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Generalized *CP* symmetries and special regions of parameter space in the two-Higgs-doublet model

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We consider the impact of imposing generalized *CP* symmetries on the Higgs sector of the two-Higgs-doublet model, and identify three classes of symmetries. Two of these classes constrain the scalar potential parameters to an exceptional region of parameter space, which respects either a Z_2 discrete flavor symmetry or a $U(1)$ symmetry. We exhibit a basis-invariant quantity that distinguishes between these two possible symmetries. We also show that the consequences of imposing these two classes of *CP* symmetry can be achieved by combining Higgs family symmetries, and that this is not possible for the usual *CP* symmetry. We comment on the vacuum structure and on renormalization in the presence of these symmetries. Finally, we demonstrate that the standard *CP* symmetry can be used to build all the models we identify, including those based on Higgs family symmetries.

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I. INTRODUCTION

Despite the fantastic successes of the standard model (SM) of electroweak interactions, its scalar sector remains largely untested [1]. An alternative to the single Higgs doublet of the SM is provided by the two-Higgs-doublet model (THDM), which can be supplemented by symmetry requirements on the Higgs fields Φ_1 and Φ_2. Symmetries leaving the kinetic terms unchanged[1] may be of two types. On the one hand, one may relate Φ_a with some unitary transformation of Φ_b. These are known as Higgs family symmetries, or HF symmetries. On the other hand, one may relate Φ_a with some unitary transformation of Φ_b^*. These are known as generalized *CP* symmetries, or GCP symmetries. In this article we consider all such symmetries that are possible in the THDM, according to their impact on the Higgs potential. We identify three classes of GCP symmetries.

The study is complicated by the fact that one may perform a basis transformation on the Higgs fields, thus hiding what might otherwise be an easily identifiable symmetry. The need to seek basis-invariant observables in models with many Higgs was pointed out by Lavoura and Silva [5], and by Botella and Silva [6], stressing applications to *CP* violation. References [6,7] indicate how to construct basis-invariant quantities in a systematic fashion for any model, including multi-Higgs-doublet models. Work on basis invariance in the THDM was much expanded upon by Davidson and Haber [8], by Gunion and Haber [9,10], by Haber and O'Neil [11], and by other authors [12]. The previous approaches highlight the role played by the Higgs fields. An alternative approach, spearheaded by Nishi [13,14], by Ivanov [3,4], and by Maniatis *et al.* [15], highlights the role played by field bilinears, which is very useful for studies of the vacuum structure of the model [16,17]. In this paper, we describe all classes of HF and GCP symmetries in both languages.

One problem with two classes of GCP identified here is that they lead to an exceptional region of parameter space (ERPS) previously identified as problematic by Gunion and Haber [9] and by Davidson and Haber [8]. Indeed, no basis-invariant quantity exists in the literature that distinguishes between the Z_2 and $U(1)$ HF symmetries in the ERPS.

If evidence for THDM physics is revealed in future experiments, then it will be critical to employ analysis techniques that are free from model-dependent assumptions. It is for this reason that a basis-independent formalism for the THDM is so powerful. Nevertheless, current experimental data already impose significant constraints on the most general THDM. In particular, we know that custodial-symmetry breaking effects, flavor changing neutral current (FCNC) constraints, and (to a lesser extent)

[1]It has been argued by Ginsburg [2] and by Ivanov [3,4] that one should also consider the effect of nonunitary global symmetry transformations of the two Higgs fields, as the most general renormalizable Higgs Lagrangian allows for kinetic mixing of the two Higgs fields. In this work, we study the possible global symmetries of the effective low-energy Higgs theory that arise *after* diagonalization of the Higgs kinetic energy terms. The nonunitary transformations that diagonalize the Higgs kinetic mixing terms also transform the parameters of the Higgs potential, and thus can determine the structure of the remnant Higgs flavor symmetries of effective low-energy Higgs scalar potential. It is the latter that constitutes the main focus of this work.

CP-violating phenomena impose some significant restrictions on the structure of the THDM (including the Higgs-fermion interactions). For example, the observed suppression of FCNCs implies that either the two heaviest neutral Higgs bosons of the THDM have masses above 1 TeV, or certain Higgs-fermion Yukawa couplings must be absent [18]. The latter can be achieved by imposing certain discrete symmetries on the THDM. Likewise, in the most general THDM, mass splittings between charged and neutral Higgs bosons can yield custodial-symmetry breaking effects at one-loop that could be large enough to be in conflict with the precision electroweak data [19]. Once again, symmetries can be imposed on the THDM to alleviate any potential disagreement with data. The implications of such symmetries for THDM phenomenology has recently been explored by Gerard and collaborators [20] and by Haber and O'Neil [21].

Thus, if THDM physics is discovered, it will be important to develop experimental methods that can reveal the presence or absence of underlying symmetries of the most general THDM. This requires two essential pieces of input. First, one must identify all possible Higgs symmetries of interest. Second, one must relate these symmetries to basis-independent observables that can be probed by experiment. In this paper, we primarily address the first step, although we also provide basis-independent characterizations of these symmetries. Our analysis focuses the symmetries of the THDM scalar potential. In principle, one can extend our study of these symmetries to the Higgs-fermion Yukawa interactions, although this lies beyond the scope of the present work.

This paper is organized as follows. In Sec. II, we introduce our notation and define an invariant that does distinguish the Z_2 and $U(1)$ HF symmetries in the ERPS. In Sec. III, we explain the role played by the vacuum expectation values (vevs) in preserving or breaking the $U(1)$ symmetry, and we comment briefly on renormalization. In Sec. IV, we introduce the GCP transformations and explain why they are organized into three classes. We summarize our results and set them in the context of the existing literature in Sec. V, and in Sec. VI, we prove a surprising result: multiple applications of the standard CP symmetry can be used to build all the models we identify, including those based on HF symmetries. We draw our conclusions in Sec. VII.

II. THE SCALAR SECTOR OF THE THDM

A. Three common notations for the scalar potential

Let us consider a $SU(2) \otimes U(1)$ gauge theory with two Higgs-doublets Φ_a, with the same hypercharge 1/2, and with vevs

$$\langle \Phi_a \rangle = \begin{pmatrix} 0 \\ v_a/\sqrt{2} \end{pmatrix}. \tag{1}$$

The index a runs from 1 to 2, and we use the standard definition for the electric charge, whereby the upper components of the $SU(2)$ doublets are charged and the lower components neutral.

The scalar potential may be written as

$$\begin{aligned} V_H = {} & m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{H.c.}] \\ & + \tfrac{1}{2}\lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \tfrac{1}{2}\lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1) \\ & \times (\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) + [\tfrac{1}{2}\lambda_5 (\Phi_1^\dagger \Phi_2)^2 \\ & + \lambda_6 (\Phi_1^\dagger \Phi_1)(\Phi_1^\dagger \Phi_2) + \lambda_7 (\Phi_2^\dagger \Phi_2)(\Phi_1^\dagger \Phi_2) + \text{H.c.}], \end{aligned} \tag{2}$$

where m_{11}^2, m_{22}^2, and $\lambda_1, \cdots, \lambda_4$ are real parameters. In general, m_{12}^2, λ_5, λ_6, and λ_7 are complex. "H.c." stands for Hermitian conjugation.

An alternative notation, useful for the construction of invariants and championed by Botella and Silva [6] is

$$V_H = Y_{ab}(\Phi_a^\dagger \Phi_b) + \tfrac{1}{2} Z_{ab,cd}(\Phi_a^\dagger \Phi_b)(\Phi_c^\dagger \Phi_d), \tag{3}$$

where Hermiticity implies

$$Y_{ab} = Y_{ba}^*, \qquad Z_{ab,cd} \equiv Z_{cd,ab} = Z_{ba,dc}^*. \tag{4}$$

The extremum conditions are

$$[Y_{ab} + Z_{ab,cd} v_d^* v_c] v_b = 0 \qquad (\text{for } a = 1, 2). \tag{5}$$

Multiplying by v_a^* leads to

$$Y_{ab}(v_a^* v_b) = -Z_{ab,cd}(v_a^* v_b)(v_d^* v_c). \tag{6}$$

One should be very careful when comparing Eqs. (2) and (3) among different authors, since the same symbol may be used for quantities, which differ by signs, factors of two, or complex conjugation. Here, we follow the definitions of Davidson and Haber [8]. With these definitions

$$\begin{aligned} Y_{11} &= m_{11}^2, & Y_{12} &= -m_{12}^2, \\ Y_{21} &= -(m_{12}^2)^* & Y_{22} &= m_{22}^2, \end{aligned} \tag{7}$$

and

$$\begin{aligned} Z_{11,11} &= \lambda_1, & Z_{22,22} &= \lambda_2, \\ Z_{11,22} = Z_{22,11} &= \lambda_3, & Z_{12,21} = Z_{21,12} &= \lambda_4, \\ Z_{12,12} &= \lambda_5, & Z_{21,21} &= \lambda_5^*, \\ Z_{11,12} = Z_{12,11} &= \lambda_6, & Z_{11,21} = Z_{21,11} &= \lambda_6^*, \\ Z_{22,12} = Z_{12,22} &= \lambda_7, & Z_{22,21} = Z_{21,22} &= \lambda_7^*. \end{aligned} \tag{8}$$

The previous two notations look at the Higgs fields Φ_a individually. A third notation is used by Nishi [13,14] and Ivanov [3,4], who emphasize the presence of field bilinears $(\Phi_a^\dagger \Phi_b)$ [17]. Following Nishi [13] we write

$$V_H = M_\mu r_\mu + \Lambda_{\mu\nu} r_\mu r_\nu, \tag{9}$$

where $\mu = 0, 1, 2, 3$ and

$$\begin{aligned} r_0 &= \frac{1}{2}[(\Phi_1^\dagger\Phi_1) + (\Phi_2^\dagger\Phi_2)], \\ r_1 &= \frac{1}{2}[(\Phi_1^\dagger\Phi_2) + (\Phi_2^\dagger\Phi_1)] = \mathrm{Re}(\Phi_1^\dagger\Phi_2), \\ r_2 &= -\frac{i}{2}[(\Phi_1^\dagger\Phi_2) - (\Phi_2^\dagger\Phi_1)] = \mathrm{Im}(\Phi_1^\dagger\Phi_2), \\ r_3 &= \frac{1}{2}[(\Phi_1^\dagger\Phi_1) - (\Phi_2^\dagger\Phi_2)]. \end{aligned} \tag{10}$$

In Eq. (9), summation of repeated indices is adopted with Euclidean metric. This differs from Ivanov's notation [3,4], who pointed out that r_μ parametrizes the gauge orbits of the Higgs fields, in a space equipped with a Minkowski metric.

In terms of the parameters of Eq. (2), the 4-vector M_μ and 4×4 matrix $\Lambda_{\mu\nu}$ are written, respectively, as

$$M_\mu = (m_{11}^2 + m_{22}^2, \quad -2\,\mathrm{Re}m_{12}^2, \quad 2\,\mathrm{Im}m_{12}^2, \quad m_{11}^2 - m_{22}^2), \tag{11}$$

and

$$\Lambda_{\mu\nu} = \begin{pmatrix} (\lambda_1+\lambda_2)/2+\lambda_3 & \mathrm{Re}(\lambda_6+\lambda_7) & -\mathrm{Im}(\lambda_6+\lambda_7) & (\lambda_1-\lambda_2)/2 \\ \mathrm{Re}(\lambda_6+\lambda_7) & \lambda_4+\mathrm{Re}\lambda_5 & -\mathrm{Im}\lambda_5 & \mathrm{Re}(\lambda_6-\lambda_7) \\ -\mathrm{Im}(\lambda_6+\lambda_7) & -\mathrm{Im}\lambda_5 & \lambda_4-\mathrm{Re}\lambda_5 & -\mathrm{Im}(\lambda_6-\lambda_7) \\ (\lambda_1-\lambda_2)/2 & \mathrm{Re}(\lambda_6-\lambda_7) & -\mathrm{Im}(\lambda_6-\lambda_7) & (\lambda_1+\lambda_2)/2-\lambda_3 \end{pmatrix}. \tag{12}$$

Equation (9) is related to Eq. (3) through

$$M^\mu = \sigma^\mu_{ab} Y_{ba}, \tag{13}$$

$$\Lambda^{\mu\nu} = \tfrac{1}{2} Z_{ab,cd}\, \sigma^\mu_{ba} \sigma^\nu_{dc}, \tag{14}$$

where the matrices σ^i are the three Pauli matrices, and σ^0 is the 2×2 identity matrix.

