mathbb{Z}_2$ symmetry previously analyzed (where the Yukawa matrices Γ_1 and Γ_2 are specified by one of the models of Table V). That is, not all models with a U(1)-symmetric scalar potential, when extended to the Yukawa sector, are equivalent to or subcases of the extensions of the $\mathbb{Z}_2$ symmetry analyzed previously. These are the models that we focus on in this section. Once again, we discard models that after imposing Eq. (68) yield $\Gamma_1 = \Gamma_2 = 0$. Then, there are only two inequivalent classes of models, $\{57, 92\}$, that cannot be related by any family permutations,

$$\text{Case IV:}\ \Gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} \mathrm{x} & \mathrm{x} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{x} \end{pmatrix}, \tag{84}$$

$$\text{Case V:}\ \Gamma_1 = \begin{pmatrix} \mathrm{x} & 0 & 0 \\ 0 & \mathrm{x} & 0 \\ 0 & 0 & \mathrm{x} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & \mathrm{x} & 0 \\ 0 & 0 & \mathrm{x} \\ \mathrm{x} & 0 & 0 \end{pmatrix}, \tag{85}$$

where x stands for an arbitrary complex number. Note that we have obtained Eqs. (84) and (85) by requiring that the Higgs Lagrangian is invariant under a $\mathbb{Z}_3$ Higgs and quark flavor symmetry transformation [the former resulting in a U (1)-symmetric scalar potential]. In particular, the $\mathbb{Z}_3$ symmetry equations [Eqs. (32) and (33)] have been applied with,[10]

$$\text{Case IV:}\ S^{(\mathbb{Z}_3)} = \mathrm{diag}\{\omega, 1\}, \qquad S_L^{(\mathbb{Z}_3)} = \mathrm{diag}\{1, \omega^2, \omega\}, \qquad S_{n_R}^{(\mathbb{Z}_3)} = \mathrm{diag}\{1, 1, \omega\}, \tag{86}$$

$$\text{Case V:}\ S^{(\mathbb{Z}_3)} = \mathrm{diag}\{1, \omega\}, \qquad S_L^{(\mathbb{Z}_3)} = S_{n_R}^{(\mathbb{Z}_3)} = \mathrm{diag}\{1, \omega^2, \omega\}, \tag{87}$$

where 1, ω, and ω^2 [with $\omega \equiv \exp(2i\pi/3)$] are the three cube roots of unity. We are now ready to find the corresponding extension of $\mathrm{U}(1) \otimes \Pi_2$ to the Yukawa interactions of the down-type quarks by imposing the Π_2 symmetry equations given in Eq. (68). We will treat the two models corresponding to Cases IV and V independently in the following section.

A. Check for massless quarks

We begin by applying to these cases the same test of looking at the effect of the Π_2 symmetry equations [Eq. (68)] on $\det(H_d)$. We again introduce $H(c_1, c_2)$ as in Eq. (71). Then, in Case IV we find that Eqs. (75)–(77) are satisfied, which implies that $\det(\Gamma_1 + \Gamma_2) = 0$. Hence, $\det H_d = H(v_1, v_2) = 0$, which implies the existence of at least one massless down-type quark.

In Case V [Eq. (85)], an explicit calculation yields

$$\det H(c_1, c_2) = |c_1^3 \det \Gamma_1 + c_2^3 \det \Gamma_2|^2, \tag{88}$$

$$\det H'(c_1, c_2) = |c_2^3 \det \Gamma_1 + c_1^3 \det \Gamma_2|^2. \tag{89}$$

[10] As noted in footnote 7 there is some additional phase freedom in defining the symmetry matrices exhibited in Eqs. (86) and (87). Without loss of generality, we have simplified the form of the symmetry matrices by setting such phases to zero.

Using

$$\det H(c_1,c_2)-\det H'(c_1,c_2)=(|c_1|^6-|c_2|^6)(|\det\Gamma_1|^2-|\det\Gamma_2|^2)-4\,\mathrm{Im}(c_1^3c_2^{*3})\,\mathrm{Im}(\det\Gamma_1\det\Gamma_2^*)=0, \tag{90}$$

it follows that

$$|\det\Gamma_1|=|\det\Gamma_2|,\qquad \mathrm{Im}(\det\Gamma_1\det\Gamma_2^*)=0, \tag{91}$$

which implies that $\det\Gamma_2=\pm\det\Gamma_1$. In particular,

$$\det H_d(v_1,v_2)=|v_1^3\pm v_2^3|^2|\det\Gamma_1|^2, \tag{92}$$

which is nonzero in general. Hence, Case V is compatible with the existence of nonzero down-type quark masses.

It is noteworthy that Eq. (92) has been derived under the assumption that a specific basis for the scalar fields and quark fields has been chosen. In particular, under a change basis of the scalar fields and quark fields, $H_d\to U_LH_dU_L^\dagger$, after making use of Eqs. (9), (16), (25), and (26). Consequently, the left-hand side of Eq. (92) is a manifestly basis independent quantity, whereas the right-hand side has been obtained in a specific basis. It is straightforward to show that one cannot flip the sign in Eq. (92) by the transformation $\Phi_2\to-\Phi_2$, since such a transformation also changes $\Gamma_2\to-\Gamma_2$ [cf. Eq. (26)], in which case $M_n=\Gamma_1v_1+\Gamma_2v_2$ (and likewise, $H_d=M_nM_n^\dagger$) are unchanged.

B. Case V

We now take a closer look at the down-type quark mass matrix in Case V and the corresponding constraints imposed by the Π_2 symmetry. We parametrize the down-type Yukawa matrices as

$$\Gamma_1=\begin{pmatrix}x_{11}&0&0\\0&x_{22}&0\\0&0&x_{33}\end{pmatrix},\qquad \Gamma_2=\begin{pmatrix}0&x_{12}&0\\0&0&x_{23}\\x_{31}&0&0\end{pmatrix}, \tag{93}$$

where the x_{ij} are complex numbers. Using the first relation given in Eq. (68), we obtain

$$\Gamma_1=\mathrm{diag}(x_{11},x_{22},x_{33})=S_L\Gamma_2S_{nR}^\dagger. \tag{94}$$

It follows that

$$\mathrm{diag}(|x_{11}|^2,|x_{22}|^2,|x_{33}|^2)=\Gamma_1\Gamma_1^\dagger=S_L(\Gamma_2\Gamma_2^\dagger)S_L^\dagger=S_L\mathrm{diag}(|x_{12}|^2,|x_{23}|^2,|x_{31}|^2)S_L^\dagger, \tag{95}$$

$$\mathrm{diag}(|x_{11}|^2,|x_{22}|^2,|x_{33}|^2)=\Gamma_1^\dagger\Gamma_1=S_{nR}(\Gamma_2^\dagger\Gamma_2)S_{nR}^\dagger=S_{nR}\mathrm{diag}(|x_{31}|^2,|x_{12}|^2,|x_{23}^2|)S_{nR}^\dagger. \tag{96}$$

Since $\Gamma_1\Gamma_1^\dagger$ and $\Gamma_2\Gamma_2^\dagger$ are related by a similarity transformation, they possess the same eigenvalues (and likewise for $\Gamma_1^\dagger\Gamma_1$ and $\Gamma_2^\dagger\Gamma_2$). Hence, S_L and S_{nR} are permutation matrices multiplied by a diagonal matrix of phases.

In general, a $n\times n$ permutation matrix P is a real orthogonal matrix that consists of matrix elements such that 1 appears exactly once in each row and column while the remaining n^2-n entries are 0. For an arbitrary $n\times n$ matrix A, the matrix AP permutes the columns of A and the matrix PA permutes the rows of A. For the $n!$ possible permutations corresponding to the symmetric group S_n, we shall employ the cycle notation [44]. In particular, S_3 is the group of permutation of three numbers $\{1,2,3\}$, whose elements are denoted by $\{Id,(12),(13),(23),(123),(132)\}$, where Id is the identity element, (12) corresponds to $1\leftrightarrow2$, $3\to3$; (123) corresponds to $1\to2$, $2\to3$, $3\to1$; etc. The corresponding permutation matrices will be denoted by P_g where $g\in S_3$. For example [44],

$$P_{(123)}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}. \tag{97}$$

One can check that $P_{(123)}(\Gamma_2\Gamma_2^\dagger)P_{(123)}^\mathsf{T}=\Gamma_2^\dagger\Gamma_2$. It then follows that

$$KS_L=S_{nR}P_{(123)}, \tag{98}$$

where K is a diagonal matrix of phases. Without loss of generality, we may choose S_{nR} to be a permutation matrix by absorbing any additional phases of S_{nR} into K.

Using the second equation of Eq. (68), the roles of Γ_1 and Γ_2 are interchanged. Thus, instead of Eq. (94), we now have $\Gamma_2=S_L\Gamma_1S_{nR}^\dagger$. Inserting this result back into Eq. (94) yields

$$\Gamma_1S_{nR}^2=S_L^2\Gamma_1. \tag{99}$$

We will see shortly that Eqs. (98) and (99) constrain the diagonal matrix of phases, K. Moreover, one can use the second equation of Eq. (68) to derive a relation analogous to Eq. (98), $(KS_L)^\dagger=S_{nR}^\dagger P_{(123)}$, which is equivalent to

$$KS_L=P_{(123)}^\mathsf{T}S_{nR}. \tag{100}$$

Combining Eqs. (98) and (100) yields

$$P_{(123)}S_{nR}P_{(123)}=S_{nR}. \tag{101}$$

Having defined S_{nR} to be a permutation matrix, one can easily check (using the S_3 group multiplication table) that only the odd permutation matrices,[11] $P_{(12)}$, $P_{(23)}$, and $P_{(13)}$, satisfy Eq. (101). Moreover, when S_{nR} is an odd permutation, KS_L given by Eq. (98) is also an odd permutation. Hence there are three possible cases:

(1) $S_{nR} = P_{(12)}$, and $KS_L = P_{(23)}$;
(2) $S_{nR} = P_{(23)}$, and $KS_L = P_{(13)}$;
(3) $S_{nR} = P_{(13)}$, and $KS_L = P_{(12)}$.

Note that these cases are related by a basis change of the quark fields where two of the generations are interchanged. Thus, without loss of generality, it suffices to consider one of the three models listed above.

Focusing on the third case above,

$$KS_L = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_{nR} = S_{nR}^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{102}$$

In light of Eq. (99), it follows that

$$K = \mathrm{diag}(e^{i\theta}, e^{-i\theta}, \pm 1), \tag{103}$$

where θ is arbitrary (mod 2π) and either choice of sign is allowed. Then, the right-hand side of Eq. (94) yields

$$\mathrm{diag}(x_{11}, x_{22}, x_{33}) = \mathrm{diag}(e^{-i\theta} x_{23}, e^{i\theta} x_{12}, \pm x_{31}). \tag{104}$$

Hence, Eqs. (93) and (104) yield

$$\Gamma_2 = \begin{pmatrix} 0 & e^{-i\theta} x_{22} & 0 \\ 0 & 0 & e^{i\theta} x_{11} \\ \pm x_{33} & 0 & 0 \end{pmatrix}. \tag{105}$$

Using Eqs. (9), (93), and (105), it follows that

$$M_n = v_1 \Gamma_1 + v_2 \Gamma_2 = \begin{pmatrix} v_1 x_{11} & v_2 x_{22} e^{-i\theta} & 0 \\ 0 & v_1 x_{22} & v_2 x_{11} e^{i\theta} \\ \pm v_2 x_{33} & 0 & v_1 x_{33} \end{pmatrix}, \tag{106}$$

and

$$\det H_d \equiv \det(M_n M_n^\dagger) = |v_1^3 \pm v_2^3|^2 |\det \Gamma_1|^2, \tag{107}$$

independently of the value of θ, in agreement with the result obtained in Eq. (92). As noted in the previous subsection, Eq. (107) generically implies that all down-type quark masses are nonzero,[12] as required for a viable candidate for a model that is invariant under a $\mathrm{U}(1) \otimes \Pi_2$ symmetry transformation.

One can now examine the consequences of Eq. (69), which governs the up-type quark Yukawa couplings. The first step would be to determine the possible structures of Δ_1 and Δ_2 after imposing the $\mathbb{Z}_3$ symmetry constraints. Using Eq. (33), we would make use of the choices of $S^{(\mathbb{Z}_3)}$ and $S_L^{(\mathbb{Z}_3)}$ employed in Eqs. (86) and (87), while surveying all possible choices for $S_{pR}^{(\mathbb{Z}_3)}$. If we now impose a Π_2 symmetry with S_{pR} given by S_{nR} in Eq. (102), we obtain Δ_1 and Δ_2 with the same textures as Γ_1 and Γ_2, respectively. This provides us with enough freedom to produce a model with nonzero CKM mixing angles and a nonvanishing J_{CP}. We conclude that *it is possible to extend $U(1) \otimes \Pi_2$ to the fermions in a way compatible with phenomenological constraints.*

VI. EQUIVALENCE OF $\mathrm{U}(1) \otimes \Pi_2$ AND GCP3 MODELS WITH THREE QUARK GENERATIONS

In Sec. II C, we recalled that the GCP3-symmetric and the $\mathrm{U}(1) \otimes \Pi_2$-symmetric scalar potentials were related by a change of scalar field basis [14]. Moreover, as noted in Sec. III, it was shown in Ref. [27] that there is only one possible extension of GCP3 to the fermions in a model with three generations. We also showed in Sec. V that it is possible to extend the $\mathrm{U}(1) \otimes \Pi_2$ symmetry of the scalar potential to the Yukawa sector. Hence, one can now pose the following question: do the Yukawa extended GCP3 model and the Yukawa extended $\mathrm{U}(1) \otimes \Pi_2$ model coincide?

In Ref. [14], it was shown that given a GCP3-invariant scalar potential $V(\Phi')$, expressed in the scalar field basis $\{\Phi'_1, \Phi'_2\}$, one can find a basis transformation,

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \Phi'_1 \\ \Phi'_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -ie^{i\lambda} & e^{i\lambda} \end{pmatrix} \begin{pmatrix} \Phi'_1 \\ \Phi'_2 \end{pmatrix}, \tag{108}$$

such that the scalar potential $V(\Phi)$, expressed in the scalar field basis $\{\Phi_1, \Phi_2\}$, is invariant with respect to $\mathrm{U}(1) \otimes \Pi_2$ transformations. In Eq. (108), we have introduced an arbitrary phase factor $e^{i\lambda}$ that will prove useful in the following. One can check that if the parameters of $V(\Phi')$ satisfy the GCP3 parameter relations specified in Table I, then the parameters of $V(\Phi)$ satisfy the $\mathrm{U}(1) \otimes \Pi_2$ parameter relations given in the same table.

