

Perturbation theory in supersymmetric QED: Infrared divergences and gauge invariance

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(Received 18 August 2016; published 3 November 2016)

We study some aspects of perturbation theory in $N = 1$ supersymmetric Abelian gauge theories with massive charged matter. In general gauges, infrared (IR) divergences and nonlocal behavior arise in one particle irreducible (1PI) diagrams, associated with a $1/k^4$ term in the propagator for the vector superfield. We examine this structure in supersymmetric QED. The IR divergences are gauge dependent and must cancel in physical quantities like the electron pole mass. We demonstrate that cancellation takes place in a nontrivial way, amounting to a reorganization of the perturbative series from powers of e^2 to powers of e . We also show how these complications are avoided in cases where a Wilsonian effective action can be defined.

DOI: [10.1103/PhysRevD.94.095003](https://doi.org/10.1103/PhysRevD.94.095003)

I. INTRODUCTION

In weakly coupled supersymmetric field theories, it is convenient for certain applications to employ a manifestly supersymmetric perturbation theory. For example, non-renormalization theorems were first proven using supergraph techniques [1]. These proofs rely on the existence of a particular infrared-safe choice of gauge, analogous at one-loop order to Feynman gauge in nonsupersymmetric QED.

However, in other supersymmetric gauges, perturbation theory is plagued by unphysical infrared divergences. Difficulties may be anticipated from the superspace propagator for the vector superfield. In this paper, we focus on the supersymmetric extension of QED [2] (henceforth denoted as SQED). In the supersymmetric R_ξ gauge [3,4], the Lagrangian of SQED is supplemented by a gauge fixing term,¹

$$\begin{aligned} \mathcal{L} = & \int d^4\theta \Phi_+^\dagger e^{2eV} \Phi_+ + \int d^4\theta \Phi_-^\dagger e^{-2eV} \Phi_- \\ & + \left[\int d^2\theta m \Phi_+ \Phi_- + \text{H.c.} \right] \\ & + \left[\frac{1}{4} \int d^2\theta \mathcal{W}^\alpha \mathcal{W}_\alpha + \text{H.c.} \right] \\ & - \frac{\xi}{8} \int d^4\theta (D^2 V)(\bar{D}^2 V), \end{aligned} \quad (1.1)$$

leading to the vector superfield propagator²

$$\begin{aligned} i\Delta_V(k, \theta_1, \theta_2) = & \frac{i}{k^4} \left(1 - \frac{1}{\xi} \right) e^{(\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1) k_\mu} \\ & - \frac{i}{4k^2} \left(1 + \frac{1}{\xi} \right) \delta^4(\theta_1 - \theta_2) e^{(\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1) k_\mu}. \end{aligned} \quad (1.2)$$

It is striking that away from $\xi = 1$, the propagator behaves as $1/k^4$ for small k . This behavior can lead to infrared (IR) divergences in loop graphs that probe the small- k modes of V .

We will exhibit such IR divergences in one-loop contributions to the two-point functions of SQED with massive charged matter. The appearance of infrared issues has been noted in the past. Reference [7] described a resolution in non-Abelian gauge theories involving the introduction of a nonlocal gauge fixing term and adjusting the gauge fixing parameter to eliminate the divergences order by order in perturbation theory. In massive Abelian theories, this procedure simplifies to the adjustment of the gauge fixing parameter without modifying the gauge fixing term itself. References [8,9] gave general proofs that the IR divergences can be regulated by a gauge-variant mass parameter and that Green functions of gauge invariant operators are independent of the regulator mass. Reference [10] computed three loop contributions to anomalous dimensions in non-Abelian gauge theories working in Feynman gauge with a suitable infrared regulator and obtained sensible results without infrared sensitivity.

On the other hand, it is also not difficult to regulate the infrared in a gauge-invariant way. Since the divergences are gauge dependent, they are unphysical and must eventually drop out of observable quantities. We study this cancellation in the pole mass of the electron chiral supermultiplet.

¹Our conventions for supersymmetric notation follow that of Ref. [5].

²Note that the normalization of this propagator differs by a factor of 2 from the vector superfield propagator given in Ref. [6].

By focusing on the explicit nature of the cancellation in a particular observable, our analysis is complementary to the general results of Refs. [8,9]. We find an interesting structure: near the pole, the perturbative series for the two-point functions reorganizes itself. Whereas the naïve one-loop graphs contribute to the series at $\mathcal{O}(e^2)$, near the pole, some—but not all—of the graphs exhibit singularities that enhance their contributions to $\mathcal{O}(e)$. We find exact cancellation between the $\mathcal{O}(e)$ one-loop graphs, including their IR-divergent pieces. We argue that the cancellation of IR-divergent terms at $\mathcal{O}(e^2)$ must occur between a combination of the remaining one-loop and enhanced two-loop graphs.

If light neutral fields are added to the theory, the charged massive fields may be integrated out to obtain a Wilsonian effective action subject to the naïve nonrenormalization theorems. The resulting wave function renormalization should be gauge invariant, and in particular infrared divergences should cancel order by order in the effective action. We will verify this at low orders in the perturbation series.

This paper is organized as follows. In Sec. II, we review and collect the SQED superspace and component propagators relevant for our analysis. In Sec. III, after recalling the gauge dependence of the mass renormalization in ordinary QED, we describe the one-loop renormalization of the electron mass in SQED. We show that there are infrared divergences and nonlocal behavior in the one particle irreducible (1PI) corrections to the helicity-flip and helicity-preserving propagators. The nonlocal factors are singular near the mass shell and lead to a mixing of loop orders at fixed order in e . We show that the leading IR-divergent gauge dependence cancels in the one-loop electron pole mass, while subleading unphysical contributions must cancel against two-loop terms. We also recover the well-known result [11] that the ultraviolet (UV) divergent part of the mass renormalization is gauge invariant. In Sec. V, we couple a massless, neutral field to the charged fields and demonstrate cancellation of infrared contributions to the self-energy at two loops. We discuss the implications for the Wilsonian effective action in this case. In Sec. VI, we demonstrate the presence of infrared divergences at higher order in the gauge $\xi = 1$, but show that it is possible to choose a gauge, order by order, in which infrared divergences are absent. In Sec. VII we summarize and conclude.

Additional background material and further results are collected in three appendixes. In Appendix A, we review mass and wave function renormalization of nonsupersymmetric QED. In Appendix B, we discuss the computation of the tree-level propagators of supersymmetric QED and examine the supersymmetric relations among the two-point functions. Finally, in Appendix C, we demonstrate that in addition to the divergences, the finite corrections to the physical electron mass also vanish in the on-shell limit.

II. PERTURBATION THEORY IN SQED

For convenience, in this section we collect the well-known propagator expressions in SQED. In superspace, the vector propagator was given in Eq. (1.2) and is repeated here for the convenience of the reader,

$$i\Delta_V(k, \theta_1, \theta_2) = \frac{i}{k^4} \left(1 - \frac{1}{\xi}\right) e^{(\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1) k_\mu} - \frac{i}{4k^2} \left(1 + \frac{1}{\xi}\right) \delta^4(\theta_1 - \theta_2) e^{(\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1) k_\mu}. \quad (2.1)$$

The corresponding propagators of the component fields are given in Appendix B.

We study the theory with massive electrons, with the superpotential given by

$$W = m\Phi_+ \Phi_-, \quad (2.2)$$

for which the superfield propagators are

$$i\Delta_{\Phi_\pm \Phi_\mp}(k, \theta_1, \theta_2) = -im\delta^{(2)}(\theta_1 - \theta_2) \exp[(\theta_1 \sigma^\mu \bar{\theta}_1 - \theta_2 \sigma^\mu \bar{\theta}_2) k_\mu] \frac{1}{k^2 - m^2}, \quad (2.3)$$

$$i\Delta_{\Phi_\pm^\dagger \Phi_\mp^\dagger}(k, \theta_1, \theta_2) = +im\delta^{(2)}(\bar{\theta}_1 - \bar{\theta}_2) \exp[(\theta_1 \sigma^\mu \bar{\theta}_1 - \theta_2 \sigma^\mu \bar{\theta}_2) k_\mu] \frac{1}{k^2 - m^2}, \quad (2.4)$$

$$i\Delta_{\Phi_\pm \Phi_\pm^\dagger}(k, \theta_1, \theta_2) = i \exp[(\theta_1 \sigma^\mu \bar{\theta}_1 - \theta_2 \sigma^\mu \bar{\theta}_2 + 2\theta_1 \sigma^\mu \bar{\theta}_2) k_\mu] \frac{1}{k^2 - m^2}. \quad (2.5)$$

It is helpful (and in many computations simpler) to work with a mixture of component and superspace formalisms. We parametrize the vector superfield components as

$$V(x, \theta, \bar{\theta}) = a(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \theta^2 M(x) + \bar{\theta}^2 \bar{M}(x) + i\theta\sigma^\mu \bar{\theta} A_\mu(x) + i\theta^2 \bar{\theta} \left(\bar{\lambda}(x) - \frac{1}{2} i\bar{\sigma}^\mu \partial_\mu \chi(x) \right) - i\bar{\theta}^2 \theta \left(\lambda(x) - \frac{1}{2} i\sigma^\mu \partial_\mu \bar{\chi}(x) \right) + \frac{1}{2} \theta^2 \bar{\theta}^2 \left(D(x) - \frac{1}{2} \square a(x) \right). \quad (2.6)$$

In Appendix B.1, we show that the component Lagrangian for the vector includes the terms

$$\mathcal{L}_V = \frac{1}{2}(1 - \xi)D^2 - \frac{1}{2}\xi(\square a)^2 + \xi D \square a. \quad (2.7)$$

Inverting the quadratic form gives for the momentum space propagators³ of a and D

$$\langle DD \rangle = i, \quad (2.8)$$

$$\langle aD \rangle = -\frac{i}{k^2}, \quad (2.9)$$

$$\langle aa \rangle = \left(1 - \frac{1}{\xi}\right) \frac{i}{k^4}. \quad (2.10)$$

We see that in components, the $1/k^4$ infrared behavior discussed in Sec. I can be traced to the kinetic term for the lowest component of V , which contains four derivatives in a general gauge. In the $\xi = 1$ gauge, the $1/k^4$ terms disappear, and severe infrared divergences are avoided in low orders of perturbation theory. However, loop corrections will reintroduce $1/k^4$ terms in the propagator. In particular, at the level of component fields, there is an $\langle aD \rangle$ propagator, and at one-loop charged fields correct the $\langle DD \rangle$ two-point function at zero momentum.