B. Basis transformations

We may rewrite the potential in terms of new fields Φ'_a, obtained from the original ones by a simple (global) basis transformation

$$\Phi_a \rightarrow \Phi'_a = U_{ab}\Phi_b, \tag{15}$$

where $U \in U(2)$ is a 2×2 unitary matrix. Under this unitary basis transformation, the gauge-kinetic terms are unchanged, but the coefficients Y_{ab} and $Z_{ab,cd}$ are transformed as

$$Y_{ab} \rightarrow Y'_{ab} = U_{a\alpha} Y_{\alpha\beta} U^*_{b\beta}, \tag{16}$$

$$Z_{ab,cd} \rightarrow Z'_{ab,cd} = U_{a\alpha} U_{c\gamma} Z_{\alpha\beta,\gamma\delta} U^*_{b\beta} U^*_{d\delta}, \tag{17}$$

and the vevs are transformed as

$$v_a \rightarrow v'_a = U_{ab} v_b. \tag{18}$$

Thus, the basis transformations U may be utilized in order to absorb some of the degrees of freedom of Y and/or Z, which implies that not all parameters of Eq. (3) have physical significance.

C. Higgs family symmetries

Let us assume that the scalar potential in Eq. (3) has some explicit internal symmetry. That is, we assume that the coefficients of V_H stay *exactly the same* under a transformation

$$\Phi_a \rightarrow \Phi^S_a = S_{ab}\Phi_b. \tag{19}$$

S is a unitary matrix, so that the gauge-kinetic couplings are also left invariant by this HF symmetry. As a result of this symmetry,

$$Y_{ab} = Y^S_{ab} = S_{a\alpha} Y_{\alpha\beta} S^*_{b\beta}, \tag{20}$$

$$Z_{ab,cd} = Z^S_{ab,cd} = S_{a\alpha} S_{c\gamma} Z_{\alpha\beta,\gamma\delta} S^*_{b\beta} S^*_{d\delta}. \tag{21}$$

Notice that this is *not* the situation considered in Eqs. (15)–(17). There, the coefficients of the Lagrangian *do change* (although the quantities that are physically measurable are invariant with respect to any change of basis). In contrast, Eqs. (19)–(21) imply the existence of a HF symmetry S of the scalar potential that leaves the coefficients of V_H unchanged.

The Higgs family symmetry group must be a subgroup of the full $U(2)$ transformation group of 2×2 unitary matrices employed in Eq. (15). Given the most general THDM scalar potential, there is always a $U(1)$ subgroup of $U(2)$ under which the scalar potential is invariant. This is the global hypercharge $U(1)_Y$ symmetry group

$$U(1)_Y\colon\ \Phi_1 \rightarrow e^{i\theta}\Phi_1, \qquad \Phi_2 \rightarrow e^{i\theta}\Phi_2, \tag{22}$$

where θ is an arbitrary angle (mod 2π). The invariance under the global $U(1)_Y$ is trivially guaranteed by the invariance under the $SU(2) \otimes U(1)$ electroweak gauge symmetry. *Since the global hypercharge $U(1)_Y$ is always present, we shall henceforth define the HF symmetries as those Higgs family symmetries that are orthogonal to $U(1)_Y$.*

We now turn to the interplay between HF symmetries and basis transformations. Let us imagine that, when written in the basis of fields Φ_a, V_H has a symmetry S. We then perform a basis transformation from the basis Φ_a to the basis Φ'_a, as given by Eq. (15). Clearly, when written in the new basis, V_H does *not* remain invariant under S. Rather, it will be invariant under

$$S' = USU^{\dagger}. \tag{23}$$

As we change basis, the form of the potential changes in a way that may obscure the presence of a HF symmetry. In particular, two HF symmetries that naively look distinct will actually yield precisely the same physical predictions if a unitary matrix U exists such that Eq. (23) is satisfied.

HF symmetries in the THDM have a long history. In papers by Glashow and Weinberg and by Paschos [18], the discrete Z_2 symmetry was introduced,

$$Z_2\colon\ \Phi_1 \to \Phi_1, \qquad \Phi_2 \to -\Phi_2, \tag{24}$$

in order to preclude flavor-changing neutral currents [18]. This is just the interchange

$$\Pi_2\colon\ \Phi_1 \leftrightarrow \Phi_2, \tag{25}$$

seen in a different basis, as shown by applying Eq. (23) in the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{26}$$

Peccei and Quinn [22] introduced the continuous $U(1)$ symmetry

$$U(1)\colon\ \Phi_1 \to e^{-i\theta}\Phi_1, \qquad \Phi_2 \to e^{i\theta}\Phi_2, \tag{27}$$

true for any value of θ, in connection with the strong CP problem. Of course, a potential invariant under $U(1)$ is also invariant under Z_2.

Finally, we examine the largest possible Higgs family symmetry group of the THDM, namely, $U(2)$. In this case, a basis transformation would have no effect on the Higgs potential parameters. Since δ_{ab} is the only $U(2)$-invariant tensor, it follows that

$$Y_{ab} = c_1\delta_{ab}, \tag{28}$$

$$Z_{ab,cd} = c_2\delta_{ab}\delta_{cd} + c_3\delta_{ad}\delta_{bc}, \tag{29}$$

where c_1, c_2, and c_3 are arbitrary real numbers.[2] One can easily check from Eqs. (16) and (17) that the unitarity of U implies that $Y' = Y$ and $Z' = Z$ for any choice of basis, as required by the $U(2)$ invariance of the scalar potential. Equations (28) and (29) impose the following constraints on the parameters of the THDM scalar potential (independently of the choice of basis):

$$\begin{aligned} m_{22}^2 &= m_{11}^2, \qquad m_{12}^2 = 0, \\ \lambda_1 = \lambda_2 &= \lambda_3 + \lambda_4, \qquad \lambda_5 = \lambda_6 = \lambda_7 = 0. \end{aligned} \tag{30}$$

As there are no nonzero potentially complex scalar potential parameters, the $U(2)$-invariant THDM is clearly CP invariant.

As previously noted, the $U(2)$ symmetry contains the global hypercharge $U(1)_Y$ as a subgroup. Thus, in order to identify the corresponding HF symmetry that is orthogonal to $U(1)_Y$, we first observe that

$$U(2) \cong SU(2) \otimes U(1)_Y/Z_2 \cong SO(3) \otimes U(1)_Y. \tag{31}$$

To prove the above isomorphism, simply note that any $U(2)$ matrix can be written as $U = e^{i\theta}\hat{U}$, where $\hat{U} \in SU(2)$. To cover the full $U(1)_Y$ group, we must take $0 \le \theta < 2\pi$. But since both $\hat{U}$ and $-\hat{U}$ are elements of $SU(2)$, whereas $+1$ and $-1 = e^{i\pi}$ are elements of $U(1)_Y$, we must identify $\hat{U}$ and $-\hat{U}$ as the same group element in order not to double cover the full $U(2)$ group. The identification of $\hat{U}$ with $-\hat{U}$ in $SU(2)$ is isomorphic to $SO(3)$, using the well-known isomorphism $SO(3) \cong SU(2)/Z_2$. Consequently, we have identified $SO(3)$ as the HF symmetry that constrains the scalar potential parameters as indicated in Eq. (30).

The impact of these symmetries on the potential parameters in Eq. (2) is shown in Sec. V. As mentioned above, if one makes a basis change, the potential parameters change and so does the explicit form of the symmetry and of its implications. For example, Eq. (26) shows that the symmetries Z_2 and Π_2 are related by a basis change. However, they have a different impact on the parameters in their respective basis. This can be seen explicitly in Table I of Sec. V. One can also easily prove that the existence of either the Z_2, Π_2 or Peccei-Quinn $U(1)$ symmetry is sufficient to guarantee the existence of a basis choice in which all scalar potential parameters are real. That is, the corresponding scalar Higgs sectors are explicitly CP conserving.

Basis-invariant signs of HF symmetries were discussed extensively in Ref. [8]. Recently, Ferreira and Silva [23] extended these methods to include Higgs models with more than two Higgs doublets.

Consider first the THDM scalar potentials that are invariant under the so-called *simple* HF symmetries of Ref. [23]. We define a simple HF symmetry to be a symmetry group G with the following property: the requirement that the THDM scalar potential is invariant under a particular element $g \in G$ (where $g \neq e$ and e is the identity element) is sufficient to guarantee invariance under the entire group G. The discrete cyclic group $Z_n = \{e, g, g^2, \ldots, g^{n-1}\}$, where $g^n = e$, is an example of a possible simple HF symmetry group. If we restrict the TDHM scalar potential to include terms of dimension four or less (e.g., the tree-level scalar potential of the THDM), then one can show that the Peccei-Quinn $U(1)$ symmetry is also a simple HF symmetry. For example, consider the matrix

$$S = \begin{pmatrix} e^{-2i\pi/3} & 0 \\ 0 & e^{2i\pi/3} \end{pmatrix}. \tag{32}$$

[2] Note that there is no $\delta_{ac}\delta_{bd}$ term contributing to $Z_{ab,cd}$, as such a term is not invariant under the transformation of Eq. (17).

Note that S is an element of the cyclic subgroup $Z_3 = \{S, S^2, S^3 = 1\}$ of the Peccei-Quinn $U(1)$ group. As shown in Ref. [23], the invariance of the tree-level THDM scalar potential under $\Phi_a \rightarrow S_{ab}\Phi_b$ automatically implies the invariance of the scalar potential under the full Peccei-Quinn $U(1)$ group. In contrast, the maximal HF symmetry, $SO(3)$, introduced above is not a simple HF symmetry, as there is no single element of $S \in SO(3)$ such that invariance under $\Phi_a \rightarrow S_{ab}\Phi_b$ guarantees invariance of the tree-level THDM scalar potential under the full $SO(3)$ group of transformations.

Typically, the simple HF symmetries take on a simple form for a particular choice of basis for the Higgs fields. We summarize here a few of the results of Ref. [23]:

(1) In the THDM, there are only two *independent* classes of *simple* symmetries: a discrete Z_2 flavor symmetry, and a continuous Peccei-Quinn $U(1)$ flavor symmetry.

(2) Other discrete flavor symmetry groups G that are subgroups of $U(1)$ are not considered independent. That is, if $S \in G$ (where $S \neq e$), then invariance under the discrete symmetry $\Phi \rightarrow S\Phi$ makes the scalar potential automatically invariant under the full Peccei-Quinn $U(1)$ group;

(3) In most regions of parameter space, one can build quantities invariant under basis transformations that detect these symmetries;

(4) There exists a so-called ERPS characterized by

$$m_{22}^2 = m_{11}^2, \qquad m_{12}^2 = 0, \qquad \lambda_2 = \lambda_1, \qquad \lambda_7 = -\lambda_6. \tag{33}$$

As shown by Davidson and Haber [8], a theory obeying these constraints does have a Z_2 symmetry, but it may or not have a $U(1)$ symmetry. Within the ERPS, the invariants in the literature cannot be used to distinguish the two cases.

The last statement above is a result of the following considerations: In order to distinguish between Z_2 and $U(1)$, Davidson and Haber [8] construct two invariant quantities given by Eqs. (46) and (50) of Ref. [8]. Outside the ERPS, these quantities are zero if and only if $U(1)$ holds. Unfortunately, in the ERPS these quantities vanish automatically independently of whether or not $U(1)$ holds. Similarly, Ferreira and Silva [23] have constructed invariants detecting HF symmetries. But their use requires the existence of a matrix, obtained by combining Y_{ab} and $Z_{ab,cd}$, which has two distinct eigenvalues. This does not occur when the ERPS is due to a symmetry. Finally, in the ERPS, Ivanov [3] states that the symmetry might be "$(Z_2)^2$ or $O(2)$" [our Z_2 *or* our $U(1)$] and does not provide a way to distinguish the two possible flavor symmetries [24].