Suppose that the down-type quark Yukawa matrices satisfy the GCP3 conditions specified in Eq. (61), which we shall henceforth denote by Γ'_1 and Γ'_2. Using the inverse of Eq. (26), we can perform basis transformations in both the

[11]The odd permutations are permutations that can be expressed as products of an odd number of 2-cycles. Thus, $\{Id, (123), (132)\}$ are even permutations whereas $\{(12), (23), (13)\}$ are odd permutations.

[12]In the special cases where $v_1^3 = \mp\, v_2^3$, the model will contain a massless down-type quark and thus must be excluded.

scalar and the Yukawa sectors to obtain the corresponding Yukawa matrices in the new basis (denoted by Γ_1 and Γ_2, respectively), which we will then compare with the results obtained in Sec. V. It then follows that

$$\Gamma_1 = U_L^\dagger \frac{1}{\sqrt{2}}(\Gamma_1' + i\Gamma_2')U_{nR}, \quad \Gamma_2 = U_L^\dagger \frac{e^{-i\lambda}}{\sqrt{2}}(i\Gamma_1' + \Gamma_2')U_{nR}. \tag{109}$$

In light of Eq. (93), we perform a singular value decomposition by choosing the matrices U_L and U_{nR} such that Γ_1 is diagonal:

$$U_L^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}, \quad U_{nR} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 0 & 1 \\ 1 & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \tag{110}$$

Equation (109) then yields

$$\begin{aligned} \Gamma_1 &= \mathrm{diag}(\sqrt{2}(a_{12} - ia_{11}), a_{13} - ia_{23}, a_{31} - ia_{32}) \\ &= \mathrm{diag}(x_{11}, x_{22}, x_{33}), \end{aligned} \tag{111}$$

where we have introduced x_{11}, x_{22}, and x_{33} to match the notation of Eq. (93), and

$$\begin{aligned} \Gamma_2 &= e^{-i\lambda}\begin{pmatrix} 0 & a_{13} + ia_{23} & 0 \\ 0 & 0 & -\sqrt{2}(a_{11} - ia_{12}) \\ a_{31} + ia_{32} & 0 & 0 \end{pmatrix} \\ &= e^{-i\lambda}\begin{pmatrix} 0 & x_{22}^* & 0 \\ 0 & 0 & ix_{11}^* \\ x_{33}^* & 0 & 0 \end{pmatrix}. \end{aligned} \tag{112}$$

Note that Γ_2 does not quite match the corresponding Yukawa matrix that appears in Eq. (93) after identifying its parameters using Eq. (104). However, we still have some phase freedom in defining the matrices U_L and U_R of the singular value decomposition. Indeed, if we make the following replacements:

$$\begin{aligned} U_L^\dagger &\to \mathrm{diag}(e^{-i\phi_1}, e^{-i\phi_2}, e^{-i\phi_3})U_L^\dagger, \\ U_R &\to U_R \mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}), \end{aligned} \tag{113}$$

then Eq. (111) is unmodified, whereas Γ_2 is transformed into

$$\Gamma_2 = \begin{pmatrix} 0 & e^{-i(\phi_{12}+\alpha_2+\lambda)}|x_{22}| & 0 \\ 0 & 0 & ie^{-i(\phi_{23}+\alpha_1+\lambda)}|x_{11}| \\ e^{-i(\phi_{31}+\alpha_3+\lambda)}|x_{33}| & 0 & 0 \end{pmatrix}, \tag{114}$$

where $\phi_{ij} \equiv \phi_i - \phi_j$ and $\alpha_i \equiv \arg x_{ii}$.

In Eq. (105), the expression obtained for Γ_2 contained the parameter θ and a choice of sign,

$$\Gamma_2 = \begin{pmatrix} 0 & |x_{22}|e^{i(\alpha_2-\theta)} & 0 \\ 0 & 0 & |x_{11}|e^{i(\alpha_1+\theta)} \\ \pm|x_{33}|e^{i\alpha_3} & 0 & 0 \end{pmatrix}. \tag{115}$$

We can therefore equate Eqs. (114) and (115) if the following equations are satisfied mod 2π:

$$2\alpha_1 = \frac{1}{2}\pi - \theta - \lambda - \phi_{23}, \tag{116}$$

$$2\alpha_2 = \theta - \lambda - \phi_{12}, \tag{117}$$

$$2\alpha_3 + m\pi = -\lambda + \phi_{12} + \phi_{23}, \tag{118}$$

where $m = 0$ or 1 and we have used $\phi_{12} + \phi_{23} + \phi_{31} = 0$ to eliminate ϕ_{31}. We can use Eqs. (116)–(118) to solve for λ, ϕ_{12}, and ϕ_{23} in terms of the α_i, m, and θ. Thus, we have demonstrated that the down-type Yukawa coupling matrices in the GCP3 model and the $\mathrm{U}(1) \otimes \Pi_2$ model can be related by an appropriate change of basis of the scalar fields and quark fields.

We conclude that *there is a one-to-one correspondence between the Yukawa extended GCP3 model and the Yukawa extended* $\mathrm{U}(1) \otimes \Pi_2$ *models that simply reflects a different choice of the scalar field and quark field basis.* The addition of the three-generation Yukawa sector did not "remove the degeneracy" of the GCP3 and $\mathrm{U}(1) \otimes \Pi_2$ models that was present when only the symmetries of the scalar potential were considered.

VII. (NON)EQUIVALENCE OF $\mathbb{Z}_2 \otimes \Pi_2$ AND GCP2 MODELS WITH TWO QUARK GENERATIONS

In previous sections, we showed that although there is no distinction between a scalar potential of the 2HDM that respects the GCP2 or $\mathbb{Z}_2 \otimes \Pi_2$ symmetry, no viable model (i.e., a model that possesses nonzero quark masses and a nontrivial CKM matrix) exists in which the GCP2 and/or the $\mathbb{Z}_2 \otimes \Pi_2$ symmetry can be consistently extended to the Yukawa sector with $n = 3$ generations of quarks. Likewise, there is no distinction between a scalar potential that respects the GCP3 or $\mathrm{U}(1) \otimes \Pi_2$ symmetry, although both these symmetries can be extended to the three-generation Yukawa sector to produce a viable model. However, we also showed that this extension does *not* "remove the degeneracy" present in the scalar sector between GCP3 and $\mathrm{U}(1) \otimes \Pi_2$, as the two models are related by a particular change of the scalar field and quark field basis.

To see whether these results are specific to the 2HDM with $n = 3$ quark generations, we shall consider a version of the 2HDM with n different from three. The case of $n = 1$ can be discarded,[13] since one would quickly conclude using the methods employed in this section that there are no consistent extensions of these symmetries to the Yukawa sector unless all Yukawa couplings vanish [48]. In this section, we consider the 2HDM with $n = 2$ quark generations. In particular, we shall examine the relation between two-generation models with GCP2 and $\mathbb{Z}_2 \otimes \Pi_2$ symmetries appropriately extended to the Yukawa sector.

A. GCP2 for two generations

In the case of two generations of fermions, the Yukawa couplings are 2×2 matrices that can be parametrized as

$$\Gamma_1 = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}, \tag{119}$$

where all parameters are complex. Furthermore, using the result in Eq. (47), we choose the quark basis where the GCP symmetry matrices take the simple form

$$X_L = \begin{pmatrix} c_\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix}, \qquad X_{n_R} = \begin{pmatrix} c_\beta & s_\beta \\ -s_\beta & c_\beta \end{pmatrix}, \tag{120}$$

where $c_\alpha \equiv \cos\alpha$, $c_\beta \equiv \cos\beta$, $s_\alpha \equiv \sin\alpha$, and $s_\beta \equiv \sin\beta$, with $0 \le \alpha, \beta \le \frac{1}{2}\pi$ [as noted below Eq. (47)]. The GCP2 symmetry equations given by Eqs. (41)–(44) are obtained by setting

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{121}$$

in light of Eqs. (37) and (54). We can then rewrite Eq. (43) as

$$X_L \Gamma_1^* + \Gamma_2 X_{n_R} = 0, \qquad X_L \Gamma_2^* - \Gamma_1 X_{n_R} = 0. \tag{122}$$

Since X_L and X_{n_R} are real orthogonal matrices, we can eliminate Γ_2 to obtain

$$X_L^2 \Gamma_1 + \Gamma_1 X_{n_R}^2 = 0. \tag{123}$$

Substituting the parametrization of Eq. (119) into Eq. (122), we obtain eight complex linear equations in eight complex variables, which one can write in matrix form as

$$A\boldsymbol{x} = \boldsymbol{0}, \quad \text{where } \boldsymbol{x} = (x_{11}^*, x_{12}^*, x_{21}^*, x_{22}^*, y_{11}, y_{12}, y_{21}, y_{22})^\mathsf{T}, \tag{124}$$

and A is a real 8×8 matrix that can be written in block matrix form as

$$A = \begin{pmatrix} c_\alpha \mathbb{1} & s_\alpha \mathbb{1} & X_{n_R}^\mathsf{T} & \boldsymbol{0} \\ -s_\alpha \mathbb{1} & c_\alpha \mathbb{1} & \boldsymbol{0} & X_{n_R}^\mathsf{T} \\ -X_{n_R}^\mathsf{T} & \boldsymbol{0} & c_\alpha \mathbb{1} & s_\alpha \mathbb{1} \\ \boldsymbol{0} & -X_{n_R}^\mathsf{T} & -s_\alpha \mathbb{1} & c_\alpha \mathbb{1} \end{pmatrix}, \tag{125}$$

where $\mathbb{1}$ is the 2×2 identity matrix and $\boldsymbol{0}$ is the eight-component zero vector in Eq. (124) and the 2×2 zero matrix in Eq. (125). Since $\boldsymbol{x} \neq 0$, it follows that $\det A = 0$, which will constrain the values of α and β as we now demonstrate. Indeed, in Appendix B, we provide an explicit evaluation of $\det A$, which yields

$$\begin{aligned} \det A &= 16(c_\alpha^2 c_\beta^2 - s_\alpha^2 s_\beta^2)^2 \\ &= 16\cos^2(\alpha+\beta)\cos^2(\alpha-\beta) = 0. \end{aligned} \tag{126}$$

Since $0 \le \alpha$, $\beta \le \frac{1}{2}\pi$, it then follows that

$$\alpha + \beta = \frac{1}{2}\pi. \tag{127}$$

Using Eq. (127) in evaluating Eq. (123) then yields

$$\begin{aligned} &X_L^2\left(\frac{1}{2}\pi - \beta\right)\Gamma_1 + \Gamma_1 X_{n_R}^2(\beta) \\ &= \begin{pmatrix} x_{21} - x_{12} & x_{11} + x_{22} \\ -x_{11} - x_{22} & -x_{12} + x_{21} \end{pmatrix} \sin 2\beta = 0, \end{aligned} \tag{128}$$

which leads to three possible cases:

$$\begin{aligned} &\mathrm{I:}\ \alpha + \beta = \frac{1}{2}\pi \quad \text{with} \quad \alpha, \qquad \beta \neq 0, \\ &\qquad \frac{1}{2}\pi \Rightarrow x_{21} = x_{12}, \qquad x_{22} = -x_{11}, \end{aligned} \tag{129}$$

$$\mathrm{II:}\ \alpha = \frac{1}{2}\pi, \quad \beta = 0, \tag{130}$$

$$\mathrm{III:}\ \alpha = 0, \qquad \beta = \frac{1}{2}\pi. \tag{131}$$

[13]If one were to expand the 2HDM Yukawa sector to include, e.g., up-type and down-type vectorlike quarks [45], then it would be possible to extend the $\mathbb{Z}_2 \otimes \Pi_2$ and $\mathrm{U}(1) \otimes \Pi_2$ symmetries of the scalar sector to a one-generation Yukawa sector [46,47]. Such models are beyond the scope of this work.

After imposing the constraints corresponding to one of these three cases, we are left with a single symmetry equation constraining the down-type Yukawa coupling matrices [cf. Eq. (122)]:

$$\Gamma_2^* = X_L^\dagger \Gamma_1 X_{n_R}. \tag{132}$$

1. Case I

In this case, α and β can take any values in the open interval $(0, \pi/2)$ that satisfy the equation $\alpha + \beta = \pi/2$. Constraining the couplings with Eqs. (129) and (132), we obtain the following Γ matrices:

$$\Gamma_1 = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & -x_{11} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} -x_{12}^* & x_{11}^* \\ x_{11}^* & x_{12}^* \end{pmatrix}. \tag{133}$$

2. Case II

In the case $\alpha = \pi/2$ and $\beta = 0$, there are no constraints on Γ_1. Using Eqs. (130) and (132), we obtain the following Γ matrices:

$$\Gamma_1 = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} -x_{21}^* & -x_{22}^* \\ x_{11}^* & x_{12}^* \end{pmatrix}. \tag{134}$$

3. Case III

In the case $\alpha = 0$ and $\beta = \pi/2$, there are no constraints on Γ_1. Using Eqs. (131) and (132), we obtain the following Γ matrices:

$$\Gamma_1 = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} -x_{12}^* & x_{11}^* \\ -x_{22}^* & x_{21}^* \end{pmatrix}. \tag{135}$$

4. Cabibbo angle and summary of cases

In contrast to Case I where Γ_1 and Γ_2 are fixed by two independent complex parameters, Cases II and III are governed by four complex parameters. This distinction will be significant when we compare the Yukawa sectors of the two-generation models constrained by the GCP2 and the $\mathbb{Z}_2 \otimes \Pi_2$ symmetry, respectively, in Sec. VII C. Moreover, it is straightforward to compute $\det H_d$ for all three cases:

$$\begin{aligned} \text{Case I:}\quad & \det H_d = |(v_1 x_{11} - v_2 x_{12}^*)^2 + (v_1 x_{12} + v_2 x_{11}^*)^2|^2, \\ \text{Case II:}\quad & \det H_d = |(v_1 x_{11} - v_2 x_{21}^*)(v_1 x_{22} + v_2 x_{12}^*) - (v_1 x_{12} - v_2 x_{22}^*)(v_1 x_{21} + v_2 x_{11}^*)|^2, \\ \text{Case III:}\quad & \det H_d = |(v_1 x_{11} - v_2 x_{12}^*)(v_1 x_{22} + v_2 x_{21}^*) - (v_1 x_{12} + v_2 x_{11}^*)(v_1 x_{21} - v_2 x_{22}^*)|^2. \end{aligned} \tag{136}$$

In all three cases, det H_d is nonzero for a generic choice of parameters,[14] which implies that the two down-type quark masses are generically nonzero. The same analysis yields the up-type quark Yukawa matrices Δ_1 and Δ_2 with precisely the same textures as the ones obtained for Γ_1 and Γ_2 above, which implies that the two up-type quark masses are also generically nonzero. Even though X_{p_R} and X_{nR} are initially unrelated, the application of the GCP2 symmetry equations analogous to Eq. (122) to the up-type Yukawa coupling matrices,

$$X_L \Delta_1^* + \Delta_2 X_{p_R} = 0, \qquad X_L \Delta_2^* - \Delta_1 X_{p_R} = 0, \tag{137}$$

yields the same three cases as in Eqs. (129)–(131), with the angle β replaced by a new angle γ that is likewise constrained by the value of the angle α. It follows that the textures of the Γ and Δ matrices must match, implying that there are three distinct classes of GCP2 models in total, which we shall denote below by I/I, II/II, and III/III to indicate the corresponding textures of the Γ and Δ matrices, respectively.