III. SELF-ENERGIES IN SQED

The naïve expectation from the nonrenormalization theorems is that there should be no renormalization of the superpotential mass m arising from the $\Phi_+ \Phi_-$ self-energy. Any renormalization of the physical mass should arise as a result of corrections to the Kähler potential.

In supersymmetric Feynman gauge, $\xi = 1$, there are indeed no one-loop 1PI contributions to $\langle \Phi_+ \Phi_- \rangle$. This can be seen directly in superspace, as in Ref. [1]. It can also be seen by working in components with explicit auxiliary fields. We take as the component expansion of the chiral superfields

$$\Phi_{\pm}(x, \theta, \bar{\theta}) = \exp(-i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu})[\phi_{\pm}(x) + \sqrt{2}\theta\psi_{\pm}(x) + \theta\theta F_{\pm}(x)]. \quad (3.1)$$

In particular, such a two-point function for the superfields would yield, in components, a nonvanishing $\langle F_+ \phi_- + F_- \phi_+ \rangle$. But it is easy to see there is no such graph at one loop. There is a wave function renormalization for Φ_+ and Φ_- which is ultraviolet divergent and corrects the physical mass.

In more general gauges, the situation is more complicated. At one loop, there are UV-finite, IR-divergent, nonlocal contributions to $\langle \Phi^+ \Phi^- \rangle$. The apparent violation

³Here $\langle DD \rangle$ is defined such that $\langle 0|TD(x)D(y)|0 \rangle_{\text{F.T.}} \equiv (2\pi)^{-4} \int d^4k \langle DD \rangle \exp[-ik \cdot (x - y)]$. Similar expressions apply to $\langle aD \rangle$ and $\langle aa \rangle$.

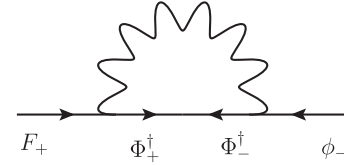


FIG. 1. One-loop contribution to the helicity-flip process.

of nonrenormalization is of the form discussed in Refs. [12–14] and attributable to the nonlocal nature of 1PI effective actions [15]. There are also corrections to $\langle \Phi^{\pm} \Phi^{\pm\dagger} \rangle$ that are both UV and IR divergent. Only suitable physical questions are expected to yield finite and gauge-invariant answers. The new feature for $\xi \neq 1$, namely the infrared divergences, arise from the $1/k^4$ term in the vector superfield propagator noted above.

To see these divergences explicitly, it is convenient to focus on two types of self-energies involving the scalar components of the electron supermultiplets: $\langle F_+ \phi_- \rangle$ corresponding to a helicity-flip process, and $\langle F_+^* F_+ \rangle$ corresponding to a helicity-preserving process.

Corrections to the $\langle F_+ \phi_- \rangle$ propagator come from the diagram shown in Fig. 1. In terms of the component fields, only a couples to $F^{\dagger} F$ and propagates along the vector line. We obtain

$$I_{F_+ \phi_-} = -e^2 m \left(1 - \frac{1}{\xi}\right) p^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4 [(p - k)^2 - m^2]}, \quad (3.2)$$

which is IR divergent and UV finite. Focusing on the small k region yields

$$(I_{F_+ \phi_-})_{\text{IR}} \sim -e^2 m \left(1 - \frac{1}{\xi}\right) \frac{p^2}{p^2 - m^2} \times \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \quad \text{for } k^2 \ll p^2. \quad (3.3)$$

At one loop we can cut off the infrared divergence at a small momentum “by hand,” or by introducing a small mass for the vector superfield. Dimensional regularization [16,17] with $d = 4 - 2\delta$ and $\delta < 0$ provides a gauge-invariant IR regulator [18].⁴ The IR-divergent part is

⁴In practice, we should employ dimensional reduction [19–21] in order to preserve the supersymmetric properties of the self-energy functions. However, dimensional reduction differs from dimensional regularization only in the treatment of the internal vector boson lines. In the computations presented in this paper, the vector boson components of internal vector superfield lines do not appear. For example, the $F_+ \Phi_+^{\dagger} V$ vertex, which appears in Figs. 1 and 2, does not involve a coupling of the vector boson field component of V . Consequently, we do not need to distinguish between dimensional regularization and dimensional reduction in this work.

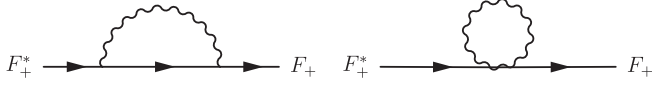


FIG. 2. One-loop contributions to the helicity-preserving process.

$$(I_{F_+ \phi_-})_{\text{IR}} = -im \frac{e^2}{16\pi^2} \left(1 - \frac{1}{\xi}\right) \frac{p^2}{p^2 - m^2} \frac{1}{\delta}. \quad (3.4)$$

The $\langle F_+^* F_+ \rangle$ propagator receives corrections from the “sunset” and “seagull” diagrams shown in Fig. 2. Both the a and M components of V propagate in the sunset diagram. The former gives rise to an IR singularity, while the latter provides a ultraviolet divergence. The Feynman integrals contributing to the sunset diagram are

$$I_{F_+^* F_+}^{\text{sun}} = -\frac{e^2}{2} \left\{ -2 \left(1 - \frac{1}{\xi}\right) \int \frac{d^4 k}{(2\pi)^4} \frac{p^2 - p \cdot k}{k^4 [(p-k)^2 - m^2]} + \frac{1}{\xi} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 [(p-k)^2 - m^2]} \right\}. \quad (3.5)$$

We isolate the IR divergence in the first integral with dimensional regularization by integrating over $d = 4 - 2\delta$ dimensions, where $\delta < 0$,

$$(I_{F_+^* F_+}^{\text{sun}})_{\text{IR}} = -i \frac{e^2}{16\pi^2} \left(1 - \frac{1}{\xi}\right) \frac{p^2}{p^2 - m^2} \frac{1}{\delta}. \quad (3.6)$$

The ultraviolet divergent part, the second integral in (3.5), may also be isolated with dimensional regularization [19–21], by taking $d = 4 - 2\epsilon$ with $\epsilon > 0$,

$$(I_{F_+^* F_+}^{\text{sun}})_{\text{UV}} = -\frac{i}{2} \frac{e^2}{16\pi^2} \frac{11}{\xi \epsilon}. \quad (3.7)$$

The a component of V propagates in the seagull loop, giving

$$I_{F_+^* F_+}^{\text{sea}} = -\frac{e^2}{2} \left(1 - \frac{1}{\xi}\right) \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4}. \quad (3.8)$$

This Feynman integral is both UV and IR divergent. Scaleless integrals may be consistently set to zero in dimensional regularization [17], so it is sometimes said that the UV and IR divergences cancel. Ultimately this property will be unimportant for our analysis. Moreover, we would like to keep these divergences separate at one loop, so we retain the ϵ, δ notation of Eqs. (3.9) and (3.10) to keep the origin of the divergences distinct.

The UV and IR divergent pieces are

$$(I_{F_+^* F_+}^{\text{sea}})_{\text{UV}} = -\frac{i}{2} \frac{e^2}{16\pi^2} \left(1 - \frac{1}{\xi}\right) \frac{1}{\epsilon}, \quad (3.9)$$

$$(I_{F_+^* F_+}^{\text{sea}})_{\text{IR}} = +\frac{i}{2} \frac{e^2}{16\pi^2} \left(1 - \frac{1}{\xi}\right) \frac{1}{\delta}. \quad (3.10)$$

We have seen that the $\Phi_+ \Phi_-$ propagator is UV finite in any R_ξ gauge. Therefore, if the physical mass is to be gauge invariant, the ultraviolet divergent pieces of the wave function renormalization must be gauge invariant [11]. This property is manifest in the sum of Eqs. (3.7) and (3.9), where terms proportional to $(1/\epsilon) \cdot (1/\xi)$ cancel.

IV. THE ELECTRON POLE MASS

In Appendix A, we review the one-loop correction to the mass of the electron in ordinary QED in the R_ξ gauges. In brief, the quadratic terms in the bare 1PI effective action have the form

$$\mathcal{L} \ni (1 + a(p)) \bar{\psi}_0 \not{p} \psi_0 - m_0 (1 + b(p)) \bar{\psi}_0 \psi_0 + \dots \quad (4.1)$$

In canonical normalization, one can define a one-loop “mass shift” for general p ,

$$\delta m(p) = m_0 [b(p) - a(p)]. \quad (4.2)$$

The ultraviolet divergence in the mass shift (giving rise to the β function for the renormalized mass parameter) is gauge invariant, with gauge-dependent terms canceling between the helicity-flip and helicity-preserving contributions to the self-energy. However, in a given renormalization scheme the finite pieces of the mass shift are only gauge invariant on shell. In the case of SQED, we might expect something similar, with gauge invariance—and now the cancellation of infrared divergences—holding only on shell.