Gunion and Haber [9] have shown that the ERPS conditions of Eq. (33) are basis independent; if they hold in one basis, then they hold in any basis. Moreover, for a model in the ERPS, a basis may be chosen such that all parameters are real.[3] Having achieved such a basis, Davidson and Haber [8] demonstrate that one may make one additional basis transformation such that

$$m_{22}^2 = m_{11}^2, \qquad m_{12}^2 = 0, \qquad \lambda_2 = \lambda_1, \qquad \lambda_7 = \lambda_6 = 0, \qquad \mathrm{Im}\lambda_5 = 0. \tag{34}$$

These conditions express the ERPS for a specific basis choice.

One might think that since this is such a special region of parameter space that it lacks any relevance. However, the fact that the conditions in Eq. (33) hold in *any* basis is a good indication that a symmetry may lie behind this condition. Indeed, as pointed out by Davidson and Haber [8], combining the two symmetries Z_2 and Π_2 *in the same basis* one is lead immediately to the ERPS in the basis of Eq. (34). Up to now, we considered the impact of imposing on the Higgs potential only one symmetry. This was dubbed a simple symmetry. Now we are considering the possibility that the potential must remain invariant under one symmetry and *also* under a second symmetry; this implies further constraints on the parameters of the Higgs potential. We refer to this possibility as a multiple symmetry. As seen from Table I of Sec. V, imposing Z_2 and Π_2 in the same basis leads to the conditions in Eq. (34). Incidentally, this example shows that a model that lies in the ERPS, is automatically invariant under Z_2.

In Sec. IV, we will show that all classes of nontrivial *CP* transformations lead directly to the ERPS, reinforcing the importance of this particular region of parameter space.

D. Requirements for $U(1)$ invariance

In the basis in which the $U(1)$ symmetry takes the form of Eq. (27), the coefficients of the potential must obey

$$m_{12}'^2 = 0, \qquad \lambda_5' = \lambda_6' = \lambda_7' = 0. \tag{35}$$

Imagine that we have a potential of Eq. (2) in the ERPS: $m_{11}^2 = m_{22}^2$, $m_{12}^2 = 0$, $\lambda_2 = \lambda_1$, and $\lambda_7 = -\lambda_6$. We now wish to know whether a transformation U may be chosen such that the potential coefficients in the new basis satisfy the $U(1)$ conditions in Eq. (35). Using the transformation rules in Eqs. (A13)–(A23) of Davidson and Haber [8], we find that such a choice of U is possible if and only if the coefficients in the original basis satisfy

$$2\lambda_6^3 - \lambda_5\lambda_6(\lambda_1 - \lambda_3 - \lambda_4) - \lambda_5^2\lambda_6^* = 0, \tag{36}$$

subject to the condition that $\lambda_5^*\lambda_6^2$ is real.

[3] Given a scalar potential whose parameters satisfy the ERPS conditions with $\mathrm{Im}(\lambda_5^*\lambda_6^2) \neq 0$, the unitary matrix required to transform into a basis in which all the scalar potential parameters are real can be determined only by numerical means.

E. The D invariant

Having established the importance of the ERPS (as it can arise from a symmetry), we will now build a basis-invariant quantity that can be used to detect the presence of a $U(1)$ symmetry in this special case.

The quadratic terms of the Higgs potential are always insensitive to the difference between Z_2 and $U(1)$. Moreover, the matrix Y is proportional to the unit matrix in the ERPS. One must thus look at the quartic terms. We were inspired by the expression of $\Lambda_{\mu\nu}$ in Eq. (12), which appears in the works of Nishi [13,14] and Ivanov [3,4]. In the ERPS of Eq. (33), $\Lambda_{\mu\nu}$ breaks into a 1×1 block (Λ_{00}), and a 3×3 block ($\tilde{\Lambda} = \{\Lambda_{ij}\}$; i, $j = 1, 2, 3$). A basis transformation U belonging to $SU(2)$ on the Φ_a fields corresponds to an orthogonal $SO(3)$ transformation in the r_i bilinears, given by

$$O_{ij} = \tfrac{1}{2}\mathrm{Tr}[U^\dagger \sigma_i U \sigma_j]. \tag{37}$$

Any matrix O of $SO(3)$ can be obtained by considering an appropriate matrix U of $SU(2)$ (unfortunately this property does not generalize for models with more than two Higgs doublets). A suitable choice of O can be made that diagonalizes the 3×3 matrix $\tilde{\Lambda}$, thus explaining Eq. (34). In this basis, the difference between the usual choices for $U(1)$ and Z_2 corresponds to the possibility that $\mathrm{Re}\lambda_5$ might vanish or not, respectively.

We will now show that, once in the ERPS, the condition for the existence of $U(1)$ is that $\tilde{\Lambda}$ has two eigenvalues, which are equal. The eigenvalues of a 3×3 matrix are the solutions to the secular equation

$$x^3 + a_2 x^2 + a_1 x + a_0 = 0, \tag{38}$$

where

$$\begin{aligned} a_0 = \det\tilde{\Lambda} &= -\tfrac{1}{3}\mathrm{Tr}(\tilde{\Lambda}^3) - \tfrac{1}{6}(\mathrm{Tr}\tilde{\Lambda})^3 + \tfrac{1}{2}(\mathrm{Tr}\tilde{\Lambda})\,\mathrm{Tr}(\tilde{\Lambda}^2) \\ &= -\tfrac{1}{3}Z_{ab,cd}(Z_{dc,gh}Z_{hg,ba} - \tfrac{3}{2}Z^{(2)}_{dc}Z^{(2)}_{ba}) \\ &\quad + \tfrac{1}{2}Z_{ab,cd}Z_{dc,ba}\,\mathrm{Tr}(Z^{(1)} - \tfrac{1}{2}Z^{(2)}) - \tfrac{1}{6}(\mathrm{Tr}Z^{(1)})^3 \\ &\quad + \tfrac{1}{4}(\mathrm{Tr}Z^{(1)})^2\,\mathrm{Tr}Z^{(2)} - \tfrac{1}{2}\,\mathrm{Tr}Z^{(1)}(\mathrm{Tr}Z^{(2)})^2, \end{aligned} \tag{39}$$

$$\begin{aligned} a_1 &= \tfrac{1}{2}(\mathrm{Tr}\tilde{\Lambda})^2 - \tfrac{1}{2}\mathrm{Tr}(\tilde{\Lambda}^2) \\ &= \tfrac{1}{2}[(\mathrm{Tr}Z^{(1)})^2 - \mathrm{Tr}Z^{(1)}\,\mathrm{Tr}Z^{(2)} + (\mathrm{Tr}Z^{(2)})^2 \\ &\quad - Z_{ab,cd}Z_{dc,ba}], \end{aligned} \tag{40}$$

$$a_2 = -\mathrm{Tr}\tilde{\Lambda} = \tfrac{1}{2}\mathrm{Tr}Z^{(2)} - \mathrm{Tr}Z^{(1)}, \tag{41}$$

and

$$Z^{(1)}_{ab} \equiv Z_{a\alpha,\alpha b} = \begin{pmatrix} \lambda_1 + \lambda_4 & \lambda_6 + \lambda_7 \\ \lambda_6^* + \lambda_7^* & \lambda_2 + \lambda_4 \end{pmatrix}, \tag{42}$$

$$Z^{(2)}_{ab} \equiv Z_{\alpha\alpha,ab} = \begin{pmatrix} \lambda_1 + \lambda_3 & \lambda_6 + \lambda_7 \\ \lambda_6^* + \lambda_7^* & \lambda_2 + \lambda_3 \end{pmatrix}. \tag{43}$$

The cubic equation, Eq. (38), has at least two degenerate solutions if [25]

$$D \equiv [\tfrac{1}{2}a_1 - \tfrac{1}{9}a_2^2]^3 + [\tfrac{1}{6}(a_1 a_2 - 3a_0) - \tfrac{1}{27}a_2^3]^2 \tag{44}$$

vanishes.

The expression of D in terms of the parameters in Eq. (2) is rather complicated, even in the ERPS. But one can show by direct computation that if the $U(1)$-symmetry condition of Eq. (36) holds (subject to $\lambda_5^*\lambda_6^2$ being real), then $D = 0$. We can simplify the expression for D by changing to a basis where all parameters are real [9], where we get

$$\begin{aligned} D = &-\tfrac{1}{27}[\lambda_5(\lambda_1 - \lambda_3 - \lambda_4 + \lambda_5) - 2\lambda_6^2]^2 \\ &\times [(\lambda_1 - \lambda_3 - \lambda_4 - \lambda_5)^2 + 16\lambda_6^2]. \end{aligned} \tag{45}$$

If $\lambda_6 \neq 0$, then $D = 0$ means

$$2\lambda_6^2 = \lambda_5(\lambda_1 - \lambda_3 - \lambda_4 + \lambda_5). \tag{46}$$

If $\lambda_6 = 0$, then $D = 0$ corresponds to one of three possible conditions:

$$\lambda_5 = 0, \qquad \lambda_5 = \pm(\lambda_1 - \lambda_3 - \lambda_4). \tag{47}$$

Notice that Eqs. (46) and (47) are equivalent to Eq. (36) in any basis where the coefficients are real.

Although D can be defined outside the ERPS, the condition $D = 0$ only guarantees that the model is invariant under $U(1)$ inside the ERPS of Eq. (33). Outside this region one can detect the presence of a $U(1)$ symmetry with the invariants proposed by Davidson and Haber [8]. This closes the last breach in the literature concerning basis-invariant signals of discrete symmetries in the THDM. Thus, in the ERPS $D = 0$ is a necessary and sufficient condition for the presence of a $U(1)$ symmetry.

III. VACUUM STRUCTURE AND RENORMALIZATION

The presence of a $U(1)$ symmetry in the Higgs potential may (or not) imply the existence of a massless scalar, the axion, depending on whether (or not) the $U(1)$ is broken by the vevs. In the previous section we related the basis-invariant condition $D = 0$ in the ERPS with the presence of a $U(1)$ symmetry. In this section we will show that, whenever the basis-invariant condition $D = 0$ is satisfied in the ERPS, there is always a stationary point for which a massless scalar, other than the usual Goldstone bosons, exists.

We start by writing the extremum conditions for the THDM in the ERPS. For simplicity, we will be working in a basis where all the parameters are real [9]. From Eqs. (5) and (8), we obtain

$$\begin{aligned} 0 &= Y_{11}v_1 + \tfrac{1}{2}[\lambda_1 v_1^3 + \lambda_{345}v_1 v_2^2 + \lambda_6(3v_1^2 v_2 - v_2^3)], \\ 0 &= Y_{11}v_2 + \tfrac{1}{2}[\lambda_1 v_2^3 + \lambda_{345}v_2 v_1^2 + \lambda_6(v_1^3 - 3v_2^2 v_1)], \end{aligned} \tag{48}$$

where we have defined $\lambda_{345} \equiv \lambda_3 + \lambda_4 + \lambda_5$. We now compute the mass matrices. As we will be considering only vacua with real vevs, there will be no mixing between the real and imaginary parts of the doublets. As such, we can define the mass matrix of the *CP*-even scalars as given by

$$[M_h^2]_{ij} = \frac{1}{2}\frac{\partial^2 V}{\partial\,\mathrm{Re}(\Phi_i^0)\partial\,\mathrm{Re}(\Phi_j^0)}, \tag{49}$$

where Φ_i^0 is the neutral (lower) component of the Φ_i doublet. Thus, we obtain, for the entries of this matrix, the following expressions:

$$\begin{aligned}
[M_h^2]_{11} &= Y_{11} + \tfrac{1}{2}(3\lambda_1 v_1^2 + \lambda_{345} v_2^2 + 6\lambda_6 v_1 v_2)\\
[M_h^2]_{22} &= Y_{11} + \tfrac{1}{2}(3\lambda_1 v_2^2 + \lambda_{345} v_1^2 + 6\lambda_6 v_1 v_2)\\
[M_h^2]_{12} &= \lambda_{345} v_1 v_2 + \tfrac{3}{2}\lambda_6 (v_1^2 - v_2^2).
\end{aligned} \tag{50}$$

Likewise, the pseudoscalar mass matrix is defined as

$$[M_A^2]_{ij} = \frac{1}{2}\frac{\partial^2 V}{\partial\,\mathrm{Im}(\Phi_i^0)\partial\,\mathrm{Im}(\Phi_j^0)} \tag{51}$$

whose entries are given by

$$\begin{aligned}
[M_A^2]_{11} &= Y_{11} + \tfrac{1}{2}[\lambda_1 v_1^2 + (\lambda_3 + \lambda_4 - \lambda_5) v_2^2 + 2\lambda_6 v_1 v_2]\\
[M_A^2]_{22} &= Y_{11} + \tfrac{1}{2}[\lambda_1 v_2^2 + (\lambda_3 + \lambda_4 - \lambda_5) v_1^2 - 2\lambda_6 v_1 v_2]\\
[M_A^2]_{12} &= \lambda_5 v_1 v_2 + \tfrac{1}{2}\lambda_6 (v_1^2 - v_2^2).
\end{aligned} \tag{52}$$

The expressions (50) and (52) are valid for all the particular cases we will now consider.