In the case of two quark generations, one can build a basis invariant quantity measuring the Cabibbo angle θ_c using

$$\begin{aligned} J_c \equiv \det\{[H_u, H_d]\} &= \det[V^\dagger D_u^2 V D_d^2 - D_d^2 V^\dagger D_u^2 V] \\ &= (m_d^2 - m_s^2)^2 (m_c^2 - m_u^2)^2 \cos^2\theta_c \sin^2\theta_c, \end{aligned} \tag{138}$$

where V is now the 2×2 Cabibbo mixing matrix

$$V = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}. \tag{139}$$

In Model I/I, H_d and H_u are given by

$$H_d = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \qquad H_u = \begin{pmatrix} C & D \\ -D & C \end{pmatrix}, \tag{140}$$

where

[14]For example, in Case 1, det $H_d = 0$ only if $v_1/v_2 = x_{11}/x_{12}^*$ and $\mathrm{Re}(v_1/v_2) = 0$.

$$A \equiv |v_1 x_{11} - v_2 x_{12}^*|^2 + |v_1 x_{12} + v_2 x_{11}^*|^2 = (|v_1|^2 + |v_2|^2)(|x_{11}|^2 + |x_{12}|^2),$$
$$B \equiv (v_1 x_{11} - v_2 x_{12}^*)(v_1^* x_{12}^* + v_2^* x_{11}) - (v_1 x_{12} + v_2 x_{11}^*)(v_1^* x_{11}^* - v_2^* x_{12}). \tag{141}$$

and C and D are obtained from A and B by replacing the corresponding elements of the Γ matrices with those of the Δ matrices. The form of H_d and H_u given in Eq. (140) immediately yield $[H_u, H_d] = 0$. Thus, $J_c = 0$, which implies that $\sin 2\theta_c = 0$, which is experimentally excluded. In contrast, in Models II/II and III/III, the four matrix elements of H_d and H_u, respectively, are independent quantities so that $[H_u, H_d] \neq 0$. In particular, $\det\{[H_u, H_d]\} \neq 0$, or equivalently $J_c \neq 0$. Thus, we are left with two possible classes of two-generation models that extend the GCP2 symmetry to the Yukawa sector, each of which is governed by four complex parameters in the down-type quark and up-type quark sectors, respectively.

B. $\mathbb{Z}_2 \otimes \Pi_2$ for two generations

Having found that GCP2 can be extended to the fermions with two generations, we now want to check whether an extension also exists for $\mathbb{Z}_2 \otimes \Pi_2$, and if such an extension corresponds to GCP2 in another choice of scalar field and quark field basis. Following the analysis of the three generation model of Sec. IV, we begin by determining all possible extensions of the $\mathbb{Z}_2$ symmetry defined in Eq. (50) to the Yukawa sector. To do this, we will employ a simplified version of the method used in Ref. [24].

1. Extensions of $\mathbb{Z}_2$ to the Yukawa sector: The basics

As noted in Ref. [24], due to the unitarity of the quark symmetry matrices, one can always choose a basis such that $\mathbb{Z}_2$ symmetry matrices, $S_L^{(\mathbb{Z}_2)}$ and $S_{nR}^{(\mathbb{Z}_2)}$, are diagonal. In this basis, the symmetry matrices can be written as

$$S^{(\mathbb{Z}_2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S_L^{(\mathbb{Z}_2)} = \begin{pmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{i\alpha_2} \end{pmatrix},$$
$$S_{nR}^{(\mathbb{Z}_2)} = \begin{pmatrix} e^{i\beta_1} & 0 \\ 0 & e^{i\beta_2} \end{pmatrix}. \tag{142}$$

This implies that the $\mathbb{Z}_2$ symmetry equation given by Eq. (32) can be written in the simple form

$$(\Gamma_1)_{ab} = (\Gamma_1)_{ab} e^{i\theta_{ab}}, \qquad (\Gamma_2)_{ab} = (\Gamma_2)_{ab} e^{i(\theta_{ab} - \pi)}, \tag{143}$$

where $a, b \in \{1, 2\}$ are left-handed quark generation indices and we have adopted the notation such that

$$\theta_{ab} \equiv \alpha_a - \beta_b. \tag{144}$$

From Eq. (143), we can readily see that there are only three possibilities:

(1) $\theta_{ab} = 0$, then $(\Gamma_1)_{ab}$ can take any value, and $(\Gamma_2)_{ab} = 0$;
(2) $\theta_{ab} = \pi$, then $(\Gamma_2)_{ab}$ can take any value, and $(\Gamma_1)_{ab} = 0$;
(3) $\theta_{ab} \neq 0, \pi$, then $(\Gamma_1)_{ab} = (\Gamma_2)_{ab} = 0$;

where all conditions on θ_{ab} are taken to be mod 2π.

2. Extensions of $\mathbb{Z}_2$ to the Yukawa sector: Left space constraints

Consider the following combinations of Yukawa matrices:

$$\Gamma_i \Gamma_j^\dagger = (-1)^{(1-\delta_{ij})} S_L \Gamma_i \Gamma_j^\dagger S_L^\dagger, \tag{145}$$

$$\Delta_i \Delta_j^\dagger = (-1)^{(1-\delta_{ij})} S_L \Delta_i \Delta_j^\dagger S_L^\dagger, \tag{146}$$

where we have used Eq. (32) with S given in Eq. (142). We can see that, contrary to the right-handed symmetries that act on either the up or the down quarks, the left-handed symmetries affect both, and are thus very constraining. For example, Eq. (145) yields the following constraint:

$$(\Gamma_i \Gamma_j^\dagger)_{ab} = (\Gamma_i \Gamma_j^\dagger)_{ab} e^{i(\alpha_a - \alpha_b) + i\pi(1-\delta_{ij})}, \tag{147}$$

after inserting the expression for S_L given in Eq. (142). In particular, the phase on the right-hand side of Eq. (147) is given by

$$\alpha_a - \alpha_b + \pi(1 - \delta_{ij}) = \begin{cases} \begin{pmatrix} 0 & \alpha_1 - \alpha_2 \\ \alpha_2 - \alpha_1 & 0 \end{pmatrix}_{ab}, & \text{for } i = j, \\ \begin{pmatrix} \pi & \pi + \alpha_1 - \alpha_2 \\ \pi + \alpha_2 - \alpha_1 & \pi \end{pmatrix}_{ab}, & \text{for } i \neq j, \end{cases} \tag{148}$$

where $a,b \in \{1,2\}$ label the elements of the 2×2 matrices above. Thus, there are three possibilities:

(1) If $\alpha_1-\alpha_2=0$, then $\Gamma_1\Gamma_2^\dagger=\Gamma_2\Gamma_1^\dagger=0$, with $\Gamma_1\Gamma_1^\dagger$, and $\Gamma_2\Gamma_2^\dagger$ unconstrained.

(2) If $\alpha_1-\alpha_2=\pi$, then $\Gamma_1\Gamma_1^\dagger$ and $\Gamma_2\Gamma_2^\dagger$ are diagonal, whereas $\Gamma_1\Gamma_2^\dagger$ and $\Gamma_2\Gamma_1^\dagger$ are off-diagonal.

(3) If $\alpha_1-\alpha_2\neq 0,\pi$, then $\Gamma_1\Gamma_1^\dagger$, $\Gamma_2\Gamma_2^\dagger$ are diagonal, whereas $\Gamma_1\Gamma_2^\dagger=\Gamma_2\Gamma_1^\dagger=0$. This implies that H_d is diagonal. A similar analysis of the up-type quark sector yields a diagonal H_u in this case. Hence, $\det\{[H_u,H_d]\}=0$ which yields $J_c=0$. It follows that $\sin 2\theta_c=0$, which is experimentally excluded.

In Case 1 above where $\alpha_1=\alpha_2$, the matrix θ_{ab} is given by

$$\theta=\begin{pmatrix}\theta_{11} & \theta_{12}\\ \theta_{11} & \theta_{12}\end{pmatrix}. \tag{149}$$

In Case 2 above where $\alpha_2=\pi+\alpha_1$, the matrix θ_{ab} is given by

$$\theta=\begin{pmatrix}\theta_{11} & \theta_{12}\\ \theta_{11}+\pi & \theta_{12}+\pi\end{pmatrix}, \tag{150}$$

where the elements of the matrix θ are evaluated mod 2π. Combining the results obtained in this section with those found below Eq. (143), we obtain all the possible extensions of $\mathbb{Z}_2$ in Table II, where x stands for an arbitrary complex number.

The cases in Table II that are related by the interchange of Γ_1 and Γ_2 are physically equivalent, as this transformation simply corresponds to a change of the scalar field basis in which Φ_1 and Φ_2 are interchanged. We can therefore choose the form of the Yukawa matrices Γ_1 and Γ_2 for the four inequivalent cases as exhibited below:

$$\textbf{Case 0}:\ \Gamma_1=\begin{pmatrix}x & x\\ x & x\end{pmatrix},\qquad \Gamma_2=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \tag{151}$$

TABLE II. Extensions of the $\mathbb{Z}_2$ symmetry to the Yukawa sector consisting of two quark generations.

θ_{11}	θ_{12}	Γ_1	Γ_2
$\alpha_1=\alpha_2$			
0	0	$\begin{pmatrix}x & x\\ x & x\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$
π	0	$\begin{pmatrix}0 & x\\ 0 & x\end{pmatrix}$	$\begin{pmatrix}x & 0\\ x & 0\end{pmatrix}$
0	π	$\begin{pmatrix}x & 0\\ x & 0\end{pmatrix}$	$\begin{pmatrix}0 & x\\ 0 & x\end{pmatrix}$
π	π	$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}x & x\\ x & x\end{pmatrix}$
$\alpha_2=\pi+\alpha_1$			
0	0	$\begin{pmatrix}x & x\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ x & x\end{pmatrix}$
π	0	$\begin{pmatrix}0 & x\\ x & 0\end{pmatrix}$	$\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix}$
0	π	$\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix}$	$\begin{pmatrix}0 & x\\ x & 0\end{pmatrix}$
π	π	$\begin{pmatrix}0 & 0\\ x & x\end{pmatrix}$	$\begin{pmatrix}x & x\\ 0 & 0\end{pmatrix}$

$$\textbf{Case 1}:\ \Gamma_1=\begin{pmatrix}x & x\\ 0 & 0\end{pmatrix},\qquad \Gamma_2=\begin{pmatrix}0 & 0\\ x & x\end{pmatrix}, \tag{152}$$

$$\textbf{Case 2}:\ \Gamma_1=\begin{pmatrix}x & 0\\ x & 0\end{pmatrix},\qquad \Gamma_2=\begin{pmatrix}0 & x\\ 0 & x\end{pmatrix}, \tag{153}$$

$$\textbf{Case 3}:\ \Gamma_1=\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix},\qquad \Gamma_2=\begin{pmatrix}0 & x\\ x & 0\end{pmatrix}. \tag{154}$$

These cases correspond, respectively, to the following $\mathbb{Z}_2$ symmetry matrices in a scalar field basis where $S^{(\mathbb{Z}_2)}=\sigma_Z$:

$$\textbf{Case 0}\,[\theta_{11}=\theta_{12}=0,\alpha_1=\alpha_2]:\ S_L^{(\mathbb{Z}_2)}=e^{i\alpha_1}\mathbb{1},\qquad S_{n_R}^{(\mathbb{Z}_2)}=e^{i\alpha_1}\mathbb{1}, \tag{155}$$

$$\textbf{Case 1}\,[\theta_{11}=\theta_{12}=0,\alpha_2=\pi+\alpha_1]:\ S_L^{(\mathbb{Z}_2)}=e^{i\alpha_1}\sigma_Z,\qquad S_{n_R}^{(\mathbb{Z}_2)}=e^{i\alpha_1}\mathbb{1}, \tag{156}$$

$$\textbf{Case 2}\,[\theta_{11}=0,\theta_{12}=\pi,\alpha_1=\alpha_2]:\ S_L^{(\mathbb{Z}_2)}=e^{i\alpha_1}\mathbb{1},\qquad S_{n_R}^{(\mathbb{Z}_2)}=e^{i\alpha_1}\sigma_Z, \tag{157}$$

$$\textbf{Case 3}\,[\theta_{11}=0,\theta_{12}=\pi,\alpha_2=\pi+\alpha_1]:\ S_L^{(\mathbb{Z}_2)}=e^{i\alpha_1}\sigma_Z,\qquad S_{n_R}^{(\mathbb{Z}_2)}=e^{i\alpha_1}\sigma_Z, \tag{158}$$

where

$$\sigma_Z\equiv\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{159}$$

Without loss of generality, we may simply set the remaining global phase $\alpha_1 = 0$, as it plays no role in constraining the forms of Γ_1 and Γ_2.

3. Extensions of $\mathbb{Z}_2$ to the two generation quark sector: Compatibility with Π_2

If the scalar potential exhibits a $\mathbb{Z}_2 \otimes \Pi_2$ symmetry, then the scalar potential is also invariant under the interchange of Φ_1 and Φ_2, corresponding to the Π_2 symmetry matrix $S^{(\Pi_2)} = \sigma_\Pi$, where

$$\sigma_\Pi \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{160}$$

When extending the $\mathbb{Z}_2$ symmetry to $\mathbb{Z}_2 \otimes \Pi_2$ in the Yukawa sector, one must satisfy the additional constraint given by Eq. (68), which we repeat here in a more explicit notation,

$$\Gamma_1 = S_L^{(\Pi_2)} \Gamma_2 [S_{nR}^{(\Pi_2)}]^\dagger, \qquad \Gamma_2 = S_L^{(\Pi_2)} \Gamma_1 [S_{nR}^{(\Pi_2)}]^\dagger. \tag{161}$$

These constraints immediately eliminate Case 0 [Eqs. (151) and (155)], since the imposition of Eq. (161) would yield $\Gamma_1 = \Gamma_2 = 0$, corresponding to vanishing couplings of the down-type quarks to the Higgs doublet fields.

