Evaluating integrals arising from Barr-Zee diagrams

Howard E. Haber

Santa Cruz Institute for Particle Physics

University of California, Santa Cruz, CA 95064, USA

March 4, 2024

Abstract

In these notes, we provide an explicit calculation of some integrals that arises in the computation of the Barr-Zee Feynman diagrams [1] that contribute to the dipole moment operator.

Section 1: The function $g(z)$

Consider the integral [1, 2],

$$g(z) = \tfrac{1}{2}z \int_0^1 \frac{dx}{x(1-x)-z} \ln\left[\frac{x(1-x)}{z}\right] , \qquad \text{for real } z > 0. \tag{1}$$

First we note that

$$x(1-x) - z = -(x-x_+)(x-x_-)\,, \tag{2}$$

where

$$x_\pm = \tfrac{1}{2}\big[1 \pm \sqrt{1-4z}\,\big]\,. \tag{3}$$

It follows that

$$x_+ + x_- = 1\,, \qquad\qquad x_+x_- = z\,. \tag{4}$$

Noting that for $z \neq \frac{1}{4}$,

$$\frac{1}{(x-x_+)(x-x_-)} = \frac{1}{x_+ - x_-}\left[\frac{1}{x-x_+} - \frac{1}{x-x_-}\right] , \tag{5}$$

eq. (1) can be rewritten as:

$$g(z) = -\frac{z}{2\sqrt{1-4z}}\left\{\int_0^1 \frac{dx}{x-x_+} \ln\left[\frac{x(1-x)}{z}\right] - \int_0^1 \frac{dx}{x-x_-} \ln\left[\frac{x(1-x)}{z}\right]\right\}. \tag{6}$$

We first analyze the case of $0 < z < \frac{1}{4}$, in which case, $0 < x_- < x_+ < 1$. We shall employ the following result [3]:

$$\begin{aligned} G(y_1; b, c) &= \int_0^1 \frac{dy}{y-y_1} \ln\left(\frac{y^2+by+c-i\epsilon}{y_1^2+by_1+c}\right) \\ &= \mathrm{Li}_2\left(\frac{-y_1}{y_+-y_1}\right) + \mathrm{Li}_2\left(\frac{-y_1}{y_--y_1}\right) - \mathrm{Li}_2\left(\frac{1-y_1}{y_+-y_1}\right) - \mathrm{Li}_2\left(\frac{1-y_1}{y_--y_1}\right) . \end{aligned} \tag{7}$$

where y_1, b and c are real parameters, $y_\pm = \frac{1}{2}\big[-b \pm \sqrt{b^2-4c}\big]$, under the assumption that $0 \leq y_\pm \leq 1$ and $b^2 > 4c$, and the dilogarithm $\mathrm{Li}_2(z)$ is defined in the Appendix. Then,

$$g(z) = -\tfrac{1}{2}z\big[G(x_+, -1, 0) - G(x_-, -1, 0)\big]\,, \tag{8}$$

after noting that $x_\pm^2 - x_\pm = x_\pm(x_\pm - 1) = -x_\pm x_\mp = -z$.

We now make use of eq. (7) with $b=-1$, $c=0$, and $y_1=x_\pm$, which yields $y_+=1$ and $y_0=0$. It follows that

$$G(x_\pm,-1,0)=\operatorname{Li}_2\left(\frac{-x_\pm}{1-x_\pm}\right)-\operatorname{Li}_2\left(\frac{1-x_\pm}{-x_\pm}\right). \tag{9}$$

It is convenient to employ the identity [4]:

$$=\operatorname{Li}_2\left(\frac{-x}{1-x}\right)=-\operatorname{Li}_2(x)-\tfrac{1}{2}\ln^2(1-x)\,,\qquad \text{for } x<1. \tag{10}$$

Letting $x\to 1-x$ yields a second identity:

$$\operatorname{Li}_2\left(\frac{1-x}{-x}\right)=-\operatorname{Li}_2(1-x)-\tfrac{1}{2}\ln^2 x\,,\qquad \text{for } x>0. \tag{11}$$