A. Case $\lambda_6 = 0$, $\{v_1, v_2\} \neq 0$

Let us first study the case $\lambda_6 = 0$, wherein we may solve the extremum conditions in an analytical manner. It is trivial to see that Eqs. (48) have three types of solutions: both vevs different from zero, one vev equal to zero (say, v_2) and both vevs zero (trivial noninteresting solution). For a solution with $\{v_1, v_2\} \neq 0$, a necessary condition must be obeyed so that there is a solution to Eqs. (48):

$$\lambda_1^2 - \lambda_{345}^2 \neq 0. \tag{53}$$

If we use the extremum conditions to evaluate $[M_h^2]$, we obtain

$$[M_h^2] = \begin{pmatrix} \lambda_1 v_1^2 & \lambda_{345} v_1 v_2 \\ \lambda_{345} v_1 v_2 & \lambda_1 v_2^2 \end{pmatrix}, \tag{54}$$

which only has a zero eigenvalue if Eq. (53) is broken. Thus, there is no axion in this matrix in this case. As for $[M_A^2]$, we get

$$[M_A^2] = -\lambda_5 \begin{pmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{pmatrix}, \tag{55}$$

which clearly has a zero eigenvalue corresponding to the Z Goldstone boson. Further, this matrix will have an axion if $\lambda_5 = 0$, which is the first condition of Eq. (47).

B. Case $\lambda_6 = 0$, $\{v_1 \neq 0, v_2 = 0\}$

Returning to Eq. (48), this case gives us

$$Y_{11} = -\tfrac{1}{2}\lambda_1 v_1^2, \tag{56}$$

which implies $Y_{11} < 0$. With this condition, the mass matrices become considerably simpler:

$$[M_h^2] = \begin{pmatrix} \lambda_1 v_1^2 & 0 \\ 0 & \frac{1}{2}(\lambda_{345} - \lambda_1) v_1^2 \end{pmatrix} \tag{57}$$

and

$$[M_A^2] = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & (\lambda_3 + \lambda_4 - \lambda_5 - \lambda_1) v_1^2 \end{pmatrix}. \tag{58}$$

So, we can have an axion in the matrix (57) if

$$\lambda_{345} - \lambda_1 = 0 \Leftrightarrow \lambda_5 = \lambda_1 - \lambda_3 - \lambda_4 \tag{59}$$

or an axion in matrix (58) if

$$\lambda_5 = -\lambda_1 + \lambda_3 + \lambda_4. \tag{60}$$

That is, we have an axion if the second or third conditions of Eq. (47) are satisfied. The other possible case, $\{v_1 = 0, v_2 \neq 0\}$, produces exactly the same conclusions.

C. Case $\lambda_6 \neq 0$

This is the hardest case to treat, since we cannot obtain analytical expressions for the vevs. Nevertheless a full analytical treatment is still possible. First, notice that with $\lambda_6 \neq 0$ Eqs. (48) imply that both vevs have to be nonzero. At the stationary point of Eqs. (48), the pseudoscalar mass matrix has a Goldstone boson and an eigenvalue given by

$$-\lambda_5 (v_1^2 + v_2^2) - \lambda_6 \frac{v_1^4 - v_2^4}{2 v_1 v_2}. \tag{61}$$

So, an axion exists if we have

$$\frac{v_1^2 - v_2^2}{v_1 v_2} = -\frac{2\lambda_5}{\lambda_6}. \tag{62}$$

On the other hand, after some algebraic manipulation, it is simple to obtain from (48) the following condition:

$$\lambda_1 - \lambda_{345} = \lambda_6 \left(\frac{v_1^2 - v_2^2}{v_1 v_2} - \frac{4 v_1 v_2}{v_1^2 - v_2^2}\right). \tag{63}$$

Substituting Eq. (62) into (63), we obtain

$$\begin{aligned}
\lambda_1 - \lambda_{345} = \lambda_6 \left(-\frac{2\lambda_5}{\lambda_6} + \frac{2\lambda_6}{\lambda_5}\right) &\Leftrightarrow 2\lambda_6^2 \\
&= \lambda_5 (\lambda_1 - \lambda_3 - \lambda_4 + \lambda_5).
\end{aligned} \tag{64}$$

Thus, we have shown that all of the conditions stemming from the basis-invariant condition $D = 0$ guarantee the existence of some stationary point for which the scalar potential yields an axion. Notice that, however, this stationary point need not coincide with the global minimum of the potential.

D. Renormalization group invariance

We now briefly examine the renormalization group (RG) behavior of our basis-invariant condition $D = 0$. It would be meaningless to say that $D = 0$ implies a $U(1)$ symmetry if that condition were only valid at a given renormalization scale. That is, it could well be that a numerical accident forces $D = 0$ at only a given scale. To avoid such a conclusion, we must verify if $D = 0$ is a RG-invariant condition (in addition to being basis invariant). For a given renormalization scale μ, the β function of a given parameter x is defined as $\beta_x = \mu\partial x/\partial\mu$. For simplicity, let us rewrite D in Eq. (45) as

$$D = -\tfrac{1}{27} D_1^2 D_2, \tag{65}$$

with

$$\begin{aligned} D_1 &= \lambda_5(\lambda_1 - \lambda_3 - \lambda_4 + \lambda_5) - 2\lambda_6^2 \\ D_2 &= (\lambda_1 - \lambda_3 - \lambda_4 - \lambda_5)^2 + 16\lambda_6^2. \end{aligned} \tag{66}$$

If we apply the operator $\mu\partial/\partial\mu$ to D, we obtain

$$\beta_D = -\tfrac{1}{27}(2D_1 D_2 \beta_{D_1} + D_1^2 \beta_{D_2}). \tag{67}$$

If $D_1 = 0$ (which corresponds to three of the conditions presented in Eqs. (46) and (47)) then we immediately have $\beta_D = 0$. That is, if $D = 0$ at a given scale, it is zero at all scales.

If $D_2 = 0$ and $D_1 \neq 0$ we will only have $\beta_D = 0$ if $\beta_{D_2} = 0$, or equivalently,

$$\begin{aligned} &2(\lambda_1 - \lambda_3 - \lambda_4 - \lambda_5)(\beta_{\lambda_1} - \beta_{\lambda_3} - \beta_{\lambda_4} - \beta_{\lambda_5}) \\ &\quad + 32\beta_{\lambda_6}\lambda_6 = 0. \end{aligned} \tag{68}$$

Given that $D_2 = 0$ implies that $\lambda_6 = 0$ and $\lambda_5 = \lambda_1 - \lambda_3 - \lambda_4$, we once again obtain $\beta_D = 0$.

Thus, the condition $D = 0$ is RG invariant. A direct verification of the RG invariance of Eqs. (46) and (47), and of the conditions that define the ERPS itself, would require the explicit form of th e β functions of the THDM involving the λ_6 coupling. That verification will be made elsewhere [26].

IV. GENERALIZED *CP* SYMMETRIES

It is common to consider the standard *CP* transformation of the scalar fields as

$$\Phi_a(t, \vec{x}) \to \Phi_a^{CP}(t, \vec{x}) = \Phi_a^*(t, -\vec{x}), \tag{69}$$

where the reference to the time (t) and space ($\vec{x}$) coordinates will henceforth be suppressed. However, in the presence of several scalars with the same quantum numbers, basis transformations can be included in the definition of the *CP* transformation. This yields GCP transformations

$$\begin{aligned} \Phi_a^{\rm GCP} &= X_{a\alpha}\Phi_\alpha^* \equiv X_{a\alpha}(\Phi_\alpha^\dagger)^\top, \\ \Phi_a^{\dagger\rm GCP} &= X_{a\alpha}^*\Phi_\alpha^\top \equiv X_{a\alpha}^*(\Phi_\alpha^\dagger)^*, \end{aligned} \tag{70}$$

where X is an arbitrary unitary matrix [27,28].[4]

Note that the transformation $\Phi_a \to \Phi_a^{\rm GCP}$, where $\Phi_a^{\rm GCP}$ is given by Eq. (70), leaves the kinetic terms invariant. The GCP transformation of a field bilinear yields

$$\Phi_a^{\dagger\rm GCP}\Phi_b^{\rm GCP} = X_{a\alpha}^* X_{b\beta}(\Phi_\alpha\Phi_\beta^\dagger)^\top. \tag{71}$$

Under this GCP transformation, the quadratic terms of the potential may be written as

$$\begin{aligned} Y_{ab}\Phi_a^{\dagger\rm GCP}\Phi_b^{\rm GCP} &= Y_{ab}X_{a\alpha}^* X_{b\beta}\Phi_\beta^\dagger\Phi_\alpha \\ &= X_{b\beta}Y_{ba}^* X_{a\alpha}^*\Phi_\beta^\dagger\Phi_\alpha \\ &= X_{\alpha a}Y_{\alpha\beta}^* X_{\beta b}^*\Phi_a^\dagger\Phi_b = (X^\dagger Y X)_{ab}^*\Phi_a^\dagger\Phi_b. \end{aligned} \tag{72}$$

We have used the Hermiticity condition $Y_{ab} = Y_{ba}^*$ in going to the second line; and changed the dummy indices $a \leftrightarrow \beta$ and $b \leftrightarrow \alpha$ in going to the third line. A similar argument can be made for the quartic terms. We conclude that the potential is invariant under the GCP transformation of Eq. (70) if and only if the coefficients obey

$$\begin{aligned} Y_{ab}^* &= X_{\alpha a}^* Y_{\alpha\beta} X_{\beta b} = (X^\dagger Y X)_{ab}, \\ Z_{ab,cd}^* &= X_{\alpha a}^* X_{\gamma c}^* Z_{\alpha\beta,\gamma\delta} X_{\beta b} X_{\delta d}. \end{aligned} \tag{73}$$

Introducing

$$\begin{aligned} \Delta Y_{ab} &= Y_{ab} - X_{\alpha a}Y_{\alpha\beta}^* X_{\beta b}^* = [Y - (X^\dagger Y X)^*]_{ab}, \\ \Delta Z_{ab,cd} &= Z_{ab,cd} - X_{\alpha a}X_{\gamma c}Z_{\alpha\beta,\gamma\delta}^* X_{\beta b}^* X_{\delta d}^*, \end{aligned} \tag{74}$$

we may write the conditions for invariance under GCP as

$$\Delta Y_{ab} = 0, \tag{75}$$

$$\Delta Z_{ab,cd} = 0. \tag{76}$$

Given Eqs. (4), it is easy to show that

[4] Equivalently, one can consider a generalized time-reversal transformation proposed in Ref. [29] and considered further in Appendix A of Ref. [9].

$$\Delta Y_{ab} = \Delta Y^*_{ba}, \qquad \Delta Z_{ab,cd} \equiv \Delta Z_{cd,ab} = \Delta Z^*_{ba,dc}. \quad (77)$$

Thus, we need only consider the real coefficients ΔY_{11}, ΔY_{22}, $\Delta Z_{11,11}$, $\Delta Z_{22,22}$, $\Delta Z_{11,22}$, $\Delta Z_{12,21}$, and the complex coefficients ΔY_{12}, $\Delta Z_{11,12}$, $\Delta Z_{22,12}$, and $\Delta Z_{12,12}$.