There is an extra subtlety in SQED due to the nonlocal, $(p^2 - m^2)^{-1}$ behavior we have seen in general R_ξ gauges. If the one-loop pole mass is shifted from the tree-level mass by a power of e^2 , as in most renormalization schemes, factors of $(p^2 - m^2)^{-1}$ can spoil the naive ordering of loop corrections in powers of e^2 . (This is analogous to issues with ordering in e in finite temperature perturbation theory in ordinary gauge theories.) In the next two subsections, we examine this subtlety and the gauge invariance of the supersymmetric electron mass in greater detail.

A. Nonlocality and the SQED loop expansion

The quadratic terms in the renormalized effective action involving the scalar component fields ϕ_\pm and F_\pm may be written as

$$\mathcal{L}_{\phi F}^{\text{eff}} = \begin{pmatrix} F_+^* & \phi_- \end{pmatrix} \Delta_{F\phi}^{-1} \begin{pmatrix} F_+ \\ \phi_-^* \end{pmatrix} + (+ \leftrightarrow -), \quad (4.3)$$

where $\Delta_{F\phi}^{-1}$ is the inverse propagator matrix [cf. Eq. (B48)]. In momentum space, we can write

$$\Delta_{F\phi}^{-1} \equiv \begin{pmatrix} 1 + A(p) & m[1 + B(p)] \\ m[1 + B(p)] & p^2[1 + A(p)] \end{pmatrix}, \quad (4.4)$$

where A and B are proportional to $\overline{\text{DR}}$ -renormalized self-energies [19,20,22], the IR-divergent pieces of which were computed above. The pole mass for the multiplet is determined by solving $\det(\Delta_{F\phi}^{-1}) = 0$,

$$p^2 - m^2 = 2F(p)m^2, \quad (4.5)$$

$$F(p) \equiv \frac{1}{2} \left[\left(\frac{1 + B(p)}{1 + A(p)} \right)^2 - 1 \right].$$

F admits an expansion in powers of e^2 . At $\mathcal{O}(e^2)$, $F = (B - A)$, and A and B correspond to the UV-subtracted one-loop diagrams of Sec. III. Thus, if $F(p)$ is well behaved near $p^2 = m^2$, the physical mass receives a one-loop correction of $\mathcal{O}(e^2)$, $m_{\text{phys}} - m = mF(m)$.

However, if A or B have singularities associated with nonlocal terms, the link between loops and powers of e^2 can break down. For a simple toy example at one-loop order, take the following form for $F(p)$:

$$F(p) = e^2 \left(f_1^s \frac{p^2}{p^2 - m^2} + f_1^n \right). \quad (4.6)$$

Here we have included a singular piece with constant coefficient f_1^s and a nonsingular piece with coefficient f_1^n .

Then the leading correction to the mass is

$$m_{\text{phys}} - m = \pm em \sqrt{f_1^s/2} + \mathcal{O}(e^2). \quad (4.7)$$

We see that the singular term contributes to the mass with one less power of e than the nonsingular term. Likewise, it is easy to see that two-loop contributions to $F(p)$ proportional to the same nonlocal singularity can contribute at $\mathcal{O}(e^2)$, the same as one-loop nonsingular terms.

The ambiguity in the sign in Eq. (4.7) can be resolved only if cancellations between $A(p)$ and $B(p)$ are such that $f_1^s = 0$, in which case the mass is not actually corrected at $\mathcal{O}(e)$. Indeed, singularities like those in this toy example appeared in the computation above of the supersymmetric electron self-energies, and in SQED we expect the $\mathcal{O}(e)$ terms in the electron pole mass to cancel for other reasons: the singularities are associated with unphysical, gauge-dependent, IR-divergent terms, and $\mathcal{O}(e)$ corrections are not present in Feynman gauge. The lessons we learn are the following:

- (1) The leading-order cancellation will take place only between one-loop graphs with nonlocal singularities.
- (2) At higher orders in e , singularities must cancel between different loop orders.

B. Cancellation of $\mathcal{O}(e)$ terms in m_{phys}

In the previous subsection, we saw that the appearance of nonlocal singularities in the supersymmetric electron self-energies, combined with the requirement of gauge invariance, implies the existence of cancellations between contributions at different loop orders. In Appendix C, we verify the exact cancellation for the terms at $\mathcal{O}(e)$, arising from the helicity-flip diagram and the helicity-preserving sunset diagram, both of which have singularities as p^2 goes on shell. Here, for brevity, we show only the cancellation of the IR-divergent pieces at $\mathcal{O}(e)$ arising from those diagrams.

From the results of Sec. III, we have

$$A(p) = -i \frac{e^2}{16\pi^2} \left(1 - \frac{1}{\xi} \right) \frac{p^2}{p^2 - m^2} \frac{1}{\delta} + \text{finite}, \quad (4.8)$$

$$B(p) = -i \frac{e^2}{16\pi^2} \left(1 - \frac{1}{\xi} \right) \frac{p^2}{p^2 - m^2} \frac{1}{\delta} + \text{finite}, \quad (4.9)$$

where A and B are defined in Eq. (4.4). Consistent with our discussion in the previous subsection, we have neglected the seagull diagram in B . The seagull contributions are nonsingular and contribute to the pole mass only at $\mathcal{O}(e^2)$. We see that the gauge-dependent IR divergences cancel in the combination $B - A$ appearing in the pole mass.

Although it enters at $\mathcal{O}(e^2)$, there is an unphysical IR divergence in the seagull diagram, which must be cancelled by a two-loop contribution to $F(p^2)$ proportional to $g^4(p^2 - m^2)^{-1}$. At two-loop order there are also double IR divergences associated with graphs with two vector superfield propagators. We expect the complete structure of cancellations to be quite intricate.

V. INTEGRATING OUT MASSIVE CHARGED FIELDS

The real power of the nonrenormalization theorems arises in situations where a Wilsonian effective action is useful. It is interesting to see how the gauge artifacts discussed above, and in particular the infrared divergences for $\xi \neq 1$, cancel when massive fields are integrated out to obtain a low energy effective action for a set of light fields.

A simple example is generated by adding a light neutral field to the massive SQED theory, with superpotential

$$W = m\Phi_+\Phi_- + \lambda\Phi_0\Phi_+\Phi_- + \lambda'\Phi_0^3. \quad (5.1)$$

Integrating out the massive Φ_{\pm} , we obtain an effective action for Φ_0 . The standard nonrenormalization theorem analysis here would indicate that the only corrections to λ arise from wave function renormalization. In this theory, it is easy to check that there are no low order corrections to the 1PI Φ_0^3 three-point function (this can be done with supergraphs, or in components, looking for an $F_0 A_0 A_0$ 1PI Green's function). This is a consequence of a holomorphy-type argument [15], treating λ as a spurion and assigning it an R charge.

There *should* be a renormalization of λ proportional to the wave function renormalization of Φ_0 . It should be gauge invariant, and free of infrared divergences and other pathologies. Gauge fields enter the wave function renormalization at two loops. While the full two-loop computation is complicated, the leading infrared divergent pieces of individual Feynman diagrams are easily isolated. There are many diagrams, but only a small set which are both infrared and ultraviolet divergent, and we examine these for illustration. In particular, diagrams which include helicity-flip (i.e. $\langle \Phi_+ \Phi_- \rangle$) propagators are ultraviolet finite, as they come with a positive power of m . This leaves five diagrams, shown in Fig. 3.

To see the cancellation of the gauge-dependent pieces, we work at zero external momentum and isolate the leading infrared and ultraviolet divergent piece. Consider the first of these diagrams. It is particularly simple to compute the $F_0 F_0^\dagger$ component propagator. The diagram is given by (we can now safely Wick rotate to Euclidean space)

$$\int \frac{d^4 p}{(2\pi^4)} \frac{d^4 k}{(2\pi^4)} \frac{p^2}{(p^2 + m^2)^3} \frac{p^2}{((p+k)^2 + m^2)} \frac{1}{k^4}. \quad (5.2)$$

The most singular part of this diagram in the infrared arises from the propagation of the lowest scalar component of the internal vector superfield and behaves as

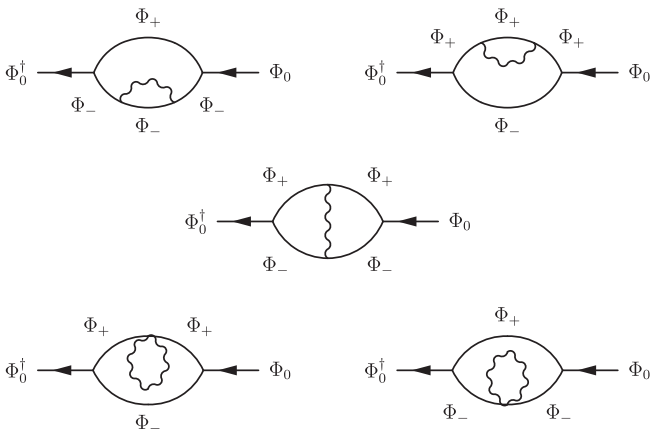


FIG. 3. Diagrams contributing to the Φ_0 effective Lagrangian at two loops.

$$\int \frac{d^4 p}{(2\pi^4)} \frac{d^4 k}{(2\pi^4)} \frac{1}{k^4} \frac{p^4}{(p^2 + m^2)^4}. \quad (5.3)$$

This expression diverges for small k and the k integral should be thought of as cut off at $|p|$. The remaining integral over p is UV divergent. For large p , the integral takes the form

$$\int_m^{|\Lambda|} \frac{d^4 p}{p^4} \int \frac{|p| d^4 k}{k^4}. \quad (5.4)$$

In the limit of small k and large Λ , all of the integrals take this form, up to constants.