To determine all possible viable models with the $\mathbb{Z}_2 \otimes \Pi_2$ symmetry extended to the Yukawa sector, one must determine all possible forms for the symmetry matrices $S_L^{(\Pi_2)}$ and $S_{nR}^{(\Pi_2)}$. We proceed as follows. We first consider the simpler problem of extending the Π_2 to the Yukawa sector. But a Π_2-symmetric scalar potential is equivalent to a $\mathbb{Z}_2$-symmetric scalar potential in another scalar field basis as shown in Eqs. (56)–(58). Thus, as a first step we perform a scalar field basis transformation so that the Π_2 symmetry matrix is $S^{(\Pi_2)} = \sigma_Z$. We can now extend the Π_2 symmetry to the Yukawa sector by using the results of Sec. VII B 1. We will then end up with three potential cases for the choice of S_L and S_{n_R} given by Eqs. (156)–(158), where we shall again set the global phase to zero with no loss of generality. We can now transform back to the basis in which the Π_2 symmetry matrix is $S^{(\Pi_2)} = \sigma_\Pi$. That is, we choose U given by Eq. (57) and

$$S_L^{(\Pi_2)} = U_L S_L U_L^\dagger, \qquad S_{n_R}^{(\Pi_2)} = U_{n_R} S_L U_{n_R}^\dagger, \tag{162}$$

where S_L and S_{n_R} are taken to be the symmetry matrices corresponding to one of the following three cases:

$$(S_L, S_{n_R}) \in \{(\sigma_Z, \mathbb{1}), (\mathbb{1}, \sigma_Z), (\sigma_Z, \sigma_Z)\}. \tag{163}$$

Having obtained the symmetry matrices for the Π_2 symmetry extended to the Yukawa sector, we can now consider the constraints on Γ_1 and Γ_2 obtained in Eqs. (152)–(154) under the $\mathbb{Z}_2 \otimes \Pi_2$ symmetry extended to the Yukawa sector. We simply insert the results of Eq. (162) into Eq. (161) to obtain

$$\Gamma_1 (U_{n_R} S_{nR} U_{n_R}^\dagger) = (U_L S_L U_L^\dagger) \Gamma_2, \tag{164}$$

$$\Gamma_2 (U_{n_R} S_{nR} U_{n_R}^\dagger) = (U_L S_L U_L^\dagger) \Gamma_1, \tag{165}$$

where the three possible cases for $\{\Gamma_1, \Gamma_2\}$ are given in Eqs. (152)–(154). Note that for any of the three sets of choices of S_L and S_{n_R} given in Eq. (163), the two equations above are equivalent.

As a result of this analysis, we can consider nine possible models, corresponding to the three possible choices for Γ_1 and Γ_2 exhibited in Eqs. (152)–(154) and the three possible sets of S_L and S_{n_R} listed in Eq. (163), which we shall henceforth denote by sets 1, 2, and 3. The resulting model corresponding to Case n for the choice of Γ_1 and Γ_2 and the mth set of possible choices for S_L and S_{n_R} will be denoted in the following by Model (n-m). Thus, there are nine possible models to consider.

At this stage, we have yet to fix the unitary matrices U_L and U_{n_R}. We shall employ the parametrization

$$U_\sigma = e^{i\phi_\sigma} \begin{pmatrix} e^{i\alpha_\sigma} \cos\theta_\sigma & e^{-i\beta_\sigma} \sin\theta_\sigma \\ -e^{i\beta_\sigma} \sin\theta_\sigma & e^{-i\alpha_\sigma} \cos\theta_\sigma \end{pmatrix}, \text{ with } \sigma \in \{L, n_R\}. \tag{166}$$

The global phase ϕ_σ has no effect on the transformation of the symmetry matrices, so we may set $\phi_\sigma = 0$ without loss of generality. In this convention, $U_\sigma \in \mathrm{SU}(2)$, and the entire SU(2) group manifold can be covered by taking the ranges of the remaining parameters to be $0 \le \theta_\sigma \le \frac{1}{2}\pi$ and $0 \le \alpha_\sigma, \beta_\sigma < 2\pi$.

As an example, consider Model (1-3), where Γ_1 and Γ_2, which take the form given by Eq. (152), can be parametrized as

$$\Gamma_1 = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 \\ x_{21} & x_{22} \end{pmatrix}, \tag{167}$$

and $S_L = S_{n_R} = \sigma_Z$, as specified in the third set of Eq. (163). Plugging these choices along with Eq. (166) into Eq. (164) yields

$$\cos 2\theta_L = 0, \tag{168}$$

$$x_{21} e^{i(\alpha_L - \beta_L)} \sin 2\theta_L = x_{12} e^{-i(\alpha_R - \beta_R)} \sin 2\theta_R - x_{11} \cos 2\theta_R, \tag{169}$$

$$x_{22} e^{i(\alpha_L - \beta_L)} \sin 2\theta_L = x_{11} e^{i(\alpha_R - \beta_R)} \sin 2\theta_R + x_{12} \cos 2\theta_R, \tag{170}$$

where we have simplified the R subscript in writing $\alpha_R \equiv \alpha_{n_R}$, $\beta_R \equiv \beta_{n_R}$, and $\theta_R \equiv \theta_{n_R}$. Since $0 \le \theta_L \le \frac{1}{2}\pi$, it follows that $\theta_L = \pi/4$. Then, Eq. (162) yields

$$S_L^{(\Pi_2)} = U_L \sigma_Z U_L^\dagger = \begin{pmatrix} 0 & -e^{i(\alpha_L - \beta_L)} \\ -e^{-i(\alpha_L-\beta_L)} & 0 \end{pmatrix}, \tag{171}$$

$$S_{n_R}^{(\Pi_2)} = U_{n_R}\sigma_Z U_{n_R}^\dagger = \begin{pmatrix} \cos 2\theta_R & -e^{i(\alpha_R-\beta_R)}\sin 2\theta_R \\ -e^{-i(\alpha_R-\beta_R)}\sin 2\theta_R & -\cos 2\theta_R \end{pmatrix}. \tag{172}$$

The $\mathbb{Z}_2 \otimes \Pi_2$ symmetry constraints do not fix the remaining free parameter, α_L, β_L, α_R, β_R, and θ_R. Indeed, one is free to transform to a different quark field basis as long as the $\mathbb{Z}_2$ symmetry matrices $S_L^{(\mathbb{Z}_2)} = \sigma_Z$ and $S_{n_R}^{(\mathbb{Z}_2)} = \mathbb{1}$ are unchanged. In light of Eqs. (35) and (36), we shall transform

$$S_L^{(\Pi_2)} \to U'_L S_L^{(\Pi_2)} U'^\dagger_L, \qquad S_{n_R}^{(\Pi_2)} \to U'_{n_R} S_{n_R}^{(\Pi_2)} U'^\dagger_{n_R}, \tag{173}$$

where $U'_L = \mathrm{diag}(e^{i(\gamma+\delta)}, e^{i(\gamma-\delta)})$ is the most general 2×2 unitary matrix that leaves $S_L^{(\mathbb{Z}_2)}$ unchanged and U'_{n_R} is an arbitrary 2×2 unitary matrix that (trivially) leaves $S_{n_R}^{(\mathbb{Z}_2)}$ unchanged. With this freedom, it is convenient to choose γ, δ, and the matrix elements of U'_{n_R} such that Eq. (173) yields

$$\alpha_L - \beta_L = \alpha_R - \beta_R = \pi, \qquad \theta_R = \frac{1}{4}\pi. \tag{174}$$

Using these results and Eq. (168) to simplify Eqs. (169) and (170), we end up with $x_{21} = x_{12}$ and $x_{22} = x_{11}$. That is, Model (1-3) is equivalent to a model in which the Yukawa coupling matrices and the Π_2 symmetry matrices are given by

$$\Gamma_1 = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 \\ x_{12} & x_{11} \end{pmatrix},$$
$$S_L^{(\Pi_2)} = \sigma_\Pi, \qquad S_{nR}^{(\Pi_2)} = \sigma_\Pi, \tag{175}$$

where σ_Π is defined in Eq. (160).

Moreover, it is straightforward to compute $H_d = (v_1\Gamma_1 + v_2\Gamma_2)(v_1\Gamma_1 + v_2\Gamma_2)^\dagger$ and its trace and determinant, which yield

$$\mathrm{Tr} H_d = [|x_{11}|^2 + |x_{12}|^2] v^2, \tag{176}$$

$$\det H_d = |v_1|^2 |v_2|^2 |x_{11}^2 - x_{12}^2|^2, \tag{177}$$

where v is defined in Eq. (6). Note that $\det H_d$ is nonzero for a generic choice of parameters, which implies that the two down-type quark masses are generically nonzero.

In Appendix C, we analyze the remaining eight models. Some of these models can be immediately excluded as they contain either a massless down-type quark or else vanishing down-type Yukawa coupling matrices. Furthermore, we find that Model (3,3) actually represents a class of models that are parametrized by the angles (θ_L, θ_R). Two of these models, corresponding to $(\theta_L, \theta_R) = (0, \frac{1}{4}\pi)$ and $(\frac{1}{4}\pi, 0)$, which shall be denoted by $(3\text{-}3)_0$ and $(3\text{-}3)_1$, respectively, are phenomenologically viable. In the remaining models, collectively denoted by $(3\text{-}3)_X$, all down-type quarks are massive. However, when the up-type quark Yukawa couplings are taken into account, the resulting models predict a vanishing Cabibbo angle (as shown in Appendix C) and hence are phenomenologically excluded.

4. Summary of viable two-generation $\mathbb{Z}_2 \otimes \Pi_2$-symmetric models

In addition to Model (1,3) analyzed above, we show in Appendix C that in Models (2-3), (3-1), (3-2), $(3\text{-}3)_0$, $(3\text{-}3)_1$, and $(3\text{-}3)_X$ all down-type quarks are massive. As noted above, the class of models denoted by $(3\text{-}3)_X$ is phenomenologically excluded. The symmetry matrices and the corresponding Yukawa coupling matrices of the remaining models are listed in Table III below.

To determine the viability of the possible two-generation models, one must now consider the corresponding results for the up-type Yukawa coupling matrices Δ_1 and Δ_2. In the analysis of the possible forms for Δ_1 and Δ_2 consistent with the $\mathbb{Z}_2 \otimes \Pi_2$ symmetry, one must use the same $S_L^{(\mathbb{Z}_2)}$ and $S_L^{(\Pi_2)}$ employed in the analysis of the down-type Yukawa sector. One is still free to fix $S_{p_R}^{(\mathbb{Z}_2)}$ and $S_{p_R}^{(\Pi_2)}$ consistent with the symmetry requirements. The end result is a table identical with Table III, with the possible forms for Δ_1 and Δ_2 coinciding with those of Γ_1 and Γ_2 but with different nonzero matrix elements (which we shall denote by y_{ij}). Consequently, none of the allowed choices for Δ_1 and Δ_2 yield massless up-type quarks for a generic choice of the parameters. The resulting Yukawa sector models are specified by a pair of model types that share the same $S_L^{(\mathbb{Z}_2)}$ and $S_L^{(\Pi_2)}$, which are listed above in Table IV. It is now straightforward to check that for all models listed in Table IV, $\det\{[H_u, H_d]\} \neq 0$, or equivalently $J_c \neq 0$ [cf. Eq. (138)]. Hence, all the models of Table IV possess a nonzero Cabibbo angle, as required by experimental data.

The models exhibited in Table IV are not all inequivalent, as some of the listed models are related by a change in the Higgs field and the quark field basis. In particular, we show in Appendix D that there are five sets of model pairs, specified in Eqs. (D9), (D11), and (D12), where the two models that make up a given pair are related by an appropriate set of basis transformations. We may take the first seven models listed in Table IV to constitute the list of inequivalent models. Each of the remaining five

TABLE III. Symmetry matrices for viable two-generation $\mathbb{Z}_2 \otimes \Pi_2$-symmetric models in a scalar field basis where $S^{(\mathbb{Z}_2)} = \sigma_Z$ and $S^{(\Pi_2)} = \sigma_\Pi$, with the corresponding forms for the $\mathbb{Z}_2$ and Π_2 symmetry matrices of the down-type Yukawa sector and the corresponding Yukawa coupling matrices.

Model	$S_L^{(\mathbb{Z}_2)}$	$S_{nR}^{(\mathbb{Z}_2)}$	$S_L^{(\Pi_2)}$	$S_{nR}^{(\Pi_2)}$	Γ_1	Γ_2
(1-3)	σ_Z	$\mathbb{1}$	σ_Π	σ_Π	$\begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ x_{12} & x_{11} \end{pmatrix}$
(2-3)	$\mathbb{1}$	σ_Z	σ_Π	σ_Π	$\begin{pmatrix} x_{11} & 0 \\ x_{21} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & x_{21} \\ 0 & x_{11} \end{pmatrix}$
(3-1)	σ_Z	σ_Z	σ_Π	$\mathbb{1}$	$\begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & x_{22} \\ x_{11} & 0 \end{pmatrix}$
(3-2)	σ_Z	σ_Z	$\mathbb{1}$	σ_Π	$\begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & x_{11} \\ x_{22} & 0 \end{pmatrix}$
$(3\text{-}3)_0$	σ_Z	σ_Z	σ_Z	σ_Π	$\begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & x_{11} \\ -x_{22} & 0 \end{pmatrix}$
$(3\text{-}3)_1$	σ_Z	σ_Z	σ_Π	σ_Z	$\begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & -x_{22} \\ x_{11} & 0 \end{pmatrix}$

models is shown in Appendix D to be equivalent to one of the seven inequivalent models in Table IV.

C. (Non)Correspondence between $\mathbb{Z}_2 \otimes \Pi_2$ and GCP2 with two quark generations

The question now arises: are the Yukawa-extended two-generation GCP2-symmetric models of Sec. VII A equivalent (i.e., the same model but expressed in different choices of the Higgs field and quark field basis) to the corresponding $\mathbb{Z}_2 \otimes \Pi_2$-symmetric models of Sec. VII B? In Sec. VII A, we classified the possible Yukawa-extended GCP2-symmetric models. We found three distinct classes of models, denoted by I/I, II/II, and III/III, all of which exhibited nonzero up-type and down-type quark masses for generic choices of the parameters. In model class I/I, the corresponding down-type and up-type Yukawa coupling matrices exhibited forms that each depended on two independent complex parameters. In light of these forms, we demonstrated that the Cabibbo angle vanished. In model classes II/II and III/III, the corresponding down-type and up-type Yukawa coupling matrices exhibited forms that each depended on four independent complex parameters, which implied a nonvanishing Cabibbo angle.