A. GCP and basis transformations

We now turn to the interplay between GCP transformations and basis transformations. Consider the potential of Eq. (3) and call it $V(\Phi)$. Now consider the potential obtained from $V(\Phi)$ by the basis transformation $\Phi_a \to \Phi'_a = U_{ab}\Phi_b$:

$$V(\Phi') = Y'_{ab}(\Phi'^{\dagger}_a \Phi'_b) + \tfrac{1}{2}Z'_{ab,cd}(\Phi'^{\dagger}_a \Phi'_b)(\Phi'^{\dagger}_c \Phi'_d), \quad (78)$$

where the coefficients in the new basis are given by Eqs. (16) and (17). We will now prove the following theorem: If $V(\Phi)$ is invariant under the GCP transformation of Eq. (70) with the matrix X, then $V(\Phi')$ is invariant under a new GCP transformation with matrix

$$X' = UXU^\top. \quad (79)$$

By hypothesis $V(\Phi)$ is invariant under the GCP transformation of Eq. (70) with the matrix X. Equation (73) guarantees that $Y^* = X^\dagger Y X$. Now, Eq. (16) relates the coefficients in the two basis through $Y = U^\dagger Y' U$. Substituting gives

$$U^\top Y'^* U^* = X^\dagger (U^\dagger Y' U) X, \quad (80)$$

or

$$Y'^* = (U^* X^\dagger U^\dagger) Y' (UXU^\top) = X'^\dagger Y' X', \quad (81)$$

as required. A similar argument holds for the quartic terms and the proof is complete.

The fact that the transpose $U^\top$ appears in Eq. (79) rather than $U^\dagger$ is crucial. In Eq. (23), applicable to HF symmetries, $U^\dagger$ appears. Consequently, a basis may be chosen where the HF symmetry is represented by a diagonal matrix S. The presence of $U^\top$ in Eq. (79) implies that, contrary to popular belief, *it is not possible to reduce all GCP transformations to the standard CP* transformation of Eq. (69) by a basis transformation. What is possible, as we shall see below, is to reduce an invariance of the THDM potential under any GCP transformation, to an invariance under the standard *CP* transformation plus some extra constraints.

To be more specific, the following result is easily established. If the unitary matrix X is symmetric, then it follows that[5] a unitary matrix U exists such that $X' = UXU^\top = 1$, in which case $Y'^* = Y'$. In this case, a basis exists in which the GCP is a standard *CP* transformation. In contrast, if the unitary matrix X is not symmetric, then no basis exists in which Y and Z are real for generic values of the scalar potential parameters. Nevertheless, as we shall demonstrate below, by *imposing* the GCP symmetry on the scalar potential, the parameters of the scalar potential are constrained in such a way that for an appropriately chosen basis change, $Y'^* = X'^\dagger Y' X' = Y'$ (with a similar result for Z').

GCP transformations were studied in Refs. [27,28]. In particular, Ecker, Grimus, and Neufeld [28] proved that for every matrix X there exists a unitary matrix U such that X' can be reduced to the form

$$X' = UXU^\top = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad (82)$$

where $0 \le \theta \le \pi/2$. Notice the restricted range for θ. The value of θ can be determined in either of two ways: (i) the eigenvalues of $(X + X^\top)^\dagger (X + X^\top)/2$ are $\cos\theta$, each of which is twice degenerate; or (ii) XX^* has the eigenvalues $e^{\pm 2i\theta}$.

B. The three classes of GCP symmetries

Having reached the special form of X' in Eq. (82), we will now follow the strategy adopted by Ferreira and Silva [23] in connection with HF symmetries. We substitute Eq. (82) for X in Eq. (73), in order to identify the constraints imposed by this reduced form of the GCP transformations on the quadratic and quartic couplings. For each value of θ, certain constraints will be forced upon the couplings. If two different values of θ enforce the same constraints, we will say that they are in the same class (since no experimental distinction between the two will then be possible). We will start by considering the special cases of $\theta = 0$ and $\theta = \pi/2$, and then turn our attention to $0 < \theta < \pi/2$.

1. CP1: θ = 0

When $\theta = 0$, X' is the unit matrix, and we obtain the standard *CP* transformation

$$\Phi_1 \to \Phi_1^*, \qquad \Phi_2 \to \Phi_2^*, \quad (83)$$

under which Eqs. (73) take the very simple form

$$Y^*_{ab} = Y_{ab}, \qquad Z^*_{ab,cd} = Z_{ab,cd}. \quad (84)$$

We denote this *CP* transformation by *CP*1. It forces all couplings to be real. Since most couplings are real by the Hermiticity of the Higgs potential, the only relevant constraints are $\mathrm{Im} m^2_{12} = \mathrm{Im}\lambda_5 = \mathrm{Im}\lambda_6 = \mathrm{Im}\lambda_7 = 0$.

[5]Here, we make use of a theorem in linear algebra that states that for any unitary symmetric matrix X, a unitary matrix V exists such that $X = VV^\top$. A proof of this result can be found, e.g., in Appendix B of Ref. [9].

2. CP2: $\theta = \pi/2$

When $\theta = \pi/2$,

$$X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{85}$$

and we obtain the CP transformation

$$\Phi_1 \to \Phi_2^*, \qquad \Phi_2 \to -\Phi_1^*, \tag{86}$$

which we denote by $CP2$. This was considered by Davidson and Haber [8] in their Eq. (37), who noted that if this symmetry holds in one basis, it holds in *all* basis choices. Under this transformation, Eq. (75) forces the matrix of quadratic couplings to obey

$$0 = \Delta Y = \begin{pmatrix} m_{11}^2 - m_{22}^2 & -2m_{12}^2 \\ -2m_{12}^{2*} & m_{22}^2 - m_{11}^2, \end{pmatrix}, \tag{87}$$

leading to $m_{22}^2 = m_{11}^2$ and $m_{12}^2 = 0$. Similarly, we may construct a matrix of matrices containing all coefficients $\Delta Z_{ab,cd}$. The uppermost-leftmost matrix corresponds to $\Delta Z_{11,cd}$. The next matrix along the same line corresponds to $\Delta Z_{12,cd}$, and so on. To enforce invariance under $CP2$, we equate it to zero,

$$0 = \begin{pmatrix} \begin{pmatrix} \lambda_1 - \lambda_2 & \lambda_6 + \lambda_7 \\ \lambda_6^* + \lambda_7^* & 0 \end{pmatrix} & \begin{pmatrix} \lambda_6 + \lambda_7 & 0 \\ 0 & \lambda_6 + \lambda_7 \end{pmatrix} \\ \begin{pmatrix} \lambda_6^* + \lambda_7^* & 0 \\ 0 & \lambda_6^* + \lambda_7^* \end{pmatrix} & \begin{pmatrix} 0 & \lambda_6 + \lambda_7 \\ \lambda_6^* + \lambda_7^* & \lambda_2 - \lambda_1 \end{pmatrix} \end{pmatrix}. \tag{88}$$

We learn that invariance under $CP2$ forces $m_{22}^2 = m_{11}^2$ and $m_{12}^2 = 0$, $\lambda_2 = \lambda_1$, and $\lambda_7 = -\lambda_6$, leading precisely to the ERPS of Eq. (33). Recall that Gunion and Haber [9] found that, under these conditions we can always find a basis where all parameters are real. As a result, if the potential is invariant under $CP2$, there is a basis where $CP2$ still holds and in which the potential is also invariant under $CP1$.

3. CP3: $0 < \theta < \pi/2$

Finally, we turn to the cases where $0 < \theta < \pi/2$. Imposing Eq. (75) yields

$$\begin{aligned} 0 &= \Delta Y_{11} = [(m_{11}^2 - m_{22}^2)s - 2\,\mathrm{Re}\,m_{12}^2 c]s, \\ 0 &= \Delta Y_{22} = -\Delta Y_{11}, \\ 0 &= \Delta Y_{12} \\ &= \mathrm{Re}\,m_{12}^2(c_2 - 1) - 2i\,\mathrm{Im}\,m_{12}^2 + \tfrac{1}{2}(m_{22}^2 - m_{11}^2)s_2, \end{aligned} \tag{89}$$

where we have used $c = \cos\theta$, $s = \sin\theta$, $c_2 = \cos 2\theta$, and $s_2 = \sin 2\theta$. Since $\theta \neq 0$, $\pi/2$, the conditions $m_{22}^2 = m_{11}^2$ and $m_{12}^2 = 0$ are imposed, as in $CP2$. Similarly, Eq. (76) yields

$$\begin{aligned} 0 = \Delta Z_{11,11} &= \lambda_1(1 - c^4) - \lambda_2 s^4 - \tfrac{1}{2}\lambda_{345}s_2^2 \\ &\quad + 4\,\mathrm{Re}\lambda_6 c^3 s + 4\,\mathrm{Re}\lambda_7 c s^3, \\ 0 = \Delta Z_{22,22} &= \lambda_2(1 - c^4) - \lambda_1 s^4 - \tfrac{1}{2}\lambda_{345}s_2^2 \\ &\quad - 4\,\mathrm{Re}\lambda_7 c^3 s - 4\,\mathrm{Re}\lambda_6 c s^3, \\ 0 = \Delta Z_{11,22} &= -\tfrac{1}{4}s_2[4\,\mathrm{Re}(\lambda_6 - \lambda_7)c_2 \\ &\quad + (\lambda_1 + \lambda_2 - 2\lambda_{345})s_2], \\ 0 = \Delta Z_{12,21} &= \Delta Z_{11,22} 0 = \mathrm{Re}\Delta Z_{11,12} \\ &= \tfrac{1}{4}s[(-3\lambda_1 + \lambda_2 + 2\lambda_{345})c \\ &\quad - (\lambda_1 + \lambda_2 - 2\lambda_{345})c_3 \\ &\quad + 4\,\mathrm{Re}\lambda_6(2s + s_3) - 4\,\mathrm{Re}\lambda_7 s_3], \\ 0 = \mathrm{Re}\Delta Z_{22,12} &= \tfrac{1}{4}s[(-\lambda_1 + 3\lambda_2 - 2\lambda_{345})c \\ &\quad + (\lambda_1 + \lambda_2 - 2\lambda_{345})c_3 - 4\,\mathrm{Re}\lambda_6 s_3 \\ &\quad + 4\,\mathrm{Re}\lambda_7(2s + s_3)], \\ 0 = \mathrm{Re}\Delta Z_{12,12} &= \Delta Z_{11,22} \end{aligned} \tag{90}$$

$$\begin{aligned} 0 &= \mathrm{Im}\Delta Z_{11,12} \\ &= \tfrac{1}{2}[\mathrm{Im}\lambda_6(3 + c_2) + \mathrm{Im}\lambda_7(1 - c_2) - \mathrm{Im}\lambda_5 s_2], \\ 0 &= \mathrm{Im}\Delta Z_{22,12} \\ &= \tfrac{1}{2}[\mathrm{Im}\lambda_6(1 - c_2) + \mathrm{Im}\lambda_7(3 + c_2) + \mathrm{Im}\lambda_5 s_2], \\ 0 &= \mathrm{Im}\Delta Z_{12,12} = 2c[\mathrm{Im}\lambda_5 c + \mathrm{Im}(\lambda_6 - \lambda_7)s], \end{aligned} \tag{91}$$

where $\lambda_{345} = \lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5$, $c_3 = \cos 3\theta$, and $s_3 = \sin 3\theta$.

The last three equations may be written as

$$0 = \begin{bmatrix} -s_2 & (3 + c_2) & (1 - c_2) \\ s_2 & (1 - c_2) & (3 + c_2) \\ (1 + c_2) & s_2 & -s_2 \end{bmatrix} \begin{bmatrix} \mathrm{Im}\lambda_5 \\ \mathrm{Im}\lambda_6 \\ \mathrm{Im}\lambda_7 \end{bmatrix}. \tag{92}$$

The determinant of this homogeneous system of three equations in three unknowns is $32c^2$, which can never be zero since we are assuming that $\theta \neq \pi/2$. As a result, λ_5, λ_6, and λ_7 are real, whatever the value of $0 < \theta < \pi/2$ chosen for the GCP transformation. Since $m_{12}^2 = 0$, all potentially complex parameters must be real. We conclude that a potential invariant under any GCP with $0 < \theta < \pi/2$ is automatically invariant under $CP1$. Combining this with what we learned from $CP2$, we conclude the following: if a potential is invariant under some GCP transformation, then a basis may be found in which it is also invariant under the standard CP transformation, with some added constraints on the parameters.