Hence,

$$G(x_\pm,-1,0)=-\operatorname{Li}_2(x_\pm)+\operatorname{Li}_2(x_\mp)-\tfrac{1}{2}\ln^2 x_\mp+\tfrac{1}{2}\ln^2 x_\pm\,, \tag{12}$$

after using $x_\mp=1-x_\pm$. In particular,

$$G(x_-,-1,0)=-G(x_+,-1,0)\,. \tag{13}$$

We therefore end up with

$$\begin{aligned} g(z)=\frac{z}{\sqrt{1-4z}}\Big\{&\operatorname{Li}_2\big(\tfrac{1}{2}\big[1+\sqrt{1-4z}\,\big]\big)-\operatorname{Li}_2\big(\tfrac{1}{2}\big[1-\sqrt{1-4z}\,\big]\big)\\ &+\tfrac{1}{2}\ln^2\big(\tfrac{1}{2}\big[1-\sqrt{1-4z}\,\big]\big)-\tfrac{1}{2}\ln^2\big(\tfrac{1}{2}\big[1+\sqrt{1-4z}\,\big]\big)\Big\}\,,\quad \text{for } 0<z<\tfrac{1}{4}. \end{aligned} \tag{14}$$

One further simplification can be made by employing the identity [4]:

$$\operatorname{Li}_2(y)+\operatorname{Li}_2(1-y)=\tfrac{1}{6}\pi^2-\ln y\ln(1-y)\,. \tag{15}$$

After making use of eqs. (4) and (15), we can rewrite eq. (14) in two equivalent forms:

$$\begin{aligned} g(z)&=\frac{z}{\sqrt{1-4z}}\Big\{2\operatorname{Li}_2\big(\tfrac{1}{2}\big[1+\sqrt{1-4z}\,\big]\big)-\ln^2\big(\tfrac{1}{2}\big[1+\sqrt{1-4z}\,\big]\big)+\tfrac{1}{2}\ln^2 z-\tfrac{1}{6}\pi^2\Big\}\\ &=\frac{z}{\sqrt{1-4z}}\Big\{-2\operatorname{Li}_2\big(\tfrac{1}{2}\big[1-\sqrt{1-4z}\,\big]\big)+\ln^2\big(\tfrac{1}{2}\big[1-\sqrt{1-4z}\,\big]\big)-\tfrac{1}{2}\ln^2 z+\tfrac{1}{6}\pi^2\Big\}\,,\\ &\qquad \text{for } 0<z<\tfrac{1}{4}. \end{aligned} \tag{16}$$

It is instructive to evaluate the limit of $z\ll 1$. One can approximate $\sqrt{1-4z}\simeq 1-2z$ in the second form of $g(z)$ given in eq. (16) to obtain

$$g(z)\simeq z\big[\tfrac{1}{6}\pi^2-2\operatorname{Li}_2(z)+\tfrac{1}{2}\ln^2 z\big]=\tfrac{1}{2}z\big[\ln^2 z+\tfrac{1}{3}\pi^2\big]+\mathcal{O}(z^2)\,. \tag{17}$$

Next, we check the result for $z=\frac{1}{4}$. In this case, Mathematica yields:

$$g(z)=\ln 2+\tfrac{2}{3}\big(4\ln 2-1\big)\big(z-\tfrac{1}{4}\big)+\mathcal{O}\big((z-\tfrac{1}{4})^2\big)\,. \tag{18}$$

One can verify the leading term of eq. (18) by evaluating eq. (1) for $z = \frac{1}{4}$. An explicit computation yields:

$$g\big(\tfrac{1}{4}\big) = -\frac{1}{2}\int_0^1 \frac{dx}{(2x-1)^2}\ln[4x(1-x)] = -\frac{1}{4}\int_{-1}^1 \frac{dy}{y^2}\ln(1-y^2) = \ln 2\,. \qquad (19)$$