In Sec. VII B, we classified the possible Yukawa-extended $\mathbb{Z}_2 \otimes \Pi_2$-symmetric models. In all cases but one, the corresponding down-type and up-type Yukawa coupling matrices exhibited forms that each depended on two independent complex parameters. Some of these models possessed at least one massless quark. Among the class of models with nonzero up-type and down-type quark masses, we showed the existence of seven inequivalent models, each of which allowed for a nonvanishing Cabibbo angle. The one exceptional case (with Yukawa coupling matrices that each depend on only one independent parameter) was shown to have a vanishing Cabibbo angle in Appendix C.

Naively, one might have expected that the GCP2-symmetric models with down-type and up-type Yukawa coupling matrices that each depended on two independent complex parameters could be related via a basis transformation to the corresponding $\mathbb{Z}_2 \otimes \Pi_2$-symmetric models with the same number of independent parameters. However, the Cabibbo angle necessarily vanishes in the former, whereas it is generically nonzero in the latter. Thus, we conclude that in contrast to the behavior of these symmetries in the scalar sector of the 2HDM, GCP2, and $\mathbb{Z}_2 \otimes \Pi_2$ are inequivalent symmetries when extended to the Yukawa sector.

Is it possible that the Yukawa-extended GCP2-symmetric models (Models II/II and III/III), with down-type and up-type Yukawa coupling matrices that each depend on four independent complex parameters, could be related via a basis transformation to the corresponding $\mathbb{Z}_2 \otimes \Pi_2$-symmetric models, which depend on half the number of independent complex parameters? The answer is clearly negative. Although the Cabibbo angle is nonvanishing in all these models, there are other physical observables that would distinguish the Yukawa-extended GCP2-symmetric models from the corresponding $\mathbb{Z}_2 \otimes \Pi_2$-symmetric models. Hence, we have demonstrated that the extension of the GCP2 and the $\mathbb{Z}_2 \otimes \Pi_2$ symmetries from the scalar sector (where the corresponding scalar potentials are related by a change in the scalar field basis) to the Yukawa sector effectively "removes the degeneracy" and yields inequivalent models.

TABLE IV. $\mathbb{Z}_2$ and Π_2 symmetry matrices for each viable Yukawa sector model that is compatible with the $\mathbb{Z}_2 \otimes \Pi_2$ symmetry of the 2HDM scalar potential in a scalar field basis where $S^{(\mathbb{Z}_2)} = \sigma_Z$ and $S^{(\Pi_2)} = \sigma_\Pi$. Of the 12 models listed below, the first seven models are inequivalent with respect to basis changes. Each of the last five models can be shown to be equivalent to one of the first seven models listed below via a particular change in the scalar field and quark field basis, as shown in Appendix D. The corresponding equivalent models are given in Eqs. (D9), (D11), and (D12).

Down/up sector models	$S_L^{(\mathbb{Z}_2)}$	$S_{nR}^{(\mathbb{Z}_2)}$	$S_{pR}^{(\mathbb{Z}_2)}$	$S_L^{(\Pi_2)}$	$S_{nR}^{(\Pi_2)}$	$S_{pR}^{(\Pi_2)}$
(1-3)/(1-3)	σ_Z	$\mathbb{1}$	$\mathbb{1}$	σ_Π	σ_Π	σ_Π
(1-3)/(3-1)	σ_Z	$\mathbb{1}$	σ_Z	σ_Π	σ_Π	$\mathbb{1}$
$(1\text{-}3)/(3\text{-}3)_1$	σ_Z	$\mathbb{1}$	σ_Z	σ_Π	σ_Π	σ_Z
(2-3)/(2-3)	$\mathbb{1}$	σ_Z	σ_Z	σ_Π	σ_Π	σ_Π
$(3\text{-}3)_0/(3\text{-}3)_0$	σ_Z	σ_Z	σ_Z	σ_Z	σ_Π	σ_Π
$(3\text{-}3)_1/(1,3)$	σ_Z	σ_Z	$\mathbb{1}$	σ_Π	σ_Z	σ_Π
$(3\text{-}3)_1/(3,3)_1$	σ_Z	σ_Z	σ_Z	σ_Π	σ_Z	σ_Z
(3-1)/(1-3)	σ_Z	σ_Z	$\mathbb{1}$	σ_Π	$\mathbb{1}$	σ_Π
(3-1)/(3-1)	σ_Z	σ_Z	σ_Z	σ_Π	$\mathbb{1}$	$\mathbb{1}$
$(3\text{-}1)/(3\text{-}3)_1$	σ_Z	σ_Z	σ_Z	σ_Π	$\mathbb{1}$	σ_Z
(3-2)/(3-2)	σ_Z	σ_Z	σ_Z	$\mathbb{1}$	σ_Π	σ_Π
$(3\text{-}3)_1/(3,1)$	σ_Z	σ_Z	σ_Z	σ_Π	σ_Z	$\mathbb{1}$

VIII. CONCLUSIONS

In this paper we have explored the curious connection between the 2HDM scalar potential obtained by imposing invariance under the Higgs-flavor symmetry, $\mathbb{Z}_2 \otimes \Pi_2$, and the scalar potential obtained by imposing invariance under the generalized *CP* symmetry, GCP2. As first noticed in Ref. [15], the resulting scalar potentials after imposing $\mathbb{Z}_2 \otimes \Pi_2$ and GCP2 were related by a transformation of the scalar field basis, and thus could be considered as physically equivalent. The objective of this paper was to extend these symmetries to the Yukawa sector, to see whether the extended $\mathbb{Z}_2 \otimes \Pi_2$-symmetric 2HDM was still equivalent to the extended GCP2-symmetric 2HDM, or whether the extension to the Yukawa sector removes the degeneracy between the two models. In Ref. [27], Ferreira and Silva proved that one could not extend the GCP2 symmetry to the Yukawa sector in a way that was consistent with nonzero quark masses and a CKM mixing angle that were consistent with experimental observations. The more difficult case of extending the $\mathbb{Z}_2 \otimes \Pi_2$ symmetry to the three-generation Yukawa sector is addressed in this paper for the first time. We find that, similar to the results obtained for the GCP2 symmetry in Ref. [27], there is no extension of the $\mathbb{Z}_2 \otimes \Pi_2$ symmetry that is consistent with experimental observations.

In analogy with the connection between $\mathbb{Z}_2 \otimes \Pi_2$ and GCP2, there is also a similar relation between the 2HDM scalar potential obtained by imposing invariance under the Higgs-flavor symmetry, $\mathrm{U}(1) \otimes \Pi_2$, and the scalar potential obtained by imposing invariance under the generalized *CP* symmetry, GCP3. It was also observed in Ref. [14] that the corresponding scalar potentials were related by a transformation of the scalar field basis. In Ref. [27], it was shown that there is a unique extension of the GCP3 symmetry to the Yukawa sector that yields a model with nonzero quark masses and a nonvanishing CKM angle. This provided the possibility of a realistic fully GCP3-symmetric 2HDM, although further analysis presented in Ref. [27] showed that the model was unable to yield a CKM mixing matrix that was fully compatible with experimental data.

In this paper, we examined for the first time all possible extensions of the $\mathrm{U}(1) \otimes \Pi_2$ symmetry to the Yukawa sector. We found that there is again a unique model with nonzero quark masses and a nonvanishing CKM angle. Moreover, we showed that the corresponding three-generation Yukawa-extended GCP3-symmetric and $\mathrm{U}(1) \otimes \Pi_2$-symmetric 2HDM are related by a simultaneous transformation of the Higgs field basis and the quark field basis.

It was tempting to conclude that the results described above imply that the physical equivalence of the models obtained by imposing a $\mathbb{Z}_2 \otimes \Pi_2$ ($\mathrm{U}(1) \otimes \Pi_2$) and GCP2 (GCP3) symmetry was a general feature of the 2HDM. In this paper, we have also proved that this conclusion is not generally correct by focusing on a 2HDM toy model with two quark generations. We have classified all such two-generation GCP2-symmetric and $\mathbb{Z}_2 \otimes \Pi_2$-symmetric models, where the corresponding symmetries have been extended to the Yukawa sector. We have found inequivalent GCP2-symmetric and $\mathbb{Z}_2 \otimes \Pi_2$-symmetric models that possess nonzero quark masses and a nonzero Cabibbo angle. For example, the corresponding down-type Yukawa coupling matrices of the GCP2-symmetric models generically depend on four complex parameters, whereas those of the $\mathbb{Z}_2 \otimes \Pi_2$-symmetric models generically depend on two complex parameters. Indeed, there are no scalar field and quark field basis transformations that can relate the phenomenologically viable GCP2-symmetric and $\mathbb{Z}_2 \otimes \Pi_2$-symmetric models. That is, the degeneracy between these two symmetry classes has been removed.

One can perform a similar classification of two-generation GCP3-symmetric and $\mathrm{U}(1) \otimes \Pi_2$-symmetric models, where the corresponding symmetries have been extended to the Yukawa sector. We again find inequivalent GCP3-symmetric and $\mathrm{U}(1) \otimes \Pi_2$-symmetric models that possess nonzero quark masses and a nonzero Cabibbo angle. Details of this analysis, which mirrors the calculations presented in Sec. VII, can be found in Ref. [48]. Once again, the degeneracy between these two symmetry classes has been removed. We conclude that the physical equivalence of the models obtained by imposing a $\mathbb{Z}_2 \otimes \Pi_2$ [$\mathrm{U}(1) \otimes \Pi_2$] and GCP2 [GCP3] symmetry is an accidental feature of the 2HDM with three quark generations.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: EXTENSIONS OF U(1) AND $\mathbb{Z}_2$ SYMMETRIES OF THE 2HDM SCALAR POTENTIAL TO THE YUKAWA SECTOR

In this appendix, we will list all the extensions of $\mathbb{Z}_2$ and U (1) symmetries of the 2HDM scalar potential to the Yukawa sector that were obtained in Ref. [24]. We shall employ the notation of Ref. [24], where the x represents the freedom to choose any complex number for that matrix element.

1. Extensions of $\mathbb{Z}_2$ to the Yukawa sector

In the list of independent forms obtained in Ref. [24] for the down-type Yukawa coupling matrices Γ_1 and Γ_2 that are compatible with the $\mathbb{Z}_2$ symmetry of the 2HDM scalar potential when extended to the Yukawa sector, extensions that were equivalent with respect to the permutations of quark flavors were excluded from the list. In contrast, extensions that are related by a permutation of the scalar doublets were not removed. In Table V, we exhibit the list as presented in Ref. [24].

TABLE V. Independent forms for the down-type Yukawa coupling matrices Γ_1 and Γ_2 that are compatible with the $\mathbb{Z}_2$ symmetry of the 2HDM scalar potential when extended to the Yukawa sector, labeled by the corresponding equation numbers of Ref. [24].

Eqs.	Γ_1	Γ_2
66	$\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
67	$\begin{pmatrix} x & x & 0 \\ x & x & 0 \\ x & x & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{pmatrix}$
68	$\begin{pmatrix} x & 0 & 0 \\ x & 0 & 0 \\ x & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & x & x \\ 0 & x & x \\ 0 & x & x \end{pmatrix}$
69	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix}$
Eqs.	Γ_1	Γ_2
71	$\begin{pmatrix} x & x & x \\ x & x & x \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & x & x \end{pmatrix}$
73	$\begin{pmatrix} x & x & 0 \\ x & x & 0 \\ 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & x \\ x & x & 0 \end{pmatrix}$
75	$\begin{pmatrix} x & 0 & 0 \\ x & 0 & 0 \\ 0 & x & x \end{pmatrix}$	$\begin{pmatrix} 0 & x & x \\ 0 & x & x \\ x & 0 & 0 \end{pmatrix}$
79	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & x & x \end{pmatrix}$	$\begin{pmatrix} x & x & x \\ x & x & x \\ 0 & 0 & 0 \end{pmatrix}$

2. Extensions of U(1) to the Yukawa sector

A list of the independent forms obtained in Ref. [24] for the down-type Yukawa coupling matrices Γ_1 and Γ_2 that are compatible with a global U(1) symmetry of the 2HDM scalar potential when extended to the Yukawa sector is given in Tables VI–IX below.

APPENDIX B: EVALUATION OF A DETERMINANT

In this appendix, we provide an explicit evaluation of the determinant of the 8×8 matrix given in Eq. (125). We begin by noticing that one can express A in 2×2 block matrix form consisting of 4×4 matrix blocks that can be written in terms of Kronecker products of 2×2 matrices,

$$A = \begin{pmatrix} c_\alpha \mathbb{1} & s_\alpha \mathbb{1} & X_{n_R}^{\mathsf{T}} & \mathbf{0} \\ -s_\alpha \mathbb{1} & c_\alpha \mathbb{1} & \mathbf{0} & X_{n_R}^{\mathsf{T}} \\ -X_{n_R}^{\mathsf{T}} & \mathbf{0} & c_\alpha \mathbb{1} & s_\alpha \mathbb{1} \\ \mathbf{0} & -X_{n_R}^{\mathsf{T}} & -s_\alpha \mathbb{1} & c_\alpha \mathbb{1} \end{pmatrix} = \begin{pmatrix} X_L \otimes \mathbb{1} & \mathbb{1}\otimes X_{n_R}^{\mathsf{T}} \\ -\mathbb{1}\otimes X_{n_R}^{\mathsf{T}} & X_L\otimes \mathbb{1} \end{pmatrix}, \tag{B1}$$

where $\mathbb{1}$ is the 2×2 identity matrix and $\mathbf{0}$ is the 2×2 zero matrix in Eq. (B1).

Using the well-known formula for the determinant of a 2×2 block matrix (e.g., see Ref. [49]),

$$\det\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \det M \det(Q - PM^{-1}N), \tag{B2}$$

where $Q - PM^{-1}N$ is the Schur complement of M (under the assumption that M is invertible), and noting that $\det(X_L\otimes \mathbb{1}) = 1$ and $X_L^{-1} = X_L^{\mathsf{T}}$, we obtain

$$\det A = \det[X_L \otimes \mathbb{1} + (\mathbb{1}\otimes X_{nR}^{\mathsf{T}})(X_L^{\mathsf{T}}\otimes \mathbb{1})(\mathbb{1}\otimes X_{nR}^{\mathsf{T}})]. \tag{B3}$$

We can manipulate Eq. (B3) into a more useful form by using the properties of the Kronecker product of two matrices,

$$\begin{aligned} \det A &= \det[X_L\otimes \mathbb{1} + (\mathbb{1}\otimes X_{nR}^{\mathsf{T}})(X_L^{\mathsf{T}}\otimes X_{nR}^{\mathsf{T}})] \\ &= \det\{(X_L\otimes \mathbb{1})[\mathbb{1}_4 + (X_L^{\mathsf{T}}\otimes X_{nR}^{\mathsf{T}})^2]\} \\ &= \det[\mathbb{1}_4 + (X_L^{\mathsf{T}}\otimes X_{nR}^{\mathsf{T}})^2], \end{aligned} \tag{B4}$$

after using $(X_L\otimes \mathbb{1})(X_L^{\mathsf{T}}\otimes \mathbb{1}) = \mathbb{1}_4$, where $\mathbb{1}_4$ is the 4×4 identity matrix. Noting that $X_L\otimes X_{n_R}$ is an orthogonal 4×4 matrix with unit determinant, it follows that

TABLE VI. Independent forms for the down-type Yukawa coupling matrices Γ_1 and Γ_2 that are compatible with the global U(1) symmetry of the 2HDM scalar potential when extended to the Yukawa sector, labeled by the corresponding equation numbers of Ref. [24]—Part 1.