To see the cancellation, then, we need only to determine the relative weights of these diagrams. The first three diagrams have the same overall weight, but the third has a sign opposite to the first two due to the opposite charges of Φ_+ and Φ_- . The fourth and fifth diagrams contain an extra factor of $1/2$, arising from the expansion of the exponential in e^{2eV} to second order, and an extra minus sign because there is one less propagator and one less vertex. As a result, the sum is of the form $1 + 1 - 1 - 1/2 - 1/2 = 0$, and the leading IR divergences cancel in the effective action.

VI. INFRARED FINITE PERTURBATION THEORY

IR divergences arise from the lowest component of the vector superfield. We could avoid the whole issue of IR divergences at one loop by choosing $\xi = 1$ for our computations, as in Ref. [1]. However, as noted in Ref. [7], even working in Feynman gauge, infrared divergences are still encountered at higher order. In terms of component fields, the problem is that with $\xi = 1$ there is still an $\langle aD \rangle$ propagator, proportional to $1/k^2$. The 1PI $\langle DD \rangle$ two-point function is nonvanishing (and UV divergent) at zero momentum, and together with $\langle aD \rangle$, gives rise to a one-loop $1/k^4$ propagator for a through diagrams like Fig. 4. This reintroduction of $1/k^4$ can be dealt with by adjusting the gauge condition order by order to cancel it off. For example, at one loop, the $\langle aa \rangle$ propagator becomes

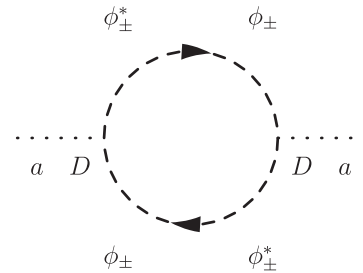


FIG. 4. Correction to the $\langle aa \rangle$ two-point function.

$$\begin{aligned} \delta\langle aa \rangle &= e^2 \frac{1}{2} \left(-\frac{2}{k^2 \xi} \right)^2 \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2} \frac{i}{(k - q)^2 - m^2} \\ &= \frac{8}{k^4} \frac{1}{\xi^2} \frac{e^2}{16\pi^2} \frac{1}{\epsilon} + \dots \end{aligned} \quad (6.1)$$

After renormalization the $1/\epsilon$ is replaced by $\log(\mu/m)$ at small k .

Letting $\xi = 1 + \delta\xi$, we can solve for $\delta\xi$ such that the $1/k^4$ term in the corrected $\langle aa \rangle$ propagator vanishes,

$$\delta\xi = 8e^2 \frac{1}{(4\pi)^2} \log(\mu/m) + \mathcal{O}(g^4). \quad (6.2)$$

This procedure may be iterated order by order in e .

Reference [7] considered non-Abelian theories, in which some one-loop corrections to the vacuum polarization are always proportional to $\log(k^2)/k^4$ for small external momentum k . This new k behavior caused further complications in the removal of IR divergences, in particular necessitating the introduction of new nonlocal gauge fixing terms. In the massive Abelian theory, all logarithms are cut in the infrared by m , corresponding to the decoupling of all charged matter and the IR freedom of the gauge coupling. Therefore we have only to cancel the $1/k^4$ behavior, as above.

VII. CONCLUSIONS

It is not surprising that there are infrared issues in perturbation theory in supersymmetric gauge theories in general gauges. From dimensional analysis alone, it follows that

$$\langle aa \rangle = \frac{C}{k^4}, \quad (7.1)$$

so to avoid infrared divergences, it is necessary that the $\langle aa \rangle$ two-point function vanish. This degree of freedom is a gauge artifact; from the simple existence of Wess-Zumino gauge [2], one expects that there can be no physical effect.

Still, we rely on covariant gauges, and particularly on the manifest supersymmetry, locality (and infrared finiteness) of the effective action, to make important statements, including proofs of nonrenormalization theorems. We have seen here that in situations in which one can integrate out massive fields, so as to obtain a Wilsonian action for light fields, infrared divergences and nonlocality cancel. In discussions of 1PI actions, it is important to consider physical questions, like the pole masses of stable particles. We have also explained how one may choose a gauge, order by order, so that infrared divergences cancel.

ACKNOWLEDGMENTS

This work was initially inspired by a question on supersymmetric gauge theories raised by John Terning.

Subsequent conversations with John Terning, Stephen Martin, and Tim Jones are gratefully acknowledged. P.D. is supported in part by the National Science Foundation Grant No. PHY13-16748. M.D., H.E.H., and L.S.H. are supported in part by U.S. Department of Energy Grant No. DE-SC0010107. Some aspects of this work were carried out at the Kavli Institute for Theoretical Physics in Santa Barbara, CA, and supported in part by the National Science Foundation under Grant No. NSF PHY11-25915.

APPENDIX A: GAUGE DEPENDENCE, WAVE FUNCTION, AND MASS RENORMALIZATION IN NONSUPERSYMMETRIC QED

In this appendix we review how gauge dependence appears in the electron mass renormalization in nonsupersymmetric QED.

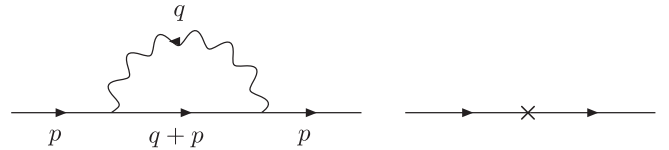
In covariant gauges, where the photon propagator is given by⁵

$$D_{\mu\nu} = -\frac{1}{k^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} (1 - \hat{a}) \right), \quad (A1)$$

we compute the 1PI electron two-point function in momentum space,

$$i\Gamma^{(2)}(p) = i(\not{p} - m) - i\Sigma(p), \quad (A2)$$

where p is the four-momentum of the electron. Here we denote the sum of the loop contributions to $i\Gamma^{(2)}(p)$ by $-i\Sigma(p)$. At one loop, the two contributing Feynman graphs are



The cross indicates the contribution of the terms $i\delta Z_2 \bar{\psi} \partial \psi - (\delta Z_m + \delta Z_2) m \bar{\psi} \psi$ of the counterterm Lagrangian, where δZ_m and δZ_2 are defined such that

$$\begin{aligned} \psi &= (1 + \delta Z_2)^{-1/2} \psi_B, \\ m &= (1 + \delta Z_m)^{-1} m_B \end{aligned} \quad (A3)$$

(with subscript B denoting bare quantities and absence thereof denoting renormalized quantities). At one loop,

⁵To make contact with the notation of Eq. (1.1), we note that $\hat{a} \equiv \xi^{-1}$. In this appendix, we prefer to employ the gauge parameter \hat{a} in order to follow the standard textbook notation employed in the treatment of QED field theory [23].

$$\begin{aligned}
-i\Sigma(p) &= (i\mu^\epsilon e)^2 \int \frac{d^n q}{(2\pi)^n} \frac{\gamma^\nu (\not{q} + \not{p} + m)\gamma^\mu}{q^2 [(q+p)^2 - m^2]} \\
&\quad \times \left(g_{\mu\nu} - (1 - \hat{a}) \frac{q_\mu q_\nu}{q^2} \right) + i\delta Z_2 \not{p} \\
&\quad - im(\delta Z_m + \delta Z_2). \tag{A4}
\end{aligned}$$

Performing the integrals, we obtain

$$\Sigma(p) = -\not{p}A(p) + mB(p), \tag{A5}$$

where

$$A(p^2) = \delta Z_2 + \frac{\alpha \hat{a}}{4\pi} (4\pi)^\epsilon \Gamma(\epsilon) + \frac{\alpha \hat{a}}{4\pi} \left\{ \left(1 + \frac{m^2}{p^2}\right) \left[1 - \left(1 - \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right)\right] - \ln\left(\frac{m^2}{\mu^2}\right) \right\} + \mathcal{O}(\epsilon), \tag{A6}$$

$$B(p^2) = \delta Z_m + \delta Z_2 + \frac{\alpha}{4\pi} (3 + \hat{a})(4\pi)^\epsilon \Gamma(\epsilon) + \frac{\alpha}{2\pi} \left\{ 2 + \hat{a} - \frac{1}{2}(3 + \hat{a}) \left[\left(1 - \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) + \ln\left(\frac{m^2}{\mu^2}\right) \right] \right\} + \mathcal{O}(\epsilon), \tag{A7}$$

and $\alpha \equiv e^2/(4\pi)$.

1. $\overline{\text{MS}}$ Renormalization

In the modified minimal subtraction scheme [16,24], the counterterms are

$$\begin{aligned}
\delta Z_2^{\overline{\text{MS}}} &= -\frac{\alpha \hat{a}}{4\pi} (4\pi)^\epsilon \Gamma(\epsilon), \\
\delta Z_m^{\overline{\text{MS}}} &= -\frac{3\alpha}{4\pi} (4\pi)^\epsilon \Gamma(\epsilon). \tag{A8}
\end{aligned}$$

Note that δZ_2 is gauge dependent, whereas δZ_m is gauge independent. Plugging the counterterms into (A6) and (A7),

$$\begin{aligned}
A(p^2)^{\overline{\text{MS}}} &= \frac{\alpha \hat{a}}{4\pi} \left\{ \left(1 + \frac{m^2}{p^2}\right) \left[1 - \left(1 - \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right)\right] - \ln\left(\frac{m^2}{\mu^2}\right) \right\}, \\
B(p^2)^{\overline{\text{MS}}} &= \frac{\alpha}{2\pi} \left\{ 2 + \hat{a} - \frac{1}{2}(3 + \hat{a}) \left[\left(1 - \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) + \ln\left(\frac{m^2}{\mu^2}\right) \right] \right\}. \tag{A9}
\end{aligned}$$

The physical pole mass, denoted by m_e , corresponds to a zero of the inverse propagator,

$$\Gamma^{(2)}(p^2)|_{p=m_e} = 0. \tag{A10}$$

At one-loop order, $\Gamma^{(2)}(p^2)$ is proportional to $\not{p} - m(1 + B(p^2)^{\overline{\text{MS}}} - A(p^2)^{\overline{\text{MS}}})$. Off shell, the quantity $B(p^2) - A(p^2)$ depends on the gauge parameter,

$$B(p^2)^{\overline{\text{MS}}} - A(p^2)^{\overline{\text{MS}}} = \frac{\alpha}{4\pi} \left\{ 4 + \hat{a} \left(1 - \frac{m^2}{p^2}\right) - 3 \ln\left(\frac{m^2 - p^2}{\mu^2}\right) + \frac{m^2}{p^2} \left[3 + \hat{a} \left(1 - \frac{m^2}{p^2}\right) \right] \ln\left(1 - \frac{p^2}{m^2}\right) \right\}. \tag{A11}$$

The electron pole mass, however, depends on $B - A$ on shell,

$$m_e = m[1 + B(m_e^2)^{\overline{\text{MS}}} - A(m_e^2)^{\overline{\text{MS}}}] . \tag{A12}$$

One can easily check that $B(m^2) - A(m^2)$ is independent of \hat{a} , demonstrating the gauge invariance of the pole mass through one-loop order. Indeed, the pole mass must be IR finite and independent of the gauge parameter \hat{a} to all orders in perturbation theory [25].