Next, we analyze the case of $z > \frac{1}{4}$. First, we can rewrite eq. (14) as

$$g(z) = \frac{z}{\sqrt{1-4z}}\left\{\mathrm{Li}_2\big(\tfrac{1}{2}\big[1+\sqrt{1-4z}\,\big]\big) - \mathrm{Li}_2\big(\tfrac{1}{2}\big[1-\sqrt{1-4z}\,\big]\big) - \tfrac{1}{2}\ln z\ln\left(\frac{1+\sqrt{1-4z}}{1-\sqrt{1-4z}}\right)\right\},$$
$$\text{for } 0 < z < \tfrac{1}{4}, \qquad (20)$$

after noting that $\ln^2 x_- - \ln^2 x_+ = \ln(x_- x_+)\ln(x_-/x_+) = \ln z\ln(x_-/x_+)$. Moreover, eq. (20) remains valid for $z > \frac{1}{4}$ if we put $\sqrt{1-4z} = i\sqrt{4z-1}$. In light of $\mathrm{Li}_2(z^*) = [\mathrm{Li}_2(z)]^*$, it follows that:

$$\begin{aligned} g(z) &= \frac{2z}{\sqrt{4z-1}}\,\mathrm{Im}\Big\{\mathrm{Li}_2\big(\tfrac{1}{2}\big[1+i\sqrt{4z-1}\,\big]\Big\} - \frac{z}{2i\sqrt{4z-1}}\ln z\ln\left(\frac{1+i\sqrt{4z-1}}{1-i\sqrt{4z-1}}\right) \\ &= \frac{2z}{\sqrt{4z-1}}\left[\mathrm{Im}\Big\{\mathrm{Li}_2\big(\tfrac{1}{2}\big[1+i\sqrt{4z-1}\,\big]\Big\} - \tfrac{1}{2}\ln z\,\tan^{-1}\sqrt{4z-1}\right], \qquad \text{for } z > \tfrac{1}{4}. \qquad (21) \end{aligned}$$

Using Ref. [4], one can express the imaginary part of the dilogarithm of a complex argument in terms of the Clausen function (which is defined in the Appendix):

$$\mathrm{Im}\,\mathrm{Li}_2(re^{i\theta}) = \omega\ln r + \tfrac{1}{2}\,\mathrm{Cl}_2(2\omega) - \tfrac{1}{2}\,\mathrm{Cl}_2(2\omega+2\theta) + \tfrac{1}{2}\,\mathrm{Cl}_2(2\theta)\,, \qquad (22)$$

where

$$\omega \equiv \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right). \qquad (23)$$

Note that if $re^{i\theta} = \frac{1}{2}\big[1+i\sqrt{4z-1}\,\big]$, then

$$r = \sqrt{z}\,, \qquad \sin\theta = \sqrt{1-\frac{1}{4z}}\,, \qquad \cos\theta = \frac{1}{2\sqrt{z}}\,. \qquad (24)$$

Hence,

$$\theta = \omega = \tan^{-1}\sqrt{4z-1}\,, \qquad (25)$$

where the principal value of the arctangent is used (i.e., $0 \le \tan^{-1}\sqrt{4z-1} \le \frac{1}{2}\pi$). It then follows that

$$\begin{aligned} &\mathrm{Im}\Big\{\mathrm{Li}_2\big(\tfrac{1}{2}\big[1+i\sqrt{4z-1}\,\big]\Big\} - \tfrac{1}{2}\ln z\,\tan^{-1}\sqrt{4z-1} \\ &\qquad = \mathrm{Cl}_2\big(2\tan^{-1}\sqrt{4z-1}\big) - \tfrac{1}{2}\,\mathrm{Cl}_2\big(4\tan^{-1}\sqrt{4z-1}\big) \\ &\qquad = \mathrm{Cl}_2\big(\pi - 2\tan^{-1}\sqrt{4z-1}\,\big)\,, \qquad (26) \end{aligned}$$

after employing the identity $\frac{1}{2}\,\mathrm{Cl}_2(4\theta) = \mathrm{Cl}_2(2\theta) - \mathrm{Cl}_2(\pi - 2\theta)$ given in Ref. [4]. Thus,

$$g(z) = \frac{2z}{\sqrt{4z-1}}\,\mathrm{Cl}_2\big(\pi - 2\tan^{-1}\sqrt{4z-1}\,\big)\,, \qquad \text{for } z > \tfrac{1}{4}. \qquad (27)$$