The other set of five independent homogeneous equations in five unknowns has a determinant equal to zero, meaning that not all parameters must vanish. We find that

$$\begin{aligned} 0 &= \Delta Z_{11,11} - \Delta Z_{22,22} \\ &= 2s[s(\lambda_1 - \lambda_2) + c2\,\mathrm{Re}(\lambda_6 + \lambda_7)], \\ 0 &= \mathrm{Re}\Delta Z_{11,12} - \mathrm{Re}\Delta Z_{22,12} \\ &= s[-c(\lambda_1 - \lambda_2) + s2\,\mathrm{Re}(\lambda_6 + \lambda_7)]. \end{aligned} \tag{93}$$

Since $s \neq 0$, we obtain the homogeneous system

$$0 = \begin{bmatrix} s & c \\ -c & s \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 \\ 2\,\mathrm{Re}(\lambda_6 + \lambda_7) \end{bmatrix}, \tag{94}$$

whose determinant is unity. We conclude that $\lambda_2 = \lambda_1$ and $\lambda_7 = -\lambda_6$. Thus, GCP invariance with any value of $0 < \theta \leq \pi/2$ leads to the ERPS of Eq. (33). Substituting back we obtain $\Delta Z_{11,11} = \Delta Z_{22,22} = -\Delta Z_{11,22}$ and $\mathrm{Re}\Delta Z_{11,12} = -\mathrm{Re}\Delta Z_{22,12}$, leaving only two independent equations:

$$\begin{aligned} 0 &= \Delta Z_{11,11} = \tfrac{1}{2} s_2[(\lambda_1 - \lambda_{345}) s_2 + 4\lambda_6 c_2], \\ 0 &= \mathrm{Re}\Delta Z_{22,12} = \tfrac{1}{2} s_2[(\lambda_1 - \lambda_{345}) c_2 - 4\lambda_6 s_2], \end{aligned} \tag{95}$$

where we have used $c + c_3 = 2cc_2$ and $s + s_3 = 2cs_2$. Since $s_2 \neq 0$, the determinant of the system does not vanish, forcing $\lambda_1 = \lambda_{345}$ and $\lambda_6 = 0$.

Notice that our results do not depend on which exact value of $0 < \theta < \pi/2$ in Eq. (82) we have chosen. If we require invariance of the potential under GCP with some particular value of $0 < \theta < \pi/2$, then the potential is immediately invariant under GCP with any other value of $0 < \theta < \pi/2$. We name this class of *CP* invariances *CP*3. Combining everything, we conclude that invariance under *CP*3 implies

$$m_{11}^2 = m_{22}^2, \qquad m_{12}^2 = 0, \qquad \lambda_2 = \lambda_1,$$
$$\lambda_7 = \lambda_6 = 0, \qquad \mathrm{Im}\lambda_5 = 0, \qquad \mathrm{Re}\lambda_5 = \lambda_1 - \lambda_3 - \lambda_4. \tag{96}$$

The results of this section are all summarized in Table I of Sec. V.

C. The square of the GCP transformation

If we apply a GCP transformation twice to the scalar fields, we will have, from Eq. (70), that

$$(\Phi_a^{\mathrm{GCP}})^{\mathrm{GCP}} = X_{a\alpha} (\Phi_\alpha^{\mathrm{GCP}})^* = X_{a\alpha} X_{\alpha b}^* \Phi_b, \tag{97}$$

so that the square of a GCP transformation is given by

$$(\mathrm{GCP})^2 = XX^*. \tag{98}$$

In particular, for a generic unitary matrix X, $(\mathrm{GCP})^2$ is a Higgs family symmetry transformation.

Usually, only GCP transformations with $(\mathrm{GCP})^2 = \mathbf{1}$ (where $\mathbf{1}$ is the unit matrix) are considered in the literature. For such a situation, $X = X^\dagger = X^*$, and one can always find a basis in which $X = \mathbf{1}$. In this case, a GCP transformation is equivalent to a standard *CP* transformation in the latter basis choice. For example, the restriction that $(\mathrm{GCP})^2 = \mathbf{1}$ (or equivalently, requiring the squared of the corresponding generalized time-reversal transformation to equal the unit matrix) was imposed in Ref. [9] and more recently in Ref. [15]. However, as we have illustrated in this section, the invariance under a GCP transformation, in which $(\mathrm{GCP})^2 \neq \mathbf{1}$ (corresponding to a unitary matrix X that is not symmetric) is a *stronger* restriction on the parameters of the scalar potential than the invariance under a standard *CP* transformation.

As we see from the results in the previous sections, X is *not* symmetric for the symmetries *CP*2 and *CP*3. In fact, this feature provides a strong distinction among the three GCP symmetries previously introduced. Let us briefly examine $(\mathrm{GCP})^2$ for the three possible cases *CP*1, *CP*2, and *CP*3.

1. $(CP1)^2$

Comparing Eqs. (70) and (83), we come to the immediate conclusion that $X_{CP1} = \mathbf{1}$, so that Eq. (98) yields

$$(CP1)^2 = \mathbf{1}. \tag{99}$$

This implies that a *CP*1-invariant scalar potential is invariant under the symmetry group $Z_2 = \{\mathbf{1}, CP1\}$.

2. $(CP2)^2$

The matrix X_{CP2} is shown in Eq. (85) so that, by Eq. (98), we obtain

$$(CP2)^2 = -\mathbf{1}. \tag{100}$$

Although this result significantly distinguished *CP*2 from *CP*1, the authors of Ref. [15] noted (in considering their $CP_g^{(i)}$ symmetries) that the transformation law for Φ_a under $(CP2)^2$ can be reduced to the identity by a global hypercharge transformation. That is, if we start with the symmetry group $Z_4 = \{\mathbf{1}, CP2, -\mathbf{1}, -CP2\}$, we can impose an equivalence relation by identifying two elements of Z_4 related by multiplication by $-\mathbf{1}$. If we denote $(Z_2)_Y = \{\mathbf{1}, -\mathbf{1}\}$ as the two-element discrete subgroup of the global hypercharge $U(1)_Y$, then the discrete symmetry group that is orthogonal to $U(1)_Y$ is given by $Z_4/(Z_2)_Y \cong Z_2$. Hence, the *CP*2-invariant scalar potential exhibits a Z_2 symmetry orthogonal to the Higgs flavor symmetries of the potential.

3. $(CP3)^2$

The matrix X_{CP3} is given in Eq. (82), with $0 < \theta < \pi/2$, so that, by Eq. (98), we obtain

$$(CP3)^2 = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}, \tag{101}$$

which once again is *not* the unit matrix. However, the transformation law for Φ_a under $(CP3)^2$ *cannot* be reduced to the identity by a global hypercharge transformation. This is the reason why Ref. [15] did not consider *CP*3. However, $(CP3)^2$ is a nontrivial HF symmetry of the

$CP3$-invariant scalar potential.[6] Thus, one can always reduce the square of $CP3$ to the identity by applying a suitable HF symmetry transformation. In particular, a $CP3$-invariant scalar potential also exhibits a Z_2 symmetry that is orthogonal to the Higgs flavor symmetries of the potential.

In this paper, we prove that there are three and only three classes of GCP transformations. Of course, within each class, one may change the explicit form of the scalar potential by a suitable basis transformation; but that will not alter its physical consequences. Similarly, one can set some parameters to zero in some *ad hoc* fashion, not rooted in a symmetry requirement. But, as we have shown, the constraints imposed on the scalar potential by a single GCP symmetry can be grouped into three classes: $CP1$, $CP2$, and $CP3$.

V. CLASSIFICATION OF THE HF AND GCP TRANSFORMATION CLASSES IN THE THDM

A. Constraints on scalar potential parameters

Suppose that one is allowed one single symmetry requirement for the potential in the THDM. One can choose an invariance under one particular Higgs family symmetry. We know that there are only two independent classes of such simple symmetries: Z_2 and Peccei-Quinn $U(1)$. One can also choose an invariance under a particular GCP symmetry. We have proved that there are three classes of GCP symmetries, named $CP1$, $CP2$, and $CP3$. If any of the above symmetries is imposed on the THDM scalar potential (in a specified basis), then the coefficients of the scalar potential are constrained, as summarized in Table I. For completeness, we also exhibit the constraints imposed by $SO(3)$, the largest possible continuous HF symmetry that is orthogonal to the global hypercharge $U(1)_Y$ transformation.

Empty entries in Table I correspond to a lack of constraints on the corresponding parameters. Table I has been constructed for those basis choices in which Z_2 and $U(1)$ have the specific forms in Eqs. (24) and (27), respectively. If, for example, the basis is changed and Z_2 acquires the form Π_2 in Eqs. (25), then the constraints on the coefficients are altered, as shown explicitly on the fourth line of Table I. However, this does not correspond to a new model. All physical predictions are the same since the specific forms of Z_2 and Π_2 differ only by the basis change in Eq. (26). The constraints for $CP1$, $CP2$, and $CP3$ shown in Table I apply to the basis in which the GCP transformation of Eq. (70) is used where X has been transformed into X' given by Eq. (82), with $\theta = 0$, $\theta = \pi/2$, and $0 < \theta < \pi/2$, respectively.

B. Multiple symmetries and GCP

We now wish to consider the possibility of simultaneously imposing more than one symmetry requirement on the Higgs potential. For example, one can require that Z_2 and Π_2 be enforced *within the same basis*. In what follows, we shall indicate that the two symmetries are enforced simultaneously by writing $Z_2 \oplus \Pi_2$. Combining the constraints from the appropriate rows of Table I, we conclude that, under these two simultaneous requirements

$$m_{22}^2 = m_{11}^2, \qquad m_{12}^2 = 0, \qquad \lambda_2 = \lambda_1, \qquad \lambda_7 = \lambda_6 = 0, \qquad \mathrm{Im}\lambda_5 = 0. \tag{102}$$

This coincides exactly with the conditions of the ERPS in a very special basis, as shown in Eq. (34). Since $CP2$ leads to the ERPS of Eq. (33), we conclude that

$$Z_2 \oplus \Pi_2 \equiv CP2 \text{ in some specific basis.} \tag{103}$$

This was noted previously by Davidson and Haber [8]. Now that we know what all classes of HF and CP symmetries can look like, we can ask whether all GCP symmetries can be written as the result of some multiple HF symmetry.

This is clearly not possible for $CP1$ because of parameter counting. Table I shows that $CP1$ reduces the scalar

TABLE I. Impact of the symmetries on the coefficients of the Higgs potential in a specified basis.

Symmetry	m_{11}^2	m_{22}^2	m_{12}^2	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
Z_2			0						0	0
$U(1)$			0					0	0	0
$SO(3)$		m_{11}^2	0		λ_1		$\lambda_1 - \lambda_3$	0	0	0
Π_2		m_{11}^2	real		λ_1			real		λ_6^*
$CP1$			real					real	real	real
$CP2$		m_{11}^2	0		λ_1					$-\lambda_6$
$CP3$		m_{11}^2	0		λ_1			$\lambda_1 - \lambda_3 - \lambda_4$ (real)	0	0

[6]In Sec. V B, we shall identify $(CP3)^2$ with the Peccei-Quinn $U(1)$ symmetry defined as in Eq. (27) and then transformed to a new basis according to the unitary matrix defined in Eq. (105).

potential to ten real parameters. We can still perform an orthogonal basis change while keeping all parameters real. This freedom can be used to remove one further parameter; for example, setting $m_{12}^2 = 0$ by diagonalizing the Y matrix. No further simplification is allowed. As a result, $CP1$ leaves nine independent parameters. The smallest HF symmetry is Z_2. Table I shows that Z_2 reduces the potential to six real and one complex parameter. The resulting eight parameters could never account for the nine needed to fully describe the most general model with the standard CP invariance $CP1$.[7]

But one can utilize two HF symmetries in order to obtain the same constraints obtained by invariance under $CP3$. Let us impose *both* $U(1)$ and Π_2 *in the same basis*. From Table I, we conclude that, under these two simultaneous requirements

$$m_{22}^2 = m_{11}^2, \qquad m_{12}^2 = 0, \qquad \lambda_2 = \lambda_1, \qquad \lambda_7 = \lambda_6 = 0, \qquad \lambda_5 = 0. \tag{104}$$

This does not coincide with the conditions for invariance under $CP3$ shown in Eq. (96). However, one can use the transformation rules in Eqs. (A13)–(A23) of Davidson and Haber [8], in order to show that a basis transformation

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \tag{105}$$

may be chosen, which takes us from Eqs. (96), where $\mathrm{Re}\lambda_5 = \lambda_1 - \lambda_3 - \lambda_4$, to Eqs. (104), where $\lambda_5 = 0$ (while maintaining the other relations among the scalar potential parameters). We conclude that

$$U(1) \oplus \Pi_2 \equiv CP3 \text{ in some specific basis.} \tag{106}$$

Note that in the basis in which the $CP3$ relations of Eq. (96) are satisfied with $\lambda_5 \neq 0$, the discrete HF symmetry Π_2 is still respected. However, using Eq. (105), it follows that the $U(1)$-Peccei-Quinn symmetry corresponds to the invariance of the scalar potential under $\Phi_a \to \mathcal{O}_{ab}\Phi_b$, where $\mathcal{O}$ is an arbitrary $SO(2)$ matrix.