2. On-shell renormalization

It is also instructive to use the on-shell (OS) subtraction scheme, where the parameter m is identified as the pole mass. Here a well-known IR divergence appears in the electron wave function counterterm. This divergence is unrelated to the IR divergences in supersymmetric QED analyzed earlier in this paper, appearing only as an artifact of the OS renormalization scheme, but it is interesting to see how it—and gauge dependence—appear in the self-energy. Writing

$$\Sigma(p) = \Sigma(m) + (\not{p} - m)\Sigma'(m) + \mathcal{O}((\not{p} - m)^2), \quad (\text{A13})$$

the OS renormalization conditions are

$$\Sigma(m)^{\text{OS}} = 0, \quad \Sigma'(m)^{\text{OS}} = 0. \quad (\text{A14})$$

It then follows that the inverse propagator can be written as

$$\begin{aligned} \Gamma^{(2)}(p)^{\text{OS}} &= \not{p} - m - \Sigma(p)^{\text{OS}} \\ &= [1 + \Sigma'(m)^{\text{OS}}](\not{p} - m) \\ &\quad - \Sigma(m)^{\text{OS}} + \mathcal{O}((\not{p} - m)^2) \\ &= \not{p} - m + \mathcal{O}((\not{p} - m)^2). \end{aligned}$$

Employing Eq. (A5), we can rewrite the boundary conditions specified in Eq. (A14) as

$$\begin{aligned} A(m^2)^{\text{OS}} &= B(m^2)^{\text{OS}}, \\ A(m^2)^{\text{OS}} &= 2m^2 \left[\left(\frac{\partial B^{\text{OS}}}{\partial p^2} \right) - \left(\frac{\partial A^{\text{OS}}}{\partial p^2} \right) \right]_{p^2=m^2}. \quad (\text{A15}) \end{aligned}$$

Using the first boundary condition and Eqs. (A6) and (A7), we conclude that

$$\begin{aligned} \delta Z_m^{\text{OS}} &= -\frac{\alpha}{4\pi} (4\pi)^\epsilon \Gamma(\epsilon) \left(\frac{m^2}{\mu^2} \right)^{-\epsilon} \left(\frac{3-2\epsilon}{1-2\epsilon} \right) \\ &= -\frac{3\alpha}{4\pi} \left[(4\pi)^\epsilon \Gamma(\epsilon) + \frac{4}{3} - \ln \left(\frac{m^2}{\mu^2} \right) \right], \end{aligned}$$

after dropping terms of $\mathcal{O}(\epsilon)$. Similarly, δZ_2^{OS} may be obtained from the second boundary condition,

$$\begin{aligned} \delta Z_2^{\text{OS}} &= -\frac{\alpha \hat{a}}{4\pi} \left(\frac{m^2}{\mu^2} \right)^{-\epsilon} \frac{(4\pi)^\epsilon \Gamma(\epsilon)}{1-2\epsilon} \\ &\quad + \frac{\alpha}{4\pi} \left(\frac{m^2}{\mu^2} \right)^{-\epsilon} \frac{(4\pi)^\epsilon \Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)} [\hat{a} - 3 + 2\epsilon]. \quad (\text{A16}) \end{aligned}$$

The term on the right hand side of Eq. (A16) proportional to $\Gamma(\epsilon)$ reflects the ultraviolet divergence in the unregulated self-energy integral [cf. Eq. (A8)]. The last term on the right hand side of Eq. (A16) which contains a pole at $\epsilon = 0$ reflects a new infrared divergence, an artifact of the OS scheme choice. (Note that this one-loop infrared divergence is absent in the Yennie gauge [26–31], $\hat{a} = 3$.)

We can determine $A(p^2)$ and $B(p^2)$ in the on-shell scheme by writing

$$A(p^2)_{\text{OS}} = A(p^2)_{\overline{\text{MS}}} + \delta Z_2^{\text{OS}} - \delta Z_2^{\overline{\text{MS}}}, \quad (\text{A17})$$

$$B(p^2)_{\text{OS}} = B(p^2)_{\overline{\text{MS}}} + \delta Z_m^{\text{OS}} + \delta Z_2^{\text{OS}} - \delta Z_m^{\overline{\text{MS}}} - \delta Z_2^{\overline{\text{MS}}}. \quad (\text{A18})$$

Equations (A8), (A16), and (A16) yield

$$\begin{aligned} \delta Z_2^{\text{OS}} - \delta Z_2^{\overline{\text{MS}}} &= -\frac{\alpha \hat{a}}{2\pi} \left[1 - \frac{1}{2} \ln \left(\frac{m^2}{\mu^2} \right) \right] \\ &\quad + \frac{\alpha}{4\pi} \left(\frac{m^2}{\mu^2} \right)^{-\epsilon} \frac{(4\pi)^\epsilon \Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)} [\hat{a} - 3 + 2\epsilon], \quad (\text{A19}) \end{aligned}$$

$$\delta Z_m^{\text{OS}} - \delta Z_m^{\overline{\text{MS}}} = -\frac{\alpha}{\pi} \left[1 - \frac{3}{4} \ln \left(\frac{m^2}{\mu^2} \right) \right]. \quad (\text{A20})$$

The infrared divergence is explicitly exhibited in Eq. (A19). Expanding about $\epsilon = 0$ yields

$$\delta Z_2^{\text{OS}} - \delta Z_2^{\overline{\text{MS}}} = \frac{\alpha(\hat{a}-3)}{4\pi} (4\pi)^\epsilon \Gamma(\epsilon) - \frac{\alpha}{\pi} \left[1 - \frac{3}{4} \ln \left(\frac{m^2}{\mu^2} \right) \right]. \quad (\text{A21})$$

Thus, both $A(p^2)_{\text{OS}}$ and $B(p^2)_{\text{OS}}$ are infrared divergent if $\hat{a} \neq 3$. Note that the difference $B(p^2)_{\text{OS}} - A(p^2)_{\text{OS}}$ is infrared finite. As in the $\overline{\text{MS}}$ scheme, $B(p^2)_{\text{OS}} - A(p^2)_{\text{OS}}$ depends on the gauge parameter \hat{a} for general p^2 , but becomes gauge invariant on shell (vanishing by construction).

APPENDIX B: THE TWO-POINT FUNCTIONS OF SQED

1. Tree-level propagators of SQED in a covariant gauge

In this subsection, we focus on the terms of the SUSY-QED Lagrangian that are independent of the chiral superfields Φ_\pm . These terms are given by the last two terms of Eq. (1.1), which can also be written in the following form:

$$\begin{aligned} \mathcal{L}_{\text{SQED}} &= \frac{1}{4} [\mathcal{W}^\alpha \mathcal{W}_\alpha]_{\theta\theta} + \frac{1}{4} [\bar{\mathcal{V}}_\alpha \bar{\mathcal{V}}^{\dot{\alpha}}]_{\bar{\theta}\bar{\theta}} \\ &\quad - \frac{1}{8} \xi [(D^2 V)(\bar{D}^2 V)]_{\theta\theta\bar{\theta}\bar{\theta}}, \quad (\text{B1}) \end{aligned}$$

where the subscript $\theta\theta$ instructs one to take the coefficient of $\theta\theta$ of the corresponding superfield, etc., and the spinor chiral superfield \mathcal{W}_α is defined by

$$\begin{aligned} \mathcal{W}_\alpha(x, \theta, \bar{\theta}) &= -\frac{1}{4} \bar{D}^2 D_\alpha V(x, \theta, \bar{\theta}) \\ &= \exp(-i\theta\sigma^\mu\bar{\theta}\partial_\mu) \left\{ -i\lambda_\alpha(x) + \theta_\alpha D(x) \right. \\ &\quad \left. - \frac{1}{2} i(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu}(x) - \theta\theta[\sigma^\mu\partial_\mu\bar{\lambda}(x)]_\alpha \right\}, \quad (\text{B2}) \end{aligned}$$

where $F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu$.