Finally, in light of eqs. (24) and (25),

$$\theta = \tan^{-1}\sqrt{4z-1} = \cos^{-1}\frac{1}{2\sqrt{z}} = \frac{\pi}{2} - \sin^{-1}\frac{1}{2\sqrt{z}}\,, \tag{28}$$

where we have made use of the identity, $\sin^{-1} y + \cos^{-1} y = \frac{1}{2}\pi$, in the last step above. Hence, it follows that:

$$\pi - 2\tan^{-1}\sqrt{4z-1} = 2\sin^{-1}\frac{1}{2\sqrt{z}}\,. \tag{29}$$

Consequently, eq. (27) can be rewritten as:

$$g(z) = \frac{2z}{\sqrt{4z-1}}\,\mathrm{Cl}_2\biggl(2\sin^{-1}\frac{1}{2\sqrt{z}}\biggr)\,, \qquad\qquad \text{for } z > \tfrac{1}{4}. \tag{30}$$

It is instructive to compute $g\bigl(\frac{1}{4}\bigr)$. For this calculation, we shall employ an equivalent form [cf. eq. (26)]:

$$g(z) = \frac{2z}{\sqrt{4z-1}}\Bigl[\mathrm{Cl}_2\bigl(2\tan^{-1}\sqrt{4z-1}\bigr) - \tfrac{1}{2}\,\mathrm{Cl}_2\bigl(4\tan^{-1}\sqrt{4z-1}\bigr)\Bigr]\,. \tag{31}$$

Using the expansion (see eq. (4.28) of Ref. [4]),

$$\mathrm{Cl}_2(\theta) = \theta(1 - \ln|\theta|) + \tfrac{1}{72}\theta^3 + \mathcal{O}(\theta^5)\,, \tag{32}$$

and approximating $\tan^{-1}\sqrt{4z-1} \simeq \sqrt{4z-1}$, it follows that

$$g\bigl(\tfrac{1}{4}\bigr) = \ln\bigl(4\tan^{-1}\sqrt{4z-1}\bigr) - \ln\bigl(2\tan^{-1}\sqrt{4z-1}\bigr) = \ln 2\,, \tag{33}$$

in agreement with eq. (19).

As a second check, consider the limit where $z \gg 1$. Then,

$$2\sin^{-1}\frac{1}{2\sqrt{z}} = \frac{1}{\sqrt{z}}\left[1 + \frac{1}{24z} + \mathcal{O}\bigl(z^{-2}\bigr)\right]\,. \tag{34}$$

Using eq. (32), we end up with

$$g(z) = 1 + \tfrac{1}{2}\ln z + \frac{5 + 3\ln z}{36z} + \mathcal{O}(z^{-2})\,. \tag{35}$$

One can verify eq. (35) by taking the limit of eq. (1) for $z \gg 1$,

$$\begin{aligned} g(z) &= \tfrac{1}{2}\ln z - \int_0^1 \ln x\,dx - \frac{1}{z}\int_0^1 x(1-x)\log x + \frac{\ln z}{2z}\int_0^1 x(1-x)dx + \mathcal{O}(z^{-2}) \\ &= \left(\frac{1}{2} + \frac{1}{12z}\right)\ln z + 1 + \frac{5}{36z} + \mathcal{O}(z^{-2})\,. \end{aligned} \tag{36}$$

It is noteworthy that the value $g(1)$ can be expressed in terms of polygamma functions. Mathematica yields:

$$g(1) = \tfrac{1}{36}\Bigl[\psi_1\bigl(\tfrac{1}{6}\bigr) + \psi_1\bigl(\tfrac{1}{3}\bigr) - \psi_1\bigl(\tfrac{2}{3}\bigr) - \psi_1\bigl(\tfrac{5}{6}\bigr)\Bigr] \simeq 1.17195\,, \tag{37}$$

where

$$\psi_1(z) \equiv \frac{d^2}{dz^2}\ln\Gamma(z) = n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^2}\,. \tag{38}$$

Section 2: The function $f(z)$

Consider the integral [1,2],

$$f(z)=\tfrac{1}{2}z\int_0^1 \frac{1-2x(1-x)}{x(1-x)-z}\,\ln\left[\frac{x(1-x)}{z}\right]\,dx\,,\qquad \text{for real } z>0. \tag{39}$$

We can rewrite eq. (39) as

$$f(z)=(1-2z)g(z)-z\int_0^1 \ln\left[\frac{x(1-x)}{z}\right]\,dx=z(2+\ln z)+(1-2z)g(z)\,. \tag{40}$$