The above results suggest that it should be possible to distinguish $CP1$, $CP2$, and $CP3$ in a basis-invariant fashion. Botella and Silva [6] have built three so-called J invariants that detect any signal of CP violation (either explicit or spontaneous) after the minimization of the scalar potential. However, in this paper we are concerned about the symmetries of the scalar potential independently of the choice of vacuum. Thus, we shall consider the four so-called I invariants built by Gunion and Haber [9] in order to detect any signal of *explicit CP* violation present (before the vacuum state is determined). If any of these invariants is nonzero, then CP is explicitly violated, and neither $CP1$, nor $CP2$, nor $CP3$ hold. Conversely, if all I invariants are zero, then CP is explicitly conserved, but we cannot tell *a priori* which GCP applies. Equations (103) and (106) provide the crucial hint. If we have CP conservation, $Z_2 \oplus \Pi_2$ holds, and $U(1)$ does not, then we have $CP2$. Alternatively, if we have CP conservation, and $U(1) \oplus \Pi_2$ also holds, then we have $CP3$. We recall that both $CP2$ and $CP3$ lead to the ERPS, and that the general conditions for the ERPS in Eq. (33) are basis independent. This allows us to distinguish $CP2$ and $CP3$ from $CP1$. But, prior to the present work, no basis-independent quantity had been identified in the literature that could distinguish Z_2 and $U(1)$ in the ERPS. The basis-independent quantity D introduced in Sec. II E is precisely the invariant required for this task. That is, in the ERPS $D \neq 0$ implies $CP2$, whereas $D = 0$ implies $CP3$.

One further consequence of the results of Table I can be seen by simultaneously imposing the $U(1)$-Peccei-Quinn symmetry and the $CP3$ symmetry *in the same basis*. The resulting constraints on the scalar potential parameters are precisely those of the $SO(3)$ HF symmetry. Thus, we conclude that

$$U(1) \oplus CP3 \equiv SO(3). \tag{107}$$

In particular, $SO(3)$ is not a simple HF symmetry, as the invariance of the scalar potential under a single element of $SO(3)$ is not sufficient to guarantee invariance under the full $SO(3)$ group of transformations.

C. Maximal symmetry group of the scalar potential orthogonal to $U(1)_Y$

The standard CP symmetry $CP1$ is a discrete Z_2 symmetry that transforms the scalar fields into their complex conjugates, and hence is not a subgroup of the $U(2)$ transformation group of Eq. (15). We have previously noted that THDM scalar potentials that exhibit *any* nontrivial HF symmetry G is automatically CP conserving. Thus, the actual symmetry group of the scalar potential is in fact the semidirect product[8] of G and Z_2, which we write as $G \rtimes Z_2$. Noting that $U(1) \rtimes Z_2 \cong SO(2) \rtimes Z_2 \cong O(2)$, and $SO(3) \otimes Z_2 \cong O(3)$, we conclude that the maximal symmetry groups of the scalar potential orthogonal to $U(1)_Y$ for the possible choices of HF symmetries are given in Table II.[9]

Finally, we reconsider $CP2$ and $CP3$. Equation (103) implies that the $CP2$ symmetry is equivalent to a $(Z_2)^2$ HF symmetry. To prove this statement, we note that in the two-dimensional flavor space of Higgs fields, the Z_2 and Π_2 discrete symmetries defined by Eqs. (24) and (25) are given by

[7] In Ivanov's language, this is clear since $CP1$ corresponds to a Z_2 transformation of the vector $\vec{r}$, which is the simplest transformation on $\vec{r}$ one could possibly make. See Sec. V D.

[8] In general, the nontrivial element of Z_2 will not commute with all elements of G, in which case the relevant mathematical structure is that of a semidirect product. In cases where the nontrivial element of Z_2 commutes with all elements of G, we denote the corresponding direct product as $G \otimes Z_2$.

[9] For ease of notation, we denote $Z_2 \otimes Z_2$ by $(Z_2)^2$ and $Z_2 \otimes Z_2 \otimes Z_2$ by $(Z_2)^3$.

TABLE II. Maximal symmetry groups [orthogonal to global $U(1)_Y$ hypercharge] of the scalar sector of the THDM.

Designation	HF symmetry group	Maximal symmetry group
Z_2	Z_2	$(Z_2)^2$
Peccei-Quinn	$U(1)$	$O(2)$
$SO(3)$	$SO(3)$	$O(3)$
$CP1$	-	Z_2
$CP2$	$(Z_2)^2$	$(Z_2)^3$
$CP3$	$O(2)$	$O(2)\otimes Z_2$

$$Z_2 = \{S_0, S_1\}, \qquad \Pi_2 = \{S_0, S_2\}, \tag{108}$$

where $S_0 \equiv \mathbf{1}$ is the 2×2 identity matrix and

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{109}$$

If we impose the Z_2 and Π_2 symmetry in the same basis, then the scalar potential is invariant under the dihedral group of eight elements

$$D_4 = \{S_0, S_1, S_2, S_3, -S_0, -S_1, -S_2, -S_3\}, \tag{110}$$

where $S_3 = S_1 S_2 = -S_2 S_1$. As before, we identify $(Z_2)_Y \equiv \{S_0, -S_0\}$ as the two-element discrete subgroup of the global hypercharge $U(1)_Y$. However, we have defined the HF symmetries to be orthogonal to $U(1)_Y$. Thus, to determine the HF symmetry group of $CP2$, we identify as equivalent those elements of D_4 that are related by multiplication by $-S_0$. Grouped theoretically, we identify the HF symmetry group of $CP2$ as

$$D_4/(Z_2)_Y \cong Z_2 \otimes Z_2. \tag{111}$$

The HF symmetry group of $CP2$ is not the maximally allowed symmetry group. In particular, the constraints of $CP2$ on the scalar potential imply the existence of a basis in which all scalar potential parameters are real. Thus, the scalar potential is explicitly CP conserving. The Z_2 symmetry associated with this CP transformation is orthogonal to the HF symmetry as previously noted. (This is easily checked explicitly by employing a four-dimensional real representation of the two complex scalar fields.) Thus, the maximal symmetry group of the $CP2$-symmetric scalar potential is $(Z_2)^3$. Similarly, Eq. (106) implies that the $CP3$ symmetry is equivalent to a $U(1)\rtimes Z_2$ HF symmetry. This is isomorphic to an $O(2)$ HF symmetry, which is a subgroup of the maximally allowed $SO(3)$ HF symmetry group. However, the constraints of $CP3$ on the scalar potential imply the existence of a basis in which all scalar potential parameters are real. Thus, the scalar potential is explicitly CP conserving. Once again, the Z_2 symmetry associated with this CP transformation is orthogonal to the HF symmetry noted above. Thus, the maximal symmetry group of the $CP3$-symmetric scalar potential is $O(2)\otimes Z_2$.

The above results are also summarized in Table II. In all cases, the maximal symmetry group is a direct product of the HF symmetry group and the Z_2 corresponding to the standard CP transformation, whose square is the identity operator.

One may now ask whether Table II exhausts all possible independent symmetry constraints that one may place on the Higgs potential. Perhaps one can choose other combinations, or maybe one can combine three, four, or more symmetries. We know of no way to answer this problem based only on the transformations of the scalar fields Φ_a. Fortunately, Ivanov has solved this problem [3] by looking at the transformation properties of field bilinears, thus obtaining for the first time the list of symmetries given in the last column of Table II.

D. More on multiple symmetries

We start by looking at the implications of the symmetries we have studied so far on the vector $\vec{r} = \{r_1, r_2, r_3\}$, whose components were introduced in Eq. (10). Notice that a unitary transformation U on the fields Φ_a induces an orthogonal transformation O on the vector of bilinears $\vec{r}$, given by Eq. (37). For every pair of unitary transformations $\pm U$ of $SU(2)$, one can find some corresponding transformation O of $SO(3)$, in a two-to-one correspondence. We then see what these symmetries imply for the coefficients of Eq. (9) [recall the $\Lambda_{\mu\nu}$ is a symmetric matrix]. Below, we list the transformation of $\vec{r}$ under which the scalar potential is invariant, followed by the corresponding constraints on the quadratic and quartic scalar potential parameters M_μ and $\Lambda_{\mu\nu}$.

Using the results of Table I, we find that Z_2 implies

$$\vec{r} \to \begin{bmatrix} -r_1 \\ -r_2 \\ r_3 \end{bmatrix}, \qquad \begin{bmatrix} M_0 \\ 0 \\ 0 \\ M_3 \end{bmatrix}, \qquad \begin{bmatrix} \Lambda_{00} & 0 & 0 & \Lambda_{03} \\ 0 & \Lambda_{11} & \Lambda_{12} & 0 \\ 0 & \Lambda_{12} & \Lambda_{22} & 0 \\ \Lambda_{03} & 0 & 0 & \Lambda_{33} \end{bmatrix}, \tag{112}$$

$U(1)$ implies

$$\vec{r} \to \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{r}, \qquad \begin{bmatrix} M_0 \\ 0 \\ 0 \\ M_3 \end{bmatrix}, \qquad \begin{bmatrix} \Lambda_{00} & 0 & 0 & \Lambda_{03} \\ 0 & \Lambda_{11} & 0 & 0 \\ 0 & 0 & \Lambda_{11} & 0 \\ \Lambda_{03} & 0 & 0 & \Lambda_{33} \end{bmatrix}, \tag{113}$$

and $SO(3)$ implies

$$\vec{r} \to \mathcal{O}\vec{r}, \qquad \begin{bmatrix} M_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \Lambda_{00} & 0 & 0 & 0 \\ 0 & \Lambda_{11} & 0 & 0 \\ 0 & 0 & \Lambda_{11} & 0 \\ 0 & 0 & 0 & \Lambda_{11} \end{bmatrix}, \tag{114}$$

where $\mathcal{O}$ is an arbitrary 3×3 orthogonal matrix of unit determinant. In the language of bilinears, a basis-invariant condition for the presence of $SO(3)$ is that the three eigenvalues of $\tilde{\Lambda}$ are equal. (Recall that $\tilde{\Lambda} = \{\Lambda_{ij}\}$; $i, j = 1, 2, 3$.)

As for the GCP symmetries, $CP1$ implies

$$\vec{r} \to \begin{bmatrix} r_1 \\ -r_2 \\ r_3 \end{bmatrix}, \qquad \begin{bmatrix} M_0 \\ M_1 \\ 0 \\ M_3 \end{bmatrix}, \qquad \begin{bmatrix} \Lambda_{00} & \Lambda_{01} & 0 & \Lambda_{03} \\ \Lambda_{01} & \Lambda_{11} & 0 & \Lambda_{13} \\ 0 & 0 & \Lambda_{22} & 0 \\ \Lambda_{03} & \Lambda_{13} & 0 & \Lambda_{33} \end{bmatrix}, \tag{115}$$

$CP2$ implies

$$\vec{r} \to \begin{bmatrix} -r_1 \\ -r_2 \\ -r_3 \end{bmatrix}, \qquad \begin{bmatrix} M_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \Lambda_{00} & 0 & 0 & 0 \\ 0 & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ 0 & \Lambda_{12} & \Lambda_{22} & \Lambda_{23} \\ 0 & \Lambda_{13} & \Lambda_{23} & \Lambda_{33} \end{bmatrix}, \tag{116}$$

and $CP3$ implies

$$\vec{r} \to \begin{bmatrix} c_2 & 0 & s_2 \\ 0 & -1 & 0 \\ -s_2 & 0 & c_2 \end{bmatrix} \vec{r}, \qquad \begin{bmatrix} M_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \Lambda_{00} & 0 & 0 & 0 \\ 0 & \Lambda_{11} & 0 & 0 \\ 0 & 0 & \Lambda_{22} & 0 \\ 0 & 0 & 0 & \Lambda_{11} \end{bmatrix}. \tag{117}$$

Notice that in $CP3$ two of the eigenvalues of Λ are equal, in accordance with our observation that D can be used to distinguish between $CP2$ and $CP3$.