The two-component spinor notation employed in this paper follows that of Refs. [5,32]. Following Ref. [5], the spinor covariant derivatives are given by

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}} \partial_\mu, \quad (\text{B3})$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i\theta^\beta \sigma_{\beta\dot{\alpha}}^\mu \partial_\mu. \quad (\text{B4})$$

Using these definitions,

$$D^2 \equiv D^\alpha D_\alpha = \epsilon^{\alpha\beta} D_\beta D_\alpha = -\partial^\alpha \partial_\alpha + 2i\bar{\theta}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\beta} \partial_\beta \partial_\mu + \bar{\theta} \bar{\theta} \square, \quad (\text{B5})$$

$$\bar{D}^2 \equiv \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}_{\dot{\beta}} \bar{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} \partial^{\dot{\alpha}} + 2i\theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}} \partial_\mu + \theta \theta \square, \quad (\text{B6})$$

where $\square \equiv \partial_\mu \partial^\mu$.

Hence, the super-QED Lagrangian including the gauge fixing term (after dropping a total derivative) is⁶

$$\begin{aligned} \mathcal{L}_{\text{SQED}} = & -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \xi (\partial_\mu V^\mu)^2 + \frac{1}{2} (1 - \xi) D^2 \\ & - \frac{1}{2} \xi [(\square a)^2 - 2D\square a + (\partial_\mu M)^2 + (\partial_\mu N)^2] \\ & + i(1 - \xi) \bar{\lambda} \bar{\sigma}^\mu \partial_\mu \lambda - \xi [i\partial_\mu \bar{\chi} \bar{\sigma}^\mu \square \chi + \lambda \square \chi + \bar{\lambda} \square \bar{\chi}]. \end{aligned} \quad (\text{B7})$$

To compute the tree-level propagators $\langle aa \rangle$, $\langle aD \rangle$, and $\langle DD \rangle$, we write

$$\begin{aligned} \mathcal{L}_{\text{SQED}} \ni & \frac{1}{2} (1 - \xi) D^2 - \frac{1}{2} \xi [(\square a)^2 - 2D\square a] \\ = & \frac{1}{2} (a \quad D) \begin{pmatrix} -\xi \square^2 & \xi \square \\ \xi \square & 1 - \xi \end{pmatrix} \begin{pmatrix} a \\ D \end{pmatrix}. \end{aligned} \quad (\text{B8})$$

We compute the inverse,

$$\begin{pmatrix} -\xi \square^2 & \xi \square \\ \xi \square & 1 - \xi \end{pmatrix}^{-1} = \begin{pmatrix} (1 - \xi^{-1}) \square^{-2} & \square^{-1} \\ \square^{-1} & 1 \end{pmatrix}. \quad (\text{B9})$$

We can also work in momentum space by taking $\partial_\mu \rightarrow -ik_\mu$. It then follows that the momentum space propagator matrix is

$$i\Delta(k) = i \begin{pmatrix} (1 - \xi^{-1})/k^4 & -1/k^2 \\ -1/k^2 & 1 \end{pmatrix}. \quad (\text{B10})$$

Hence,

$$\langle aa \rangle = i(1 - \xi^{-1})/k^4, \quad \langle aD \rangle = -i/k^2, \quad \langle DD \rangle = i. \quad (\text{B11})$$

The tree-level fermionic propagators are obtained by writing

$$\mathcal{L}_{\text{SQED}} \ni \frac{1}{2} (\lambda \quad \bar{\lambda} \quad \chi \quad \bar{\chi}) \begin{pmatrix} 0 & i(1 - \xi) \sigma^\mu \partial_\mu & -\xi \square & 0 \\ i(1 - \xi) \bar{\sigma}^\mu \partial_\mu & 0 & 0 & -\xi \square \\ -\xi \square & 0 & 0 & i\xi \sigma^\mu \partial_\mu \square \\ 0 & -\xi & i\xi \bar{\sigma}^\mu \partial_\mu \square & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \bar{\lambda} \\ \chi \\ \bar{\chi} \end{pmatrix}, \quad (\text{B12})$$

which differs from the fermionic part of Eq. (B7) by a total derivative which is subsequently dropped. The inverse of the matrix that appears in Eq. (B12) is

$$-\frac{1}{\square^2} \begin{pmatrix} 0 & i\sigma^\mu \partial_\mu \square & \square & 0 \\ i\bar{\sigma}^\mu \partial_\mu \square & 0 & 0 & \square \\ \square & 0 & 0 & i(\xi^{-1} - 1) \sigma^\mu \partial_\mu \\ 0 & \xi^{-1} \square & i(\xi^{-1} - 1) \bar{\sigma}^\mu \partial_\mu & 0 \end{pmatrix}. \quad (\text{B13})$$

We can now read off the propagator matrix in momentum space by taking $\partial_\mu \rightarrow -ik_\mu$,

⁶The explicit form of the bosonic part of $\mathcal{L}_{\text{SQED}}$ in the R_ξ gauge can be found in Ref. [4].

$$i\Delta(k) = \frac{i}{k^2} \begin{pmatrix} 0 & \sigma \cdot k & 1 & 0 \\ \bar{\sigma} \cdot k & 0 & 0 & 1 \\ 1 & 0 & 0 & (1 - \xi^{-1})\sigma \cdot k/k^2 \\ 0 & 1 & (1 - \xi^{-1})\bar{\sigma} \cdot k/k^2 & 0 \end{pmatrix}. \quad (\text{B14})$$

It follows that

$$\langle \lambda \bar{\lambda} \rangle = \frac{i\sigma \cdot k}{k^2}, \quad \langle \bar{\lambda} \lambda \rangle = \frac{i\bar{\sigma} \cdot k}{k^2}, \quad (\text{B15})$$

$$\begin{aligned} \langle \chi \bar{\chi} \rangle &= \frac{i(1 - \xi^{-1})\sigma \cdot k}{k^4}, \\ \langle \bar{\chi} \chi \rangle &= \frac{i(1 - \xi^{-1})\bar{\sigma} \cdot k}{k^4}, \end{aligned} \quad (\text{B16})$$

$$\langle \lambda \chi \rangle = \langle \chi \lambda \rangle = \langle \bar{\lambda} \bar{\chi} \rangle = \langle \bar{\chi} \bar{\lambda} \rangle = \frac{i}{k^2}, \quad (\text{B17})$$

$$\langle \lambda \lambda \rangle = \langle \bar{\lambda} \bar{\lambda} \rangle = \langle \chi \chi \rangle = \langle \bar{\chi} \bar{\chi} \rangle = \langle \lambda \bar{\chi} \rangle = \langle \chi \bar{\lambda} \rangle = \langle \bar{\lambda} \chi \rangle = \langle \bar{\chi} \lambda \rangle = 0. \quad (\text{B18})$$

Note that the propagators for the gauginos (λ and $\bar{\lambda}$) are standard fermionic propagators for massless two-component fermions [32].

Finally, the inverse of the terms quadratic in the vector boson fields is the well-known QED expression,

$$\frac{1}{\square} \left(g_{\mu\nu} - (1 - \xi^{-1}) \frac{\partial_\mu \partial_\nu}{\square} \right). \quad (\text{B19})$$

That is, in momentum space, we obtain the standard tree-level photon propagator in a covariant gauge,

$$\langle V_\mu V_\nu \rangle = \frac{i}{k^2} \left(-g_{\mu\nu} + (1 - \xi^{-1}) \frac{k_\mu k_\nu}{k^2} \right). \quad (\text{B20})$$

The tree-level propagators can be obtained directly from a single master formula written in terms of the vector superfield,⁷

$$\begin{aligned} &\langle V(x, \theta, \bar{\theta}) V(y, \zeta, \bar{\zeta}) \rangle \\ &= \frac{i}{\square} \exp[i(\theta \sigma^\mu \bar{\zeta} - \zeta \sigma^\mu \bar{\theta}) \partial_\mu] \\ &\quad \times \left\{ \frac{1 - \xi^{-1}}{\square} + \frac{1}{4} (1 + \xi^{-1}) \delta^4(\theta - \zeta) \right\} \delta^4(x - y), \end{aligned} \quad (\text{B21})$$

where

⁷In this notation for the propagator, the time ordered product symbol T is suppressed.

$$\delta^4(\theta - \zeta) \equiv (\theta - \zeta)^\alpha (\theta - \zeta)_\alpha (\bar{\theta} - \bar{\zeta})_\beta (\bar{\theta} - \bar{\zeta})^\beta. \quad (\text{B22})$$

In momentum space, Eq. (B21) yields

$$\begin{aligned} &\langle V(\theta, \bar{\theta}) V(\zeta, \bar{\zeta}) \rangle \\ &= \frac{i}{k^2} \exp[\theta \sigma \cdot k \bar{\zeta} - \zeta \sigma \cdot k \bar{\theta}] \\ &\quad \times \left\{ \frac{1 - \xi^{-1}}{k^2} - \frac{1}{4} (1 + \xi^{-1}) \delta^4(\theta - \zeta) \right\}, \end{aligned} \quad (\text{B23})$$

which is the result quoted in Eq. (1.2). It is straightforward to check that Eq. (B21) reproduces the tree-level propagators of the component fields obtained above.

The renormalization of SQED coupled to matter is highly nontrivial, in light of the fact that the supersymmetric gauge-invariant Lagrangian is inherently nonlinear. Supersymmetric procedures for the renormalization of gauge theories (that do not impose the Wess-Zumino gauge) have been proposed in Ref. [33].