For $z\ll 1$, we see that $f(z)\simeq g(z)\simeq \frac{1}{2}z\ln^2 z$. Using eqs. (19) and (40), it follows that $f\left(\frac{1}{4}\right)=\frac{1}{2}$. For $z=1$, we have $f(1)=2-g(1)=0.828046$. Finally, the leading contributions to $f(z)$ as $z\to\infty$ are obtained by employing eq. (35):

$$\begin{aligned} f(z)&=z(2+\ln z)+(1-2z)\Big[1+\tfrac{1}{2}\ln z+\frac{5+3\ln z}{36z}+\mathcal{O}(z^{-2})\Big] \\ &=\tfrac{13}{18}+\tfrac{1}{3}\ln z+\mathcal{O}(z^{-1})\,. \end{aligned} \tag{41}$$

Section 3: The function $h(z)$

Consider the integral [2],

$$h(z)=-\tfrac{1}{2}z\int_0^1 \frac{dx}{x(1-x)-z}\left\{1-\frac{z}{x(1-x)-z}\,\ln\left[\frac{x(1-x)}{z}\right]\right\}\,,\qquad \text{for real } z>0. \tag{42}$$

To evaluate $h(z)$, we first compute

$$\begin{aligned} \frac{d}{dz}\left(\frac{2g(z)}{z}\right)&=\frac{\partial}{\partial z}\int_0^1\frac{dx}{x(1-x)-z}\,\ln\left[\frac{x(1-x)}{z}\right] \\ &=\int_0^1\frac{dx}{\big[x(1-x)-z\big]^2}\,\ln\left[\frac{x(1-x)}{z}\right]-\frac{1}{z}\int_0^1\frac{dx}{x(1-x)-z} \\ &=\frac{2h(z)}{z^2}\,. \end{aligned} \tag{43}$$

Hence, it follows that

$$h(z)=z^2\frac{d}{dz}\left(\frac{g(z)}{z}\right)\,. \tag{44}$$

We now use the results of Section 1. The following derivatives will be needed:

$$\frac{dx_\pm}{dz}=\mp\,(1-4z)^{-1/2}\,, \tag{45}$$

$$\frac{d}{dy}\,\mathrm{Li}_2(y)=-\frac{\ln(1-y)}{y}\,, \tag{46}$$

$$\frac{d}{d\theta}\,\mathrm{Cl}_2(\theta)=-\ln\big[2\sin\big(\tfrac{1}{2}\theta\big)\big]\,. \tag{47}$$

Using eq. (14), it follows that

$$\frac{d}{dz}\left(\frac{g(z)}{z}\right) = \frac{2g(z)}{z(1-4z)} + \frac{1}{1-4z}\left\{\big[\ln x_+ + \ln x_-\big]\left(\frac{1}{x_+}+\frac{1}{x_-}\right)\right\}$$
$$= \frac{2g(z)+\ln z}{z(1-4z)}\,, \qquad \text{for } 0<z<\tfrac{1}{4}. \tag{48}$$

Likewise, using eq. (30), it follows that

$$\frac{d}{dz}\left(\frac{g(z)}{z}\right) = -\frac{2g(z)}{z(4z-1)} - \frac{4}{\sqrt{4z-1}}\ln\Big[2\sin\Big(\sin^{-1}\frac{1}{2\sqrt{z}}\Big)\Big]\frac{d}{dz}\sin^{-1}\left(\frac{1}{2\sqrt{z}}\right)$$
$$= -\frac{2g(z)}{z(4z-1)} + \frac{2\ln z}{\sqrt{4z-1}}\frac{1}{\sqrt{1-1/(4z)}}\left(-\tfrac{1}{4}z^{-3/2}\right). \tag{49}$$

Hence, we end up with:

$$\frac{d}{dz}\left(\frac{g(z)}{z}\right) = \frac{2g(z)+\ln z}{z(1-4z)}\,, \qquad \text{for } z>\tfrac{1}{4}. \tag{50}$$

Not surprisingly, the results obtained in eqs. (48) and (50) coincide (and thus are valid for all real values of $z>0$).