Eqs.	Γ_1	Γ_2
57	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ x & x & 0 \end{pmatrix}$	$\begin{pmatrix} x & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{pmatrix}$
58	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & x & 0 \end{pmatrix}$	$\begin{pmatrix} x & x & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$
59	$\begin{pmatrix} 0 & 0 & x \\ x & x & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & x & 0 \end{pmatrix}$
60	$\begin{pmatrix} 0 & 0 & 0 \\ x & x & 0 \\ 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ x & x & 0 \end{pmatrix}$
Eqs.	Γ_1	Γ_2
61	$\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & x & 0 \end{pmatrix}$
62	$\begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & x \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix}$
63	$\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix}$
64	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ x & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix}$

TABLE VII. Independent forms for the down-type Yukawa coupling matrices Γ_1 and Γ_2 that are compatible with the global U (1) symmetry of the 2HDM scalar potential when extended to the Yukawa sector, labeled by the corresponding equation numbers of Ref. [24]—Part 2.

Eqs.	Γ_1	Γ_2
66	$\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
67	$\begin{pmatrix} x & x & 0 \\ x & x & 0 \\ x & x & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{pmatrix}$
68	$\begin{pmatrix} x & 0 & 0 \\ x & 0 & 0 \\ x & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & x & x \\ 0 & x & x \\ 0 & x & x \end{pmatrix}$
69	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix}$
Eqs.	Γ_1	Γ_2
71	$\begin{pmatrix} x & x & x \\ x & x & x \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & x & x \end{pmatrix}$
72	$\begin{pmatrix} x & x & 0 \\ x & x & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & x \\ x & x & 0 \end{pmatrix}$
74	$\begin{pmatrix} x & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & x & x \\ 0 & x & x \\ x & 0 & 0 \end{pmatrix}$
76	$\begin{pmatrix} x & x & 0 \\ x & x & 0 \\ 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & x & 0 \end{pmatrix}$

$\mathbb{1}_4 = (X_L \otimes X_{n_R})(X_L^\mathsf{T} \otimes X_{n_R}^\mathsf{T})$ and

$$\begin{aligned} \det A &= \det[(X_L^\mathsf{T} \otimes X_{nR}^\mathsf{T})(X_L \otimes X_{nR} + X_L^\mathsf{T} \otimes X_{nR}^\mathsf{T})] \\ &= \det[X_L \otimes X_{nR} + X_L^\mathsf{T} \otimes X_{nR}^\mathsf{T}]. \end{aligned} \tag{B5}$$

Explicitly, we have

$$\begin{aligned} &X_L \otimes X_{nR} + X_L^\mathsf{T} \otimes X_{nR}^\mathsf{T} \\ &= \begin{pmatrix} c_\alpha(X_{n_R} + X_{n_R}^\mathsf{T}) & s_\alpha(X_{n_R} - X_{n_R}^\mathsf{T}) \\ -s_\alpha(X_{n_R} - X_{n_R}^\mathsf{T}) & c_\alpha(X_{n_R} + X_{n_R}^\mathsf{T}) \end{pmatrix} \\ &= \begin{pmatrix} 2c_\alpha c_\beta \mathbb{1} & 2s_\alpha s_\beta \boldsymbol{J} \\ -2s_\alpha s_\beta \boldsymbol{J} & 2c_\alpha c_\beta \mathbb{1} \end{pmatrix}, \end{aligned} \tag{B6}$$

where $\mathbf{J} \equiv \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Using Eq. (B2) to evaluate the determinant of Eq. (B6) with $\boldsymbol{J}^2 = -\mathbb{1}$, one quickly obtains

$$\det A = 16(c_\alpha^2 c_\beta^2 - s_\alpha^2 s_\beta^2)^2, \tag{B7}$$

as indicated in Eq. (126).

APPENDIX C: EXTENSIONS OF $\mathbb{Z}_2 \otimes \Pi_2$ TO THE YUKAWA SECTOR OF THE 2HDM WITH TWO QUARK GENERATIONS

In Sec. VII B 3, we identified nine classes of models that are compatible with the extension of the $\mathbb{Z}_2 \otimes \Pi_2$ symmetry to the two-generation Yukawa sector. These models were obtained by determining the unitary matrices U_L and U_{n_R} that satisfy

$$\Gamma_1(U_{n_R} S_{nR} U_{n_R}^\dagger) = (U_L S_L U_L^\dagger)\Gamma_2, \tag{C1}$$

where the Yukawa coupling matrices $\{\Gamma_1, \Gamma_2\}$ have the form given by Cases 1, 2, or 3 specified in Eqs. (152)–(154) and the choices of the symmetry matrices S_L and S_{n_R} correspond to one of the following three choices exhibited in Eq. (163), which we repeat here for the convenience of the reader:

$$(S_L, S_{n_R}) \in \{(\sigma_Z, \mathbb{1}), (\mathbb{1}, \sigma_Z), (\sigma_Z, \sigma_Z)\}, \tag{C2}$$

TABLE VIII. Independent forms for the down-type Yukawa coupling matrices Γ_1 and Γ_2 that are compatible with the global U(1) symmetry of the 2HDM scalar potential when extended to the Yukawa sector, labeled by the corresponding equation numbers of Ref. [24]—Part 3.

Eqs.	Γ_1	Γ_2
77	$\begin{pmatrix} x & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} 0 & x & 0 \\ 0 & x & 0 \\ x & 0 & 0 \end{pmatrix}$
78	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} x & x & 0 \\ x & x & 0 \\ 0 & 0 & 0 \end{pmatrix}$
79	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & x & x \end{pmatrix}$	$\begin{pmatrix} x & x & x \\ x & x & x \\ 0 & 0 & 0 \end{pmatrix}$
81	$\begin{pmatrix} x & x & 0 \\ x & x & 0 \\ 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$

Eqs.	Γ_1	Γ_2
82	$\begin{pmatrix} x & 0 & 0 \\ x & 0 & 0 \\ 0 & x & x \end{pmatrix}$	$\begin{pmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{pmatrix}$
83	$\begin{pmatrix} x & x & 0 \\ x & x & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{pmatrix}$
84	$\begin{pmatrix} x & 0 & 0 \\ x & 0 & 0 \\ 0 & x & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & x & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$
85	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & x & x \end{pmatrix}$	$\begin{pmatrix} x & x & x \\ x & x & x \\ 0 & 0 & 0 \end{pmatrix}$

TABLE IX. Independent forms for the down-type Yukawa coupling matrices Γ_1 and Γ_2 that are compatible with the global U(1) symmetry of the 2HDM scalar potential when extended to the Yukawa sector, labeled by the corresponding equation numbers of Ref. [24]—Part 4.

Eqs.	Γ_1	Γ_2
86	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & x & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$
89	$\begin{pmatrix} x & x & 0 \\ 0 & 0 & x \\ 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ x & x & 0 \end{pmatrix}$
90	$\begin{pmatrix} x & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ x & x & 0 \end{pmatrix}$
91	$\begin{pmatrix} x & 0 & 0 \\ 0 & x & x \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & x & x \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}$

Eqs.	Γ_1	Γ_2
92	$\begin{pmatrix} x & 0 & 0 \\ x & x & 0 \\ 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} 0 & x & 0 \\ 0 & 0 & x \\ x & 0 & 0 \end{pmatrix}$
93	$\begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & x & x \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & x & x \\ x & 0 & 0 \end{pmatrix}$
94	$\begin{pmatrix} 0 & 0 & 0 \\ x & x & 0 \\ 0 & 0 & x \end{pmatrix}$	$\begin{pmatrix} x & x & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$
95	$\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & x & x \end{pmatrix}$	$\begin{pmatrix} x & 0 & 0 \\ 0 & x & x \\ 0 & 0 & 0 \end{pmatrix}$

where σ_Z is defined in Eq. (159). This yields nine possible model types, denoted by (n-m), corresponding to the Case n Yukawa coupling matrices and the mth set of possible choices for S_L and S_{n_R} given in Eq. (C2).

To solve Eq. (C1), we parametrize the unitary matrices U_L and U_{n_R} as in Eq. (166),

$$U_\sigma = \begin{pmatrix} e^{i\alpha_\sigma}\cos\theta_\sigma & e^{-i\beta_\sigma}\sin\theta_\sigma \\ -e^{i\beta_\sigma}\sin\theta_\sigma & e^{-i\alpha_\sigma}\cos\theta_\sigma \end{pmatrix}, \quad \text{with} \quad \sigma \in \{L, n_R\}, \tag{C3}$$

where an arbitrary global phase has been set to zero, $0 \le \theta_\sigma \le \frac{1}{2}\pi$, and $0 \le \alpha_\sigma, \beta_\sigma < 2\pi$. In the analysis of the possible model types below, we shall make use of the following quantity:

$$U_\sigma \sigma_Z U_\sigma^\dagger = \begin{pmatrix} \cos 2\theta_\sigma & -e^{i(\alpha_\sigma-\beta_\sigma)}\sin 2\theta_\sigma \\ -e^{-i(\alpha_\sigma-\beta_\sigma)}\sin 2\theta_\sigma & -\cos 2\theta_\sigma \end{pmatrix}. \tag{C4}$$

1. Model (1-1)

$$\Gamma_1 = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 \\ x_{21} & x_{22} \end{pmatrix},$$
$$S_L = \sigma_Z, \qquad S_{nR} = \mathbb{1}. \tag{C5}$$

Plugging these results into Eq. (C1) and using Eq. (C4) yields

$$\cos 2\theta_L = 0, \tag{C6}$$

$$x_{11} = -x_{21} e^{i(\alpha_L-\beta_L)} \sin 2\theta_L, \tag{C7}$$

$$x_{12} = -x_{22} e^{i(\alpha_L-\beta_L)} \sin 2\theta_L. \tag{C8}$$

It follows that $\theta_L = \pi/4$ and

$$\Gamma_2 = -e^{-i(\alpha_L-\beta_L)} \begin{pmatrix} 0 & 0 \\ x_{11} & x_{12} \end{pmatrix}. \tag{C9}$$

Finally, we compute

$$H_d = (v_1\Gamma_1 + v_2\Gamma_2)(v_1\Gamma_1 + v_2\Gamma_2)^\dagger = [|x_{11}|^2 + |x_{12}|^2]\begin{pmatrix} |v_1|^2 & -v_1 v_2^* e^{i(\alpha_L-\beta_L)} \\ -v_1^* v_2 e^{-i(\alpha_L-\beta_L)} & |v_2|^2 \end{pmatrix}. \tag{C10}$$

Thus, $\det H_d = 0$, corresponding to the existence of a massless down-type quark. Thus, we discard this model.

2. Model (1-2)

$$\Gamma_1 = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 \\ x_{21} & x_{22} \end{pmatrix}, \qquad S_L = \mathbb{1}, \qquad S_{nR} = \sigma_Z. \tag{C11}$$

Plugging these results into Eq. (C1) and using Eq. (C3) yields $x_{21} = x_{22} = 0$. In light of Eq. (C1), $\Gamma_1 = \Gamma_2 = 0$, and the model is discarded.

3. Model (1-3)

$$\Gamma_1 = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 \\ x_{21} & x_{22} \end{pmatrix}, \quad S_L = S_{n_R} = \sigma_Z. \tag{C12}$$

This case has been treated explicitly in Sec. VII B 3. We found that $x_{21} = x_{12}$ and $x_{22} = x_{11}$, and

$$\mathrm{Tr}\, H_d = [|x_{11}|^2 + |x_{12}|^2]v^2, \tag{C13}$$

$$\det\, H_d = |v_1|^2|v_2|^2|x_{11}^2 - x_{12}^2|^2, \tag{C14}$$

where $v^2 \equiv |v_1|^2 + |v_2|^2$. Thus, the two down-type quarks are generically nonzero. In addition, the corresponding Π_2 symmetry matrices obtained from Eq. (162) are

$$S_L^{(\Pi_2)} = \sigma_\Pi, \qquad S_{nR}^{(\Pi_2)} = \sigma_\Pi, \tag{C15}$$

where σ_Π is defined in Eq. (160).

4. Model (2-1)

We start in Case 2 of $\mathbb{Z}_2$:

$$\Gamma_1 = \begin{pmatrix} x_{11} & 0 \\ x_{21} & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & x_{12} \\ 0 & x_{22} \end{pmatrix}, \quad S_L = \sigma_Z, \quad S_{nR} = \mathbb{1}. \tag{C16}$$

Plugging these results into Eq. (C1) and using Eq. (C4) yields $x_{11} = x_{21} = 0$. In light of Eq. (C1), $\Gamma_1 = \Gamma_2 = 0$, and the model is discarded.

5. Model (2-2)

$$\Gamma_1 = \begin{pmatrix} x_{11} & 0 \\ x_{21} & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & x_{12} \\ 0 & x_{22} \end{pmatrix}, \quad S_L = \mathbb{1}, \quad S_{nR} = \sigma_Z. \tag{C17}$$

Plugging these results into Eq. (C1) and using Eq. (C4) yields

$$\cos 2\theta_R = 0, \tag{C18}$$

$$x_{12} = -x_{11} e^{i(\alpha_R-\beta_R)} \sin 2\theta_R, \tag{C19}$$

$$x_{22} = -x_{21} e^{i(\alpha_R-\beta_R)} \sin 2\theta_R, \tag{C20}$$

where we have simplified the R subscript in writing $\alpha_R \equiv \alpha_{n_R}$, $\beta_R \equiv \beta_{n_R}$, and $\theta_R \equiv \theta_{n_R}$. It follows that $\theta_R = \pi/4$ and

$$\Gamma_2 = -e^{i(\alpha_R-\beta_R)}\begin{pmatrix} 0 & x_{11} \\ 0 & x_{21} \end{pmatrix}. \tag{C21}$$

Finally, we compute

$$H_d = (v_1\Gamma_1 + v_2\Gamma_2)(v_1\Gamma_1 + v_2\Gamma_2)^\dagger = v^2\begin{pmatrix} |x_{11}|^2 & x_{11}x_{21}^* \\ x_{11}^* x_{21} & |x_{21}|^2 \end{pmatrix}. \tag{C22}$$

Thus, $\det H_d = 0$, corresponding to the existence of a massless down-type quark. Thus, we discard this model.