2. Relations among the SQED two-point functions

Consider a chiral supermultiplet,

$$\Phi(x, \theta, \bar{\theta}) = \exp(-i\theta \sigma^\mu \bar{\theta} \partial_\mu) [\phi(x) + \sqrt{2}\theta \psi(x) + \theta \theta F(x)]. \quad (\text{B24})$$

The component fields transform as

$$\begin{aligned} \delta_\eta \phi &= \sqrt{2}\eta \psi, \\ \delta_\eta \psi_\alpha &= -i\sqrt{2}(\sigma^\mu \bar{\eta})_\alpha \partial_\mu \phi + \sqrt{2}\eta_\alpha F, \\ \delta_\eta F &= -i\sqrt{2}\bar{\eta} \bar{\sigma}^\mu \partial_\mu \psi, \end{aligned} \quad (\text{B25})$$

where η and $\bar{\eta}$ are anticommuting parameters. By Hermitian conjugation,

$$\begin{aligned} \delta_\eta \phi^* &= \sqrt{2}\bar{\eta} \bar{\psi}, \\ \delta_\eta \bar{\psi}_{\dot{\alpha}} &= i\sqrt{2}(\eta \sigma^\mu)_{\dot{\alpha}} \partial_\mu \phi^* + \sqrt{2}\bar{\eta}_{\dot{\alpha}} F^*, \\ \delta_\eta F^* &= i\sqrt{2}(\partial_\mu \bar{\psi}) \bar{\sigma}^\mu \eta. \end{aligned} \quad (\text{B26})$$

The transformed fields $A = \phi, \psi$ or F (or their corresponding complex conjugated fields) can be expressed in terms of the commutators

$$\delta_\eta A(x) = i[\eta Q + \bar{\eta} \bar{Q}, A(x)], \quad (\text{B27})$$

where Q and \bar{Q} generate supersymmetric translations.

Consider first the identity

$$i\langle 0 | [\eta Q + \bar{\eta} \bar{Q}, \psi_\alpha(x) \phi(y)] | 0 \rangle = 0, \quad (\text{B28})$$

which follows under the assumption that the supersymmetry generators annihilate the vacuum (i.e., supersymmetry is an unbroken symmetry). In light of Eq. (B27), we obtain

$$\begin{aligned} 0 &= \langle 0 | \delta_\eta [\psi_\alpha(x) \phi(y)] | 0 \rangle \\ &= \langle 0 | [\psi_\alpha(x) + \delta_\eta \psi_\alpha(x)] [\phi(y) + \delta_\eta \phi(y)] - \psi_\alpha(x) \phi(y) | 0 \rangle \\ &= \langle 0 | [\delta_\eta \psi_\alpha(x)] \phi(y) + \psi_\alpha(x) [\delta_\eta \phi(y)] | 0 \rangle. \end{aligned} \quad (\text{B29})$$

Plugging in the transformation laws given above,

$$\begin{aligned} \eta^\beta \{ \epsilon_{\alpha\beta} \langle 0 | F(x) \phi(y) | 0 \rangle - \langle 0 | \psi_\alpha(x) \psi_\beta(y) | 0 \rangle \} \\ - i \bar{\eta}^{\dot{\beta}} \sigma_{\alpha\dot{\beta}}^\mu \partial_\mu^x \langle 0 | \phi(x) \phi(y) | 0 \rangle = 0, \end{aligned} \quad (\text{B30})$$

where $\partial_\mu^x \equiv \partial / \partial x^\mu$. The coefficients of η and $\bar{\eta}$ must separately vanish. Thus, we conclude that⁸

$$\langle 0 | \psi_\alpha(x) \psi_\beta(y) | 0 \rangle = \epsilon_{\alpha\beta} \langle 0 | F(x) \phi(y) | 0 \rangle. \quad (\text{B31})$$

Similarly, the identity,

$$i\langle 0 | [\eta Q + \bar{\eta} \bar{Q}, \phi^*(x) \bar{\psi}_{\dot{\beta}}(x)] | 0 \rangle = 0, \quad (\text{B32})$$

yields

$$\langle 0 | \bar{\psi}_{\dot{\alpha}}(x) \bar{\psi}_{\dot{\beta}}(y) | 0 \rangle = -\epsilon_{\dot{\alpha}\dot{\beta}} \langle 0 | \phi^*(x) F^*(y) | 0 \rangle. \quad (\text{B33})$$

Next, we consider the identity,

$$i\langle 0 | [\eta Q + \bar{\eta} \bar{Q}, \psi_\alpha(x) \phi^*(y)] | 0 \rangle = 0. \quad (\text{B34})$$

A similar computation yields

$$\begin{aligned} \bar{\eta}^{\dot{\beta}} \{ -i \sigma_{\alpha\dot{\beta}}^\mu \partial_\mu \langle 0 | \phi(x) \phi^*(y) | 0 \rangle + \langle 0 | \psi_\alpha(x) \bar{\psi}_{\dot{\beta}}(y) | 0 \rangle \} \\ + \eta_\alpha \langle 0 | F(x) \phi^*(y) | 0 \rangle. \end{aligned} \quad (\text{B35})$$

It follows that⁹

$$\langle 0 | \psi_\alpha(x) \bar{\psi}_{\dot{\beta}}(y) | 0 \rangle = i \sigma_{\alpha\dot{\beta}}^\mu \partial_\mu \langle 0 | \phi(x) \phi^*(y) | 0 \rangle. \quad (\text{B36})$$

⁸Note that Eq. (B30) also implies that $\langle 0 | \phi(x) \phi(y) | 0 \rangle$ is a constant (independent of position) after noting that the two-point function is translationally invariant.

⁹Note that Eq. (B35) also implies that $\langle 0 | F(x) \phi^*(y) | 0 \rangle = 0$.

It is convenient to rewrite Eq. (B36) in momentum space,¹⁰

$$\langle 0 | \psi_\alpha(x) \bar{\psi}_{\dot{\beta}}(y) | 0 \rangle_{\text{FT}} = p \cdot \sigma_{\alpha\dot{\beta}} \langle 0 | \phi(x) \phi^*(y) | 0 \rangle. \quad (\text{B37})$$

Finally, we consider the identity,

$$i\langle 0 | [\eta Q + \bar{\eta} \bar{Q}, F^*(x) \psi_\beta(y)] | 0 \rangle = 0. \quad (\text{B38})$$

Once again, a similar computation yields

$$\begin{aligned} \{ \delta_\beta^\alpha \langle 0 | F^*(x) F(y) | 0 \rangle + i \bar{\sigma}^{\mu\dot{\alpha}\alpha} \langle 0 | (\partial_\mu \bar{\psi}_{\dot{\alpha}}(x) \psi_\beta(y)) | 0 \rangle \} \eta_\alpha \\ - i \bar{\eta}^{\dot{\alpha}} \sigma_{\beta\dot{\alpha}}^\mu \partial_\mu^y \langle 0 | F^*(x) \phi(y) | 0 \rangle = 0. \end{aligned} \quad (\text{B39})$$

It follows that

$$\delta_\beta^\alpha \langle 0 | F^*(x) F(y) | 0 \rangle = -i \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \langle 0 | \bar{\psi}_{\dot{\alpha}}(x) \psi_\beta(y) | 0 \rangle. \quad (\text{B40})$$

After raising the spinor indices, we can manipulate Eq. (B40) into the following form:

$$\langle 0 | (\bar{\psi}^{\dot{\alpha}}(x) \psi^\beta(y)) | 0 \rangle = \frac{i \bar{\sigma}^{\mu\dot{\alpha}\beta} \partial_\mu}{\square} \langle 0 | F^*(x) F(y) | 0 \rangle. \quad (\text{B41})$$

In momentum space, Eq. (B41) takes the following form:

$$\langle 0 | \bar{\psi}^{\dot{\alpha}}(x) \psi^\beta(y) | 0 \rangle_{\text{FT}} = \frac{p \cdot \bar{\sigma}^{\dot{\alpha}\beta}}{p^2} \langle 0 | F^*(x) F(y) | 0 \rangle_{\text{FT}}. \quad (\text{B42})$$

One further relation of interest can be found by comparing Eqs. (B36) and (B42). In particular, if we lower the spinor indices in Eq. (B42), anticommute the two fermion fields, and interchange the position coordinates, then it follows that

$$\langle 0 | \psi_\beta(x) \bar{\psi}_{\dot{\alpha}}(y) | 0 \rangle_{\text{FT}} = \frac{p \cdot \sigma_{\beta\dot{\alpha}}}{p^2} \langle 0 | F^*(x) F(y) | 0 \rangle_{\text{FT}}. \quad (\text{B43})$$

Hence,

$$\langle 0 | F^*(x) F(y) | 0 \rangle_{\text{FT}} = p^2 \langle 0 | \phi(x) \phi^*(y) | 0 \rangle_{\text{FT}}. \quad (\text{B44})$$

Note that the supersymmetric relations obtained above also apply to the corresponding time-ordered two-point functions; i.e., they apply to the corresponding propagators

¹⁰The Fourier transform of a translationally invariant function $f(x, y) \equiv f(x - y)$ is given by

$$\begin{aligned} f(x, y) &= \int \frac{d^4 p}{(2\pi)^4} \hat{f}(p) e^{-ip \cdot (x-y)}, \quad \text{where} \\ \hat{f}(p) &= \int d^4 x f(x, 0) e^{ip \cdot x}. \end{aligned}$$

In the notation of the text above, $f(x, y)_{\text{FT}} \equiv \hat{f}(p)$. Moreover, we note that $f(y, x)_{\text{FT}} = \hat{f}(-p)$.

to all orders in perturbation theory.¹¹ More general supersymmetric Ward identities that relate two- and three-point 1PI Green functions in SQED can be found in Ref. [34].