Because each unitary transformation on the fields Φ_a induces an $SO(3)$ transformation on the vector of bilinears $\vec{r}$, and because the standard CP transformation corresponds to an inversion of r_2 (a Z_2 transformation on the vector $\vec{r}$), Ivanov [3] considers all possible proper and improper transformations of $O(3)$ acting on $\vec{r}$. He identifies the following six classes of transformations: (i) Z_2; (ii) $(Z_2)^2$; (iii) $(Z_2)^3$; (iv) $O(2)$; (v) $O(2) \otimes Z_2$; and (vi) $O(3)$. Note that these symmetries are all orthogonal to the global $U(1)_Y$ hypercharge symmetry, as the bilinears r_0 and $\vec{r}$ are all singlets under a $U(1)_Y$ transformation. The six classes above identified by Ivanov correspond precisely to the six possible maximal symmetry groups identified in Table II. No other independent symmetry transformations are possible.

Our work permits one to identify the abstract transformation of field bilinears utilized by Ivanov in terms of transformations on the scalar fields themselves, as needed for model building. Combining our work with Ivanov's, we conclude that there is only one new type of symmetry requirement that one can place on the Higgs potential via multiple symmetries. Combining this with our earlier results, we conclude that all possible symmetries on the scalar sector of the THDM can be reduced to multiple HF symmetries, with the exception of the standard CP transformation ($CP1$).

VI. BUILDING ALL SYMMETRIES WITH THE STANDARD *CP*

We have seen that there are only six independent symmetry requirements, listed in Table II, that one can impose on the Higgs potential. We have shown that all possible symmetries of the scalar sector of the THDM can be reduced to multiple HF symmetries, with the exception of the standard CP transformation ($CP1$). Now we wish to show a dramatic result: *all possible symmetries on the scalar sector of the THDM can be reduced to multiple applications of the standard CP symmetry.*

Using Eq. (79), we see that the basis transformation of Eq. (15), changes the standard CP symmetry of Eq. (69) into the GCP symmetry of Eq. (70), with

$$X = UU^{\top}. \tag{118}$$

In particular, an orthogonal basis transformation does not affect the form of the standard CP transformation. Since we wish to generate $X \neq 1$, we will need complex matrices U.

Now we wish to consider the following situation: We have a basis (call it the original basis) and impose the standard CP symmetry $CP1$ on that original basis. Next we consider the same model in a different basis (call it M) and impose the standard CP symmetry on that basis M. In general, this procedure of imposing the standard CP symmetry in the original basis *and also* in the rotated basis M leads to two independent impositions. The first imposition makes all parameters real in the original basis. One way to combine the second imposition with the first is to consider the basis transformation U_M taking us from basis M into the original basis. As we have seen, the standard CP symmetry in basis M turns, when written in the original basis, into a symmetry under

$$\Phi_a^{CP} = (X_M)_{a\alpha}\Phi_\alpha^*, \qquad \Phi_a^{\dagger CP} = (X_M)_{a\alpha}^*(\Phi_\alpha^\dagger)^*, \quad (119)$$

with $X_M = U_M U_M^\top$. Next we consider several such possibilities.

We start with

$$U_A = \begin{pmatrix} c_{\pi/4} & -is_{\pi/4} \\ -is_{\pi/4} & c_{\pi/4} \end{pmatrix}, \qquad X_A = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \quad (120)$$

Here and henceforth c (s) with a subindex indicates the cosine (sine) of the angle given in the subindex. We denote by $CP1_A$ the imposition of the CP symmetry in Eq. (119) with $X_M = X_A$ (which coincides with the imposition of the standard CP symmetry in the basis $M = A$).

Next we consider

$$U_B = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}, \qquad X_B = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \quad (121)$$

We denote by $CP1_B$ the imposition of the CP symmetry in Eq. (119) with $X_M = X_B$ (which coincides with the imposition of the standard CP symmetry in the basis $M = B$).

A third possible choice is

$$U_C = \begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix}, \qquad X_C = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{pmatrix}, \quad (122)$$

where $\delta \neq n\pi/2$ with n integer. We denote by $CP1_C$ the imposition of the CP symmetry in Eq. (119) with $X_M = X_C$ (which coincides with the imposition of the standard CP symmetry in the basis $M = C$).

Finally, we consider

$$U_D = \begin{pmatrix} c_{\delta/2} & is_{\delta/2} \\ is_{\delta/2} & c_{\delta/2} \end{pmatrix}, \qquad X_D = \begin{pmatrix} c_\delta & is_\delta \\ is_\delta & c_\delta \end{pmatrix}, \quad (123)$$

where $\delta \neq n\pi/2$ with n integer. We denote by $CP1_D$ the imposition of the CP symmetry in Eq. (119) with $X_M = X_D$ (which coincides with the imposition of the standard CP symmetry in the basis $M = D$).

The impact of the first three symmetries on the coefficients of the Higgs potential are summarized in Table III.

Imposing $CP1_D$ on the Higgs potential leads to the more complicated set of equations

$$\begin{aligned} 2\,\mathrm{Im}(m_{12}^2)c_\delta + (m_{22}^2 - m_{11}^2)s_\delta &= 0, \\ 2\,\mathrm{Im}(\lambda_6 - \lambda_7)c_{2\delta} + \lambda_{12345}s_{2\delta} &= 0, \\ 2\,\mathrm{Im}(\lambda_6 + \lambda_7)c_\delta + (\lambda_1 - \lambda_2)s_\delta &= 0, \\ \mathrm{Im}\lambda_5 c_\delta + \mathrm{Re}(\lambda_6 - \lambda_7)s_\delta &= 0, \end{aligned} \quad (124)$$

where

$$\lambda_{12345} = \frac{1}{2}(\lambda_1 + \lambda_2) - \lambda_3 - \lambda_4 + \mathrm{Re}\lambda_5. \quad (125)$$

Combining these results with those in Table I, we have shown that

$$\begin{aligned} CP1 \oplus CP1_B &= Z_2 \text{ in some specific basis}, \\ CP1 \oplus CP1_C &= U(1), \\ CP1 \oplus CP1_A \oplus CP1_B &= CP2 \text{ in some specific basis}, \\ CP1 \oplus CP1_A \oplus CP1_C &= CP3 \text{ in some specific basis}, \\ CP1 \oplus CP1_C \oplus CP1_D &= SO(3). \end{aligned} \quad (126)$$

Let us comment on the "specific basis choices" needed. Imposing $CP1 \oplus CP1_B$ leads to $m_{12}^2 = \lambda_6 = \lambda_7 = 0$ and $\mathrm{Im}\lambda_5 = 0$, while imposing Z_2 leads to $m_{12}^2 = \lambda_6 = \lambda_7 = 0$ with no restriction on λ_5. However, when Z_2 holds, one may rephase Φ_2 by the exponential of $-i\arg(\lambda_5)/2$, thus making λ_5 real. In this basis, the restrictions of Z_2 coincide with the restrictions of $CP1 \oplus CP1_B$. Similarly, imposing $CP1 \oplus CP1_A \oplus CP1_C$ leads to $m_{12}^2 = \lambda_5 = \lambda_6 = \lambda_7 = 0$,

TABLE III. Impact of the $CP1_M$ symmetries on the coefficients of the Higgs potential. The notation "imag" means that the corresponding entry is purely imaginary. $CP1$ in the original basis has been included for reference.

Symmetry	m_{11}^2	m_{22}^2	m_{12}^2	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
$CP1$			real					real	real	real
$CP1_A$		m_{11}^2			λ_1					λ_6
$CP1_B$			imag					real	imag	imag
$CP1_C$			$\lvert m_{12}^2\rvert e^{i\delta}$					$\lvert\lambda_5\rvert e^{2i\delta}$	$\lvert\lambda_6\rvert e^{i\delta}$	$\lvert\lambda_7\rvert e^{i\delta}$

$m_{22}^2 = m_{11}^2$, and $\lambda_2 = \lambda_1$. We see from Table I that $CP3$ has these features, except that λ_5 need not vanish; it is real and $\mathrm{Re}\lambda_5 = \lambda_1 - \lambda_3 - \lambda_4$. Starting from the $CP3$ conditions and using the transformation rules in Eqs. (A13)–(A23) of Davidson and Haber [8], we find that a basis choice is possible such that $\mathrm{Re}\lambda_5 = 0$.[10] Perhaps it is easier to prove the equality

$$CP1 \oplus CP1_B \oplus CP1_D = CP3 \text{ in some specific basis.} \tag{127}$$

In this case, the only difference between the impositions from the two sides of the equality come from the sign of $\mathrm{Re}\lambda_5$, which is trivial to flip through the basis change $\Phi_2 \rightarrow -\Phi_2$. Finally, imposing $CP1 \oplus CP1_A \oplus CP1_B$ we obtain $m_{12}^2 = \mathrm{Im}\lambda_5 = \lambda_6 = \lambda_7 = 0$, $m_{22}^2 = m_{11}^2$, and $\lambda_2 = \lambda_1$. This does not coincide with the conditions of $CP2$, which lead to the ERPS of Eq. (33). Fortunately, and as we mentioned before, Davidson and Haber [8] proved that one may make a further basis transformation such that Eq. (34) holds, thus coinciding with the conditions imposed by $CP1 \oplus CP1_A \oplus CP1_B$.

Notice that our description of $CP2$ in terms of several $CP1$ symmetries is in agreement with the results found by the authors of Ref. [15]. These authors also showed a very interesting result, concerning spontaneous symmetry breaking in 2HDM models possessing a $CP2$ symmetry. Namely, they prove (their Theorem 4) that electroweak symmetry breaking will *necessarily* spontaneously break $CP2$. However, they also show that the vacuum will respect at least one of the $CP1$ symmetries, which compose $CP2$. Which is to say, in a model that has a $CP2$ symmetry, spontaneous symmetry breaking necessarily respect the $CP1$ symmetry.

In summary, we have proved that all possible symmetries on the scalar sector of the THDM, including Higgs family symmetries, can be reduced to multiple applications of the standard *CP* symmetry.

VII. CONCLUSIONS

We have studied the application of generalized *CP* symmetries to the THDM, and found that there are only two independent classes ($CP2$ and $CP3$), in addition to the standard *CP* symmetry ($CP1$). These two classes lead to an exceptional region of parameter, which exhibits either a Z_2 discrete symmetry or a larger $U(1)$-Peccei-Quinn symmetry. We have succeeded in identifying a basis-independent invariant quantity that can distinguish between the Z_2 and $U(1)$ symmetries. In particular, such an invariant is required in order to distinguish between $CP2$ and $CP3$, and completes the description of all symmetries in the THDM in terms of basis-invariant quantities. Moreover, $CP2$ and $CP3$ can be obtained by combining two Higgs family symmetries and that this is not possible for $CP1$.

We have shown that all symmetries of the THDM previously identified by Ivanov [3] can be achieved through simple symmetries, with the exception of $SO(3)$. However, the $SO(3)$ Higgs family symmetry can be achieved by imposing a $U(1)$-Peccei-Quinn symmetry and the $CP3$ symmetry in the same basis. Finally, we have demonstrated that all possible symmetries of the scalar sector of the THDM can be reduced to multiple applications of the standard *CP* symmetry. Our complete description of the symmetries on the scalar fields can be combined with symmetries in the quark and lepton sectors, to aid in model building.

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[10] Notice that, in the new basis, λ_1 differs in general from $\lambda_3 + \lambda_4$; otherwise the larger $SO(3)$ Higgs family symmetry would hold.

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