6. Model (2-3)

$$\Gamma_1 = \begin{pmatrix} x_{11} & 0 \\ x_{21} & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & x_{12} \\ 0 & x_{22} \end{pmatrix}, \quad S_L = S_{n_R} = \sigma_Z. \tag{C23}$$

Plugging these results into Eq. (C1) and using Eq. (C4) yields

$$\cos\, 2\theta_R = 0, \tag{C24}$$

$$x_{11} e^{i(\alpha_R-\beta_R)} \sin\, 2\theta_R = x_{22} e^{-i(\alpha_L-\beta_L)} \sin\, 2\theta_L - x_{12} \cos\, 2\theta_L, \tag{C25}$$

$$x_{21} e^{i(\alpha_R-\beta_R)} \sin\, 2\theta_R = x_{12} e^{i(\alpha_L-\beta_L)} \sin\, 2\theta_L + x_{22} \cos\, 2\theta_L. \tag{C26}$$

It follows that $\theta_R = \pi/4$. Then, Eq. (162) yields

$$S_L^{(\Pi_2)} = U_L \sigma_Z U_L^\dagger = \begin{pmatrix} \cos 2\theta_L & -e^{i(\alpha_L-\beta_L)}\sin 2\theta_L \\ -e^{-i(\alpha_L-\beta_L)}\sin 2\theta_L & -\cos 2\theta_L \end{pmatrix}, \tag{C27}$$

$$S_{n_R}^{(\Pi_2)} = U_{n_R}\sigma_Z U_{n_R}^\dagger = \begin{pmatrix} 0 & -e^{i(\alpha_R-\beta_R)} \\ -e^{-i(\alpha_R-\beta_R)} & 0 \end{pmatrix}. \tag{C28}$$

The $\mathbb{Z}_2 \otimes \Pi_2$ symmetry constraints do not fix the remaining free parameter, α_L, β_L, α_R, β_R, and θ_L. Indeed, one is free to transform to a different quark field basis as long as the $\mathbb{Z}_2$ symmetry matrices $S_L^{(\mathbb{Z}_2)} = e^{i\alpha_1}\sigma_Z$ and $S_{n_R}^{(\mathbb{Z}_2)} = e^{\alpha_1}\mathbb{1}$ are unchanged. In light of Eqs. (35) and (36), we shall transform

$$S_L^{(\Pi_2)} \to U'_L S_L^{(\Pi_2)} U_L'^\dagger, \qquad S_{n_R}^{(\Pi_2)} \to U'_{n_R} S_{n_R}^{(\Pi_2)} U_{n_R}'^\dagger, \tag{C29}$$

where $U'_L = \mathrm{diag}(e^{i(\gamma+\delta)}, e^{i(\gamma-\delta)})$ is the most general 2×2 unitary matrix that leaves $S_L^{(\mathbb{Z}_2)}$ unchanged and U'_{n_R} is an arbitrary 2×2 unitary matrix that (trivially) leaves $S_{n_R}^{(\mathbb{Z}_2)}$ unchanged. With this freedom, it is convenient to set

$$\alpha_L - \beta_L = \alpha_R - \beta_R = \pi, \qquad \theta_L = \frac{1}{4}\pi. \tag{C30}$$

With this choice, Eqs. (C24)–(C26) yield $x_{12} = x_{21}$ and $x_{22} = x_{11}$. That is, Model (2-3) corresponds to

$$\Gamma_1 = \begin{pmatrix} x_{11} & 0 \\ x_{21} & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & x_{21} \\ 0 & x_{11} \end{pmatrix},$$
$$S_L^{(\Pi_2)} = e^{i\xi_2}\sigma_\Pi, \qquad S_{nR}^{(\Pi_2)} = e^{i\xi_2}\sigma_\Pi. \tag{C31}$$

It is straightforward to compute $H_d = (v_1\Gamma_1 + v_2\Gamma_2)(v_1\Gamma_1 + v_2\Gamma_2)^\dagger$ and its trace and determinant, which yield

$$\mathrm{Tr}\, H_d = [|x_{11}|^2 + |x_{21}|^2]v^2, \tag{C32}$$

$$\det H_d = |v_1|^2|v_2|^2|x_{11}^2 - x_{21}^2|^2. \tag{C33}$$

Note that $\det H_d$ is nonzero for a generic choice of parameters, which implies that the two down-type quark masses are generically nonzero.

7. Model (3-1)

$$\Gamma_1 = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix},$$
$$S_L = \sigma_Z, \qquad S_{nR} = \mathbb{1}. \tag{C34}$$

Plugging these results into Eq. (C1) and using Eq. (C4) yields

$$\cos 2\theta_L = 0, \tag{C35}$$

$$x_{11} = -x_{21}e^{i(\alpha_L-\beta_L)}\sin 2\theta_L, \tag{C36}$$

$$x_{22} = -x_{12}e^{-i(\alpha_L-\beta_L)}\sin 2\theta_L. \tag{C37}$$

It follows that $\theta_L = \pi/4$ and

$$\Gamma_2 = \begin{pmatrix} 0 & -x_{22}e^{i(\alpha_L-\beta_L)} \\ -x_{11}e^{-i(\alpha_L-\beta_L)} & 0 \end{pmatrix}. \tag{C38}$$

Then, Eq. (162) yields

$$S_L^{(\Pi_2)} = U_L\sigma_Z U_L^\dagger = \begin{pmatrix} 0 & -e^{i(\alpha_L-\beta_L)} \\ -e^{-i(\alpha_L-\beta_L)} & 0 \end{pmatrix},$$
$$S_{n_R}^{(\Pi_2)} = U_{n_R}U_{n_R}^\dagger = \mathbb{1}. \tag{C39}$$

Once again, we are free to change the quark field basis using Eq. (C29) where $U'_L = \mathrm{diag}(e^{i(\gamma+\delta)}, e^{i(\gamma-\delta)})$ is the most general 2×2 unitary matrix that leaves $S_L^{(\mathbb{Z}_2)}$ unchanged and U'_{n_R} is an arbitrary 2×2 unitary matrix that (trivially) leaves $S_{n_R}^{(\mathbb{Z}_2)}$ unchanged. With this freedom, it is convenient to set $\alpha_L - \beta_L = \pi$, which yields

$$\Gamma_1 = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & x_{22} \\ x_{11} & 0 \end{pmatrix},$$
$$S_L^{(\Pi_2)} = \sigma_\Pi, \qquad S_{nR}^{(\Pi_2)} = \mathbb{1}. \tag{C40}$$

It is straightforward to compute $H_d = (v_1\Gamma_1 + v_2\Gamma_2)(v_1\Gamma_1 + v_2\Gamma_2)^\dagger$ and its trace and determinant, which yield

$$\mathrm{Tr}\, H_d = [|x_{11}|^2 + |x_{22}|^2]v^2, \tag{C41}$$

$$\det H_d = |x_{11}|^2|x_{22}|^2|v_1^2 - v_2^2|^2. \tag{C42}$$

Note that $\det H_d$ is nonzero for a generic choice of parameters, which implies that the two down-type quark masses are generically nonzero.

8. Model (3-2)

$$\Gamma_1 = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix},$$
$$S_L = \mathbb{1}, \qquad S_{nR} = \sigma_Z. \tag{C43}$$

Plugging these results into Eq. (C1) and using Eq. (C4) yields

$$\cos 2\theta_R = 0, \tag{C44}$$

$$x_{12} = -x_{11}e^{i(\alpha_R-\beta_R)}\sin 2\theta_R, \tag{C45}$$

$$x_{21} = -x_{22} e^{-i(\alpha_R-\beta_R)} \sin 2\theta_R. \qquad \text{(C46)}$$

It follows that $\theta_R = \pi/4$ and

$$\Gamma_2 = \begin{pmatrix} 0 & -x_{11} e^{i(\alpha_R-\beta_R} \\ -x_{22} e^{-i(\alpha_R-\beta_R} & 0 \end{pmatrix}. \qquad \text{(C47)}$$

Then, Eq. (162) yields

$$S_L^{(\Pi_2)} = U_{n_L} U_{n_L}^\dagger = \mathbb{1},$$
$$S_{n_R}^{(\Pi_2)} = U_R \sigma_Z U_R^\dagger = \begin{pmatrix} 0 & -e^{i(\alpha_R-\beta_R)} \\ -e^{-i(\alpha_R-\beta_R)} & 0 \end{pmatrix}. \qquad \text{(C48)}$$

We are free again to change the quark field basis using Eq. (C29) where $U_R' = \mathrm{diag}(e^{i(\gamma+\delta)}, e^{i(\gamma-\delta)})$ is the most general 2×2 unitary matrix that leaves $S_{n_R}^{(\mathbb{Z}_2)}$ unchanged and U_L' is an arbitrary 2×2 unitary matrix that (trivially) leaves $S_L^{(\mathbb{Z}_2)}$ unchanged. With this freedom, it is convenient to set $\alpha_R - \beta_R = \pi$, which yields

$$\Gamma_1 = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & x_{11} \\ x_{22} & 0 \end{pmatrix},$$
$$S_L^{(\Pi_2)} = \mathbb{1}, \qquad S_{nR}^{(\Pi_2)} = \sigma_\Pi. \qquad \text{(C49)}$$

It is straightforward to compute $H_d = (v_1\Gamma_1 + v_2\Gamma_2)(v_1\Gamma_1 + v_2\Gamma_2)^\dagger$ and its trace and determinant, which yield

$$\mathrm{Tr}\, H_d = [|x_{11}|^2 + |x_{22}|^2] v^2, \qquad \text{(C50)}$$

$$\det H_d = |x_{11}|^2 |x_{22}|^2 |v_1^2 - v_2^2|^2. \qquad \text{(C51)}$$

Note that $\det H_d$ is nonzero for a generic choice of parameters, which implies that the two down-type quark masses are generically nonzero.

9. Models (3-3)

We shall see below that there are a number of possible submodels within the class of Models (3-3). This model class is defined by the following Yukawa coupling matrices and symmetry matrices:

$$\Gamma_1 = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix}, \quad S_L = S_{n_R} = \sigma_Z. \qquad \text{(C52)}$$

Plugging these results into Eq. (C1) and using Eq. (C4) yields

$$x_{11} \cos 2\theta_R + x_{21} e^{i(\alpha_L-\beta_L)} \sin 2\theta_L = 0, \qquad \text{(C53)}$$

$$x_{11} e^{i(\alpha_R-\beta_R)} \sin 2\theta_R + x_{12} \cos 2\theta_L = 0, \qquad \text{(C54)}$$

$$x_{22} e^{-i(\alpha_R-\beta_R)} \sin 2\theta_R - x_{21} \cos 2\theta_L = 0, \qquad \text{(C55)}$$

$$x_{22} \cos 2\theta_R - x_{12} e^{-i(\alpha_L-\beta_L)} \sin 2\theta_L = 0. \qquad \text{(C56)}$$

This is a homogeneous system of four linear equations in the variables x_{11}, x_{12}, x_{21}, and x_{22}. Nontrivial solutions exist only if

$$\det \begin{pmatrix} \cos 2\theta_R & 0 & e^{i(\alpha_L-\beta_L)} \sin 2\theta_L & 0 \\ e^{i(\alpha_R-\beta_R)} \sin 2\theta_R & \cos 2\theta_L & 0 & 0 \\ 0 & 0 & -\cos 2\theta_L & e^{-i(\alpha_R-\beta_R)} \sin 2\theta_R \\ 0 & -e^{-i(\alpha_L-\beta_L)} \sin 2\theta_L & 0 & \cos 2\theta_R \end{pmatrix} = 0, \qquad \text{(C57)}$$

which simplifies to

$$\sin^2 2\theta_L \sin^2 2\theta_R - \cos^2 2\theta_L \cos^2 2\theta_R = 0. \qquad \text{(C58)}$$

The solution to this equation is

$$\cos[2(\theta_L \pm \theta_R)] = 0. \qquad \text{(C59)}$$

We now consider separately the following submodels.

10. Model $(3,3)_0$: $\theta_L = 0$ and $\theta_R = \pi/4$

Equation (C53)–(C56) yields $x_{12} = -x_{11} e^{i(\alpha_R-\beta_R)}$ and $x_{21} = x_{22} e^{-i(\alpha_R-\beta_R)}$. Using Eq. (162) yields

$$S_L^{(\Pi_2)} = U_L \sigma_Z U_L^\dagger = \sigma_Z,$$
$$S_{n_R}^{(\Pi_2)} = U_{n_R} \sigma_Z U_{n_R}^\dagger = \begin{pmatrix} 0 & -e^{i(\alpha_R-\beta_R)} \\ -e^{-i(\alpha_R-\beta_R)} & 0 \end{pmatrix}. \qquad \text{(C60)}$$

As in previous cases, one can transform to another quark field basis where $\alpha_R - \beta_R = \pi$, which yields

TABLE X. Submodels within the class of Models (3-3). Cases (i)–(iv) comprise models where neither θ_L nor θ_R is equal to $\pi/4$. Since $0 \le \theta_L,\ \theta_R \le \pi/2$, it follows that $0 < \theta_L < \pi/4$ in Cases (i) and (ii) above and $\pi/4 < \theta_L < \pi/2$ in Cases (iii) and (iv) above. The relations among the x_{ij} are obtained from Eqs. (C53)–(C56) in a quark field basis where $\alpha_L - \beta_L = \alpha_R - \beta_R = \pi$. For Cases (i)–(iv), the ξ_{ij} are defined such that $x_{ij} \equiv \xi_{ij} x_{11}$ with $\xi_{11} = 1$.