To make contact with the analysis of Appendix A, we convert from two-component to four-component fermion notation.¹² The four-component propagator function for fermions in momentum space is given by

$$\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle_{\text{FT}} = \begin{pmatrix} \langle 0|T\psi_\alpha(x)\psi^\beta(y)|0\rangle_{\text{FT}} & \langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle_{\text{FT}} \\ \langle 0|T\bar{\psi}^{\dot{\alpha}}(x)\psi^\beta(y)|0\rangle_{\text{FT}} & \langle 0|T\bar{\psi}^{\dot{\alpha}}(x)\bar{\psi}_\beta(y)|0\rangle_{\text{FT}} \end{pmatrix}, \quad (\text{B45})$$

where $\Psi(x)$ is a four-component spinor. Using the supersymmetric relations obtained above, it follows that

$$\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle_{\text{FT}} = \begin{pmatrix} -\delta_\alpha^\beta \langle 0|TF(x)\phi(y)|0\rangle_{\text{FT}} & \frac{p \cdot \sigma_{\dot{\alpha}\beta}}{p^2} \langle 0|TF^*(x)F(y)|0\rangle_{\text{FT}} \\ p \cdot \bar{\sigma}^{\dot{\alpha}\beta} \langle 0|T\phi(x)\phi^*(y)|0\rangle_{\text{FT}} & -\delta^{\dot{\alpha}\beta} \langle 0|T\phi^*(x)F^*(y)|0\rangle_{\text{FT}} \end{pmatrix}. \quad (\text{B46})$$

As a check, we apply the above results to SQED. The matter fields correspond to two chiral multiplets, $\Phi_+ = (\phi_+, \psi_+, F_+)$ and $\Phi_- = (\phi_-, \psi_-, F_-)$, with the corresponding superpotential, $W = m\Phi_+\Phi_-$. The scalar field contributions to the Lagrangian are

$$\begin{aligned} \mathcal{L} &= |\partial_\mu \phi_+|^2 + |\partial_\mu \phi_-|^2 + |F_+|^2 + |F_-|^2 \\ &\quad + m(F_+\phi_- + F_-\phi_+ + \text{H.c.}) \\ &= \begin{pmatrix} F_+^* & \phi_- \end{pmatrix} \Delta_{0F\phi}^{-1} \begin{pmatrix} F_+ \\ \phi_-^* \end{pmatrix} + (+ \leftrightarrow -), \end{aligned} \quad (\text{B47})$$

where we have dropped terms that are a total derivative and we have defined the inverse tree-level propagator matrix

$$\Delta_{0F\phi}^{-1} \equiv \begin{pmatrix} 1 & m \\ m & -\square \end{pmatrix}. \quad (\text{B48})$$

Inverting this matrix and passing to momentum space yields

$$i\Delta_{0F\phi}(p) = \frac{i}{p^2 - m^2} \begin{pmatrix} p^2 & -m \\ -m & 1 \end{pmatrix}. \quad (\text{B49})$$

Defining the Dirac electron field by

$$\Psi = \begin{pmatrix} \psi_+ \\ \bar{\psi}_- \end{pmatrix}, \quad (\text{B50})$$

the four-component electron propagator is given by

$$\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle_{\text{FT}} = \begin{pmatrix} -\delta_\alpha^\beta \langle 0|TF_+(x)\phi_-(y)|0\rangle_{\text{FT}} & \frac{p \cdot \sigma_{\dot{\alpha}\beta}}{p^2} \langle 0|TF_+^*(x)F_+(y)|0\rangle_{\text{FT}} \\ p \cdot \bar{\sigma}^{\dot{\alpha}\beta} \langle 0|T\phi_-(x)\phi_-^*(y)|0\rangle_{\text{FT}} & -\delta^{\dot{\alpha}\beta} \langle 0|T\phi_-^*(x)F_+(y)|0\rangle_{\text{FT}} \end{pmatrix}. \quad (\text{B51})$$

In light of Eq. (B49), we end up with the usual tree-level electron propagator,

$$\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle_{\text{FT}}^{\text{tree}} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}. \quad (\text{B52})$$

¹¹Strictly speaking, we should make use of the T^* product which has the property that one can freely move total derivatives from inside of the vacuum expectation value of the product of fields to outside. This is equivalent to defining the T product via its functional integral representation.

¹²The relation between the two-component spinor and four-component spinor notation is discussed in Ref. [32].

In principle, the radiatively corrected electron pole mass is obtained by inverting the 4×4 propagator matrix given by Eq. (B51), computing its determinant, and finding the value of p^2 at which the determinant vanishes (details can be found in Ref. [22]). However, it is significantly simpler to perform the computations by analyzing the radiative corrections in the scalar (ϕ - F) sector, as discussed in Sec. IV A.

APPENDIX C: CANCELLATION OF FINITE CORRECTIONS

In the text, we focused on cancellation of the leading IR-divergent pieces from the correction to the physical

electron mass in SQED with $\overline{\text{DR}}$ renormalization. Here we demonstrate that the finite pieces also cancel.

Using dimensional regularization with $d = 4 - 2\delta$, $\delta < 0$, the correction to the mass term, given by Eq. (3.2), may be expressed as

$$I_{F_+\phi_-} + \text{c.t.} = -2me^2 \left(1 - \frac{1}{\xi}\right) \frac{p^2}{p^2 - m^2} \frac{1}{(4\pi)^2} (4\pi)^\delta \times \frac{\Gamma(\delta)}{1 - \delta} F\left(1 + \delta, -\delta; 2 - \delta; \frac{p^2}{p^2 - m^2}\right) \times \frac{1}{(m^2 - p^2)^\delta}, \quad (\text{C1})$$

where $F \equiv {}_2F_1$ is the Gauss hypergeometric function. The correction to the kinetic term at leading order in e is

given by Eq. (3.5). The UV divergence gets canceled by the counterterm in $\overline{\text{DR}}$, leaving only the finite and IR-divergent pieces,

$$I_{F_+^*F_+} + \text{c.t.} = \frac{-2e^2}{(4\pi)^2} \left(1 - \frac{1}{\xi}\right) \frac{(4\pi)^\delta}{(m^2 - p^2)^\delta} \times \frac{p^2}{p^2 - m^2} \frac{2\Gamma(\delta)}{(2 - \delta)(1 - \delta)} \times F\left(1 + \delta, -\delta; 3 - \delta; \frac{p^2}{p^2 - m^2}\right). \quad (\text{C2})$$

The quantity relevant to the physical mass correction is therefore given by

$$\frac{I_{F_+\phi_-}}{m} - I_{F_+^*F_+} = \frac{-2e^2(1 - 1/\xi)}{(4\pi)^2} \frac{p^2}{p^2 - m^2} \frac{(4\pi)^\delta}{(m^2 - p^2)^\delta} \frac{\Gamma(\delta)}{1 - \delta} \times \left\{ F\left(1 + \delta, -\delta; 2 - \delta; \frac{p^2}{p^2 - m^2}\right) - \frac{2}{2 - \delta} F\left(1 + \delta, -\delta; 3 - \delta; \frac{p^2}{p^2 - m^2}\right) \right\}. \quad (\text{C3})$$

The goal of this exercise is to evaluate the expression,

$$\mathcal{F}(\epsilon) = \frac{\Gamma(\epsilon)}{1 - \epsilon} \left\{ F\left(1 + \epsilon, -\epsilon; 2 - \epsilon; \frac{p^2}{p^2 - m^2}\right) - \frac{2}{2 - \epsilon} F\left(1 + \epsilon, -\epsilon; 3 - \epsilon; \frac{p^2}{p^2 - m^2}\right) \right\}, \quad (\text{C4})$$

in the limit of $\epsilon \rightarrow 0$.

The relevant formulas taken from Ref. [35] are as follows. First, we make use of formula (42) on p. 103 of Ref. [35],

$$(c - b - 1)F(a, b; c; z) + bF(a, b + 1; c; z) - (c - 1)F(a, b; c - 1; z) = 0. \quad (\text{C5})$$

Choosing $a = 1 + \epsilon$, $b = -\epsilon$, $c = 3 - \epsilon$, it follows that

$$(2 - \epsilon)F(1 + \epsilon, -\epsilon; 2 - \epsilon; z) - 2F(1 + \epsilon, -\epsilon; 3 - \epsilon; z) = -\epsilon F(1 + \epsilon, 1 - \epsilon; 3 - \epsilon; z). \quad (\text{C6})$$

Using this result in Eq. (C4) with $z \equiv p^2/(p^2 - m^2)$ yields

$$\mathcal{F}(\epsilon) = -\frac{\Gamma(1 + \epsilon)}{(1 - \epsilon)(2 - \epsilon)} F\left(1 + \epsilon, 1 - \epsilon; 3 - \epsilon; \frac{p^2}{p^2 - m^2}\right), \quad (\text{C7})$$

after using $\epsilon\Gamma(\epsilon) = \Gamma(1 + \epsilon)$. Taking the $\epsilon \rightarrow 0$ limit, we end up with

$$\mathcal{F}(0) = -\frac{1}{2} F\left(1, 1; 3; \frac{p^2}{p^2 - m^2}\right). \quad (\text{C8})$$

Next, we make use of formula (15) on p. 102 of Ref. [35], which implies that

$$F(1, 1; 2; z) = -\frac{\ln(1 - z)}{z}. \quad (\text{C9})$$

We then use formula (24) of p. 102 of Ref. [35], which gives (for $n = 1$)

$$\frac{(c - a)(c - b)}{c} (1 - z)^{a+b-c-1} F(a, b; c + 1; z) = \frac{d}{dz} \left[(1 - z)^{a+b-c} F(a, b; c; z) \right], \quad (\text{C10})$$

to derive

$$F(1, 1; 3; z) = \frac{2}{z} \left[1 + \frac{(1 - z) \ln(1 - z)}{z} \right]. \quad (\text{C11})$$

Making use of Eq. (C11), we arrive at our final result,

$$\mathcal{F}(0) = \left(\frac{m^2}{p^2} - 1\right) \left[1 + \frac{m^2}{p^2} \ln\left(1 - \frac{p^2}{m^2}\right)\right]. \quad (\text{C12})$$

The limit of $p^2 \rightarrow m^2$ then yields

$$\lim_{p^2 \rightarrow m^2} \mathcal{F}(0) = 0. \quad (\text{C13})$$

Thus, finite corrections to the physical mass vanish in the on-shell limit.

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