In light of eq. (44), it follows that

$$h(z) = \frac{z\big[2g(z)+\ln z\big]}{1-4z}\,. \tag{51}$$

Finally, we can check some limiting cases. Using eq. (17), it follows that for $z \ll 1$,

$$h(z) = z\ln z + z^2\big[\ln^2 z + 4\ln z + \tfrac{1}{3}\pi^2\big] + \mathcal{O}(z^3)\,. \tag{52}$$

Using eqs. (37) and (51), $h(1) = -0.781302$. For $z \gg 1$, eq. (35) yields:

$$h(z) = -\tfrac{1}{2}\big(1+\ln z\big) - \frac{7+6\ln z}{36z} + \mathcal{O}(z^{-2})\,. \tag{53}$$

Using Mathematica, one can check eq. (53) by integrating the leading terms of $h(z)$ in the limit of $z \gg 1$. One can also verify either by explicit integration or by employing eqs. (18) and (51) that $h\big(\tfrac{1}{4}\big) = -\tfrac{1}{6}(1+2\ln 2)$. Note that the first order correction to this result requires a more accurate version of eq. (18):

$$g(z) = \ln 2 + \tfrac{1}{6}(1-4z)(1-4\ln 2) - \tfrac{1}{60}(1-4z)^2(1+8\ln 2) + \mathcal{O}\big((1-4z)^3\big)\,. \tag{54}$$

Then,

$$2zg(z) = \tfrac{1}{2}\ln 2 + \tfrac{1}{12}(1-10\ln 2)(1-4z) - \tfrac{1}{120}(11-32\ln 2)(1-4z)^2 + \mathcal{O}\big((1-4z)^3\big)\,,$$
$$z\ln z = -\tfrac{1}{2}\ln 2 - \tfrac{1}{4}(1-2\ln 2)(1-4z) + \tfrac{1}{8}(1-4z)^2 + \mathcal{O}\big((1-4z)^3\big)\,. \tag{55}$$

Hence, eq. (51) yields

$$h(z) = -\tfrac{1}{6}(1+2\ln 2) + \tfrac{1}{30}(1+8\ln 2)(1-4z) + \mathcal{O}\big((1-4z)^2\big)\,. \tag{56}$$

Section 4: Summary of results

We define three integrals for real positive values of z [1, 2]:

$$g(z) = \tfrac{1}{2}z \int_0^1 \frac{dx}{x(1-x)-z}\,\ln\left[\frac{x(1-x)}{z}\right], \tag{57}$$

$$f(z) = \tfrac{1}{2}z \int_0^1 \frac{1-2x(1-x)}{x(1-x)-z}\,\ln\left[\frac{x(1-x)}{z}\right] dx\,, \tag{58}$$

$$h(z) = -\tfrac{1}{2}z \int_0^1 \frac{dx}{x(1-x)-z}\left\{1-\frac{z}{x(1-x)-z}\ln\left[\frac{x(1-x)}{z}\right]\right\}. \tag{59}$$

Then, one can derive the following expressions for $f(z)$ and $h(z)$ in terms of $g(z)$:

$$f(z) = z(2+\ln z) + (1-2z)g(z)\,, \tag{60}$$

$$h(z) = \frac{z\big[2g(z)+\ln z\big]}{1-4z}\,. \tag{61}$$

An explicit expression for $g(z)$ is given by:

$$g(z) = \begin{cases} \dfrac{z}{\sqrt{1-4z}}\left\{\mathrm{Li}_2(x_+) - \mathrm{Li}_2(x_-) - \tfrac{1}{2}\ln z \ln\left(\dfrac{x_+}{x_-}\right)\right\}, & \text{for } 0 < z \leq \tfrac{1}{4}\,, \\[2ex] \dfrac{2z}{\sqrt{4z-1}}\,\mathrm{Cl}_2\left(2\sin^{-1}\dfrac{1}{2\sqrt{z}}\right), & \text{for } z > \tfrac{1}{4}\,, \end{cases} \tag{62}$$

where $x_\pm \equiv \frac{1}{2}\big[1 \pm \sqrt{1-4z}\,\big]$ and $0 \leq \sin^{-1}[1/(2\sqrt{z})] \leq \frac{1}{2}\pi$ (for $z \geq \frac{1}{4}$). In Fig. 1, we have employed Mathematica (Version 14.0) to produce plots of the functions $g(z)$, $f(z)$ and $-h(z)$ for $0.01 \leq z \leq 100$. This figure reproduces the results first shown in Fig. 3 of Ref. [2].

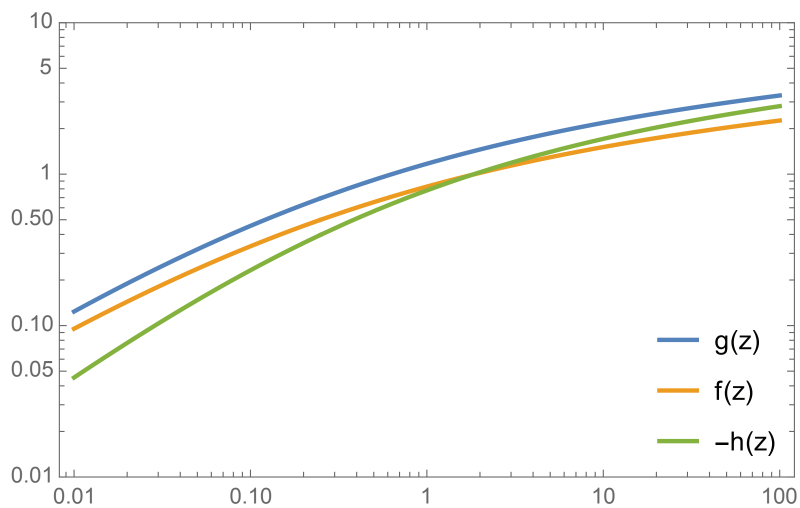

Figure 1: Plots of $g(z)$ given by eq. (62), $f(z)$ given by eq. (60), and $h(z)$ given by eq. (61) as a function of the variable z for $0.01 \leq z \leq 100$. These plots were produced using Version 14.0 of Mathematica.

Alternative expressions for $g(z)$ for $0 < z < \frac{1}{4}$ that involve only one dilogarithm can be found in eqs. (16) and (20). Note that the function $g(z)$ is continuous at $z = \frac{1}{4}$ with a value given by $g\big(\frac{1}{4}\big) = \ln 2$. Likewise, the function of $h(z)$ is also continuous (and finite) at $z = \frac{1}{4}$ with a value given by $h\big(\frac{1}{4}\big) = -\frac{1}{6}(1+2\ln 2)$. For $z \ll 1$, the leading behavior of the functions $f(z)$, $g(z)$, and $h(z)$ are given by: $f(z) \simeq g(z) \simeq \frac{1}{2}z\ln^2 z$ and $h(z) \simeq z\ln z$. For $z \gg 1$, the leading behavior of the functions are given by: $g(z) \simeq 1 + \frac{1}{2}\ln z$, $f(z) \simeq \frac{13}{18} + \frac{1}{3}\ln z$, and $h(z) \simeq -\frac{1}{2}(1 + \ln z)$. More accurate approximations for $g(z)$ and $h(z)$ are provided in Sections 1 and 3, respectively.

Appendix: Definitions of the dilogarithm and the Clausen function

In these notes, we follow the definitions given by Lewin in Ref. [4]. The dilogarithm is defined as

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-w)}{w}\, dw\,, \tag{A.1}$$

where $z = x + iy$ is any complex number excluding $z = x \in \mathbb{R}$ for $1 < x < \infty$. By expanding $\ln(1 - w)$ in a power series around $w = 0$ and integrating term by term, one can derive a convergent series representation of the dilogarithm:

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}\,, \quad \text{for } |z| \leq 1. \tag{A.2}$$

It is conventional to define a single-valued dilogarithm function on the cut complex plane, where the branch cut is located on the real axis for $1 \leq x \leq \infty$. Although $z = x = 1$ is a branch point, the integral in eq. (A.1) is well defined there[1] and yields $\mathrm{Li}_2(1) = \frac{1}{6}\pi^2$. Lewin chooses to define $\ln(-x) = \lim_{\varepsilon\to 0}\ln(-x + i\varepsilon)$ for real $x > 0$ and $\mathrm{Li}_2(x) = \lim_{\varepsilon\to 0}\mathrm{Li}_2(x - i\varepsilon)$ for real $x > 1$, in which case $\mathrm{