Model	θ_R	$\sin 2\theta_R$	$\cos 2\theta_R$	x_{ij} relations	ξ_{ij}
$\theta_L = 0$	$\frac{1}{4}\pi$	1	0	$x_{12} = x_{11}$ and $x_{21} = -x_{22}$	
$\theta_L = \frac{1}{4}\pi$	0	0	1	$x_{21} = x_{11}$ and $x_{12} = -x_{22}$	
Case (i)	$\frac{1}{4}\pi - \theta_L$	$\cos 2\theta_L$	$\sin 2\theta_L$	$x_{11} = -x_{22} = x_{12} = x_{21}$	$\xi_{12} = \xi_{21} = 1,\ \xi_{22} = -1$
Case (ii)	$\frac{1}{4}\pi + \theta_L$	$\cos 2\theta_L$	$-\sin 2\theta_L$	$x_{11} = x_{22} = x_{12} = -x_{21}$	$\xi_{12} = \xi_{22} = 1,\ \xi_{21} = -1$
Case (iii)	$\frac{3}{4}\pi - \theta_L$	$-\cos 2\theta_L$	$-\sin 2\theta_L$	$x_{11} = -x_{22} = -x_{12} = -x_{21}$	$\xi_{12} = \xi_{21} = \xi_{22} = -1$
Case (iv)	$-\frac{1}{4}\pi + \theta_L$	$-\cos 2\theta_L$	$\sin 2\theta_L$	$x_{11} = x_{22} = -x_{12} = x_{21}$	$\xi_{21} = \xi_{22} = 1,\ \xi_{12} = -1$

$$\Gamma_1 = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & x_{11} \\ -x_{22} & 0 \end{pmatrix},$$
$$S_L^{(\Pi_2)} = \sigma_Z, \qquad S_{nR}^{(\Pi_2)} = \sigma_\Pi. \tag{C61}$$

It is straightforward to compute $H_d = (v_1\Gamma_1 + v_2\Gamma_2)(v_1\Gamma_1 + v_2\Gamma_2)^\dagger$ and its trace and determinant, which yield[15]

$$\mathrm{Tr}\, H_d = [|x_{11}|^2 + |x_{22}|^2] v^2, \tag{C62}$$

$$\det\, H_d = |x_{11}|^2 |x_{22}|^2 |v_1^2 + v_2^2|^2. \tag{C63}$$

Note that $\det H_d$ is nonzero for a generic choice of parameters, which implies that the two down-type quark masses are generically nonzero.

11. Model $(3,3)_1$: $\theta_L = \pi/4$ and $\theta_R = 0$

Equation (C53)–(C56) yields $x_{21} = -x_{11} e^{-i(\alpha_L - \beta_L)}$ and $x_{12} = x_{22} e^{i(\alpha_L - \beta_L)}$. The computation is similar to the case of Model $(3,3)_0$. Transforming to another quark field basis where $\alpha_L - \beta_L = \pi$, we end up with

$$\Gamma_1 = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & -x_{22} \\ x_{11} & 0 \end{pmatrix},$$
$$S_L^{(\Pi_2)} = \sigma_\Pi, \qquad S_{nR}^{(\Pi_2)} = \sigma_Z. \tag{C64}$$

We again obtain

$$\mathrm{Tr}\, H_d = [|x_{11}|^2 + |x_{22}|^2] v^2, \tag{C65}$$

$$\det\, H_d = |x_{11}|^2 |x_{22}|^2 |v_1^2 + v_2^2|^2, \tag{C66}$$

corresponding to nonzero down-type quark masses for a generic choice of parameters.

[15]Note that in general $v^2 \equiv |v_1|^2 + |v_2|^2 \ne v_1^2 + v_2^2$, since v_1 and v_2 are generically complex [cf. Eq. (6)].

12. Models $(3\text{-}3)_X$: $\theta_L, \theta_R \ne 0, \pi/4$

The class of submodels under consideration here correspond to solutions of Eq. (C59) where neither θ_L nor θ_R is equal to 0 or $\pi/4$. There are four possible cases, denoted by Cases (i)–(iv), which are defined in Table X. Using Eqs. (C53)–(C56), one easily obtains the constraints on the elements of the Yukawa coupling matrices in Cases (i)–(iv), which are listed in Table X. It is convenient to write

$$x_{ij} = \xi_{ij} x_{12}, \tag{C67}$$

where $\xi_{12} = 1$ and $\xi_{ij} = \pm 1$ for $ij = 12$, 21, and 22, where the signs are determined from the x_{ij} listed in Table X.

As previously noted, one can simplify the analysis by a judicious choice of the quark field basis. Following Eqs. (C29) and (C30), we shall fix the phases $\alpha_L - \beta_L = \alpha_R - \beta_R = \pi$. With this choice, the Yukawa coupling matrices are given by

$$\Gamma_1 = x_{11} \begin{pmatrix} 1 & 0 \\ 0 & \xi_{22} \end{pmatrix}, \qquad \Gamma_2 = x_{11} \begin{pmatrix} 0 & \xi_{12} \\ \xi_{21} & 0 \end{pmatrix}, \tag{C68}$$

and the Π_2 symmetry matrices, which are obtained from Eq. (162), are given by

$$S_L^{(\Pi_2)} = \begin{pmatrix} \cos 2\theta_L & \sin 2\theta_L \\ \sin 2\theta_L & -\cos 2\theta_L \end{pmatrix},$$
$$S_{nR}^{(\Pi_2)} = \begin{pmatrix} \xi_{21} \sin 2\theta_L & \xi_{12} \cos 2\theta_L \\ \xi_{12} \cos 2\theta_L & -\xi_{21} \sin 2\theta_L \end{pmatrix}. \tag{C69}$$

It is straightforward to compute $H_d = (v_1\Gamma_1 + v_2\Gamma_2)(v_1\Gamma_1 + v_2\Gamma_2)^\dagger$ and its trace and determinant:

$$H_d = (\Gamma_1 v_1 + \Gamma_2 v_2)(\Gamma_1 v_1 + \Gamma_2 v_2)^\dagger$$
$$= |x_{11}^2| \begin{pmatrix} v^2 & 2i\xi_{21}\,\mathrm{Im}(v_1 v_2^*) \\ -2i\xi_{21}\,\mathrm{Im}(v_1 v_2^*) & v^2 \end{pmatrix}, \tag{C70}$$

$$\mathrm{Tr}\, H_d = 2|x_{11}|^2 v^2, \tag{C71}$$

$$\det H_d = |x_{11}|^4 v^4, \tag{C72}$$

where $\xi_{21} = \pm 1$, with the sign depending on the case chosen. As in the previous (3-3) submodels, the down-type quark masses are nonzero. However, in contrast to Models $(3,3)_0$ and $(3,3)_1$, we note that the down-type quark masses are degenerate if $\mathrm{Im}(v_1 v_2^*) = 0$.

The analysis of the up-type quark Yukawa coupling matrices yields the same textures for Δ_1 and Δ_2 exhibited in Eq. (C68), with the matrix element x_{11} replaced by y_{11} and sign factors ξ_{ij} replaced by ξ'_{ij}. In addition, $S_L^{(\Pi_2)}$ is given by Eq. (C69) and $S_{p_R}^{(\Pi_2)}$ is obtained from $S_{n_R}^{(\Pi_2)}$ by replacing ξ_{ij} with ξ'_{ij}. It therefore follows that

$$H_u = (\Delta_1 v_1 + \Delta_2 v_2)(\Delta_1 v_1 + \Delta_2 v_2)^\dagger = |y_{11}|^2 \begin{pmatrix} v^2 & 2i\xi'_{21}\,\mathrm{Im}(v_1 v_2^*) \\ -2i\xi'_{21}\,\mathrm{Im}(v_1 v_2^*) & v^2 \end{pmatrix}, \tag{C73}$$

where $\xi'_{21} = \pm 1$, with the sign depending on the case chosen. Since $\det H_u = |y_{11}|^4 v^2$, it follows that the up-type quark masses are nonzero. However, note that

$$[H_u, H_d] = 0, \tag{C74}$$

for either choice of $\xi_{21}\xi'_{21} = \pm 1$. In particular, $\det\{[H_u, H_d]\} = 0$ or equivalently $J_c = 0$ [cf. Eq. (138)], corresponding to a vanishing Cabibbo angle. Thus, it follows that all Models $(3\text{-}3)_X$ are phenomenologically excluded.

APPENDIX D: EQUIVALENT TWO-GENERATION $\mathbb{Z}_2 \otimes \Pi_2$-SYMMETRIC MODELS

In Table IV, we classified all phenomenologically viable two-generation Yukawa-extended $\mathbb{Z}_2 \otimes \Pi_2$-symmetric models (i.e., models with nonzero quark masses and a nonzero Cabibbo angle). However, the models that appear in Table IV, are not all inequivalent, as certain pairs of models are related by a particular change in the Higgs field and the quark field basis. Suppose we transform to a new basis characterized by the basis transformation matrices

$$U = U_L = U_{n_R} = U_{p_R} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{D1}$$

Using Eq. (26), it follows that the Model (1-3) down-type Yukawa coupling matrices listed in Table III transform into

$$\Gamma'_1 = \begin{pmatrix} x'_{11} & 0 \\ 0 & x'_{22} \end{pmatrix}, \qquad \Gamma'_2 = \begin{pmatrix} 0 & x'_{22} \\ x'_{11} & 0 \end{pmatrix}, \tag{D2}$$

where

$$x'_{11} = x'_{21} = \frac{x_{11} + x_{12}}{\sqrt{2}}, \qquad x'_{12} = x'_{22} = \frac{x_{11} - x_{12}}{\sqrt{2}}. \tag{D3}$$

That is, the textures of Γ'_1 and Γ'_2 precisely match those of Model (3-1) listed in Table III. Moreover, in light of Eq. (25), the VEVs are transformed into

$$v'_1 = \frac{1}{\sqrt{2}}(v_1 + v_2), \qquad v'_2 = \frac{1}{\sqrt{2}}(v_1 - v_2). \tag{D4}$$

It follows that the trace and determinant of H_d given in Eqs. (176) and (177) are transformed into

$$\mathrm{Tr}\, H_d = [|x'_{11}|^2 + |x'_{12}|^2] v^2, \tag{D5}$$

$$\det H_d = |x'_{11}|^2 |x'_{22}|^2 |v'^2_1 - v'^2_2|^2, \tag{D6}$$

in agreement with the Model (3-1) results obtained in Eqs. (C41) and (C42). Finally, if we make use of Eqs. (35) and (36), we obtain the corresponding symmetry matrices

$$S'^{(\mathbb{Z}_2)} = \sigma_\Pi, \qquad S'^{(\mathbb{Z}_2)}_L = \sigma_\Pi, \qquad S'^{(\mathbb{Z}_2)}_{nR} = \mathbb{1}, \qquad S'^{(\Pi_2)} = \sigma_Z, \qquad S'^{(\Pi_2)}_L = \sigma_Z, \qquad S'^{(\Pi_2)}_{nR} = \sigma_Z. \tag{D7}$$

We cannot directly compare this result to Table III, as the results of this table have assumed a scalar field basis where $S^{(\mathbb{Z}_2)} = \sigma_Z$ and $S^{(\Pi_2)} = \sigma_\Pi$. But, we can overcome this impediment simply by interchanging the roles of the $\mathbb{Z}_2$ and Π_2 symmetries, in which case Eq. (D7) becomes

$$S'^{(\mathbb{Z}_2)} = \sigma_Z, \qquad S'^{(\mathbb{Z}_2)}_L = \sigma_Z, \qquad S'^{(\mathbb{Z}_2)}_{nR} = \sigma_Z, \qquad S'^{(\Pi_2)} = \sigma_\Pi, \qquad S'^{(\Pi_2)}_L = \sigma_\Pi, \qquad S'^{(\Pi_2)}_{nR} = \mathbb{1}, \tag{D8}$$

which precisely matches the symmetry matrices of Model (3-1) listed in Table III. Hence, we conclude that Models (1-3) and (3-1) are simply related by a basis change and hence can be regarded as equivalent.

Note that the basis transformation matrices given in Eq. (D1) are equal to their inverses. Thus, if we start with the Model (3-1) down-type Yukawa coupling matrices and apply the same procedure as outlined above, one ends up with corresponding Yukawa coupling matrices of Model (1-3). Likewise, applying the same basis transformation matrices followed by interchanging the identification of the $\mathbb{Z}_2$ and Π_2 symmetries also converts down-type Yukawa coupling matrices of Model (2-3) into those of Model (3-2) and vice versa.

The same arguments also apply to the up-type Yukawa coupling matrices. In particular, the textures of Δ_1 and Δ_2 mirror those of Γ_1 and Γ_2 given in Table III. For the basis transformation characterized by the matrices specified in Eq. (D1), we can use the same treatment that was employed above in the analysis of the down-type Yukawa coupling matrices to conclude that the two models in each of the following model pairs that appear in Table IV,

$$\{(1\text{-}3)/(1\text{-}3);(3\text{-}1)/(3\text{-}1)\},\qquad \{(1\text{-}3)/(3\text{-}1);(3\text{-}1)/(1\text{-}3)\},\qquad \{(2\text{-}3)/(2\text{-}3);(3\text{-}2)/(3\text{-}2)\},\tag{D9}$$

are equivalent (i.e., they correspond to the same model with a different choice of basis). However, the above analysis also shows that, e.g., Models (1-3)/(3-1) and (1-3)/(1-3) are *not* equivalent, since one must employ the *same* basis transformation matrices to both the down-type and the up-type Yukawa coupling matrices.

One can also consider a change of basis for which $U_{p_R} \neq U_{n_R}$. Suppose we take U, U_L, and U_{p_R} given in Eq. (D1) and $U_{n_R} = \sigma_Z$. Using Eq. (26), it follows that Model $(3,3)_1$ is the unique model of Table III such that $\Gamma_i' = \Gamma_i$ in the transformed basis. The transformed symmetry matrices of Model $(3\text{-}3)_1$ are obtained using Eqs. (35) and (36),

$$S'^{(\mathbb{Z}_2)} = \sigma_\Pi,\qquad S_L'^{(\mathbb{Z}_2)} = \sigma_\Pi,\qquad S_{nR}'^{(\mathbb{Z}_2)} = \sigma_Z,\qquad S'^{(\Pi_2)} = \sigma_Z,\qquad S_L'^{(\Pi_2)} = \sigma_Z,\qquad S_{nR}'^{(\Pi_2)} = \sigma_Z.\tag{D10}$$

Finally, after interchanging the identification of the $\mathbb{Z}_2$ and Π_2 symmetries, we reproduce the original symmetry matrices of Model $(3\text{-}3)_1$. Consequently, it follows that the two models in the following model pair are equivalent:

$$\{(3\text{-}3)_1/(1\text{-}3);(3\text{-}3)_1/(3\text{-}1)\}.\tag{D11}$$

Similarly, suppose we take U, U_L, and U_{n_R} given in Eq. (D1) and $U_{p_R} = \sigma_Z$. Using the same analysis as the one just given, where Eq. (27) is now employed, it follows that $\Delta_i' = \Delta_i$ in the transformed basis. Hence, the two models in the following model pair are equivalent:

$$\{(1\text{-}3)/(3\text{-}3)_1;(3\text{-}1)/(3\text{-}3)_1\}.\tag{D12}$$

This completes the search for equivalent models. Taking Eqs. (D9), (D11), and (D12) into account, there are seven inequivalent models for the Yukawa sector in Table IV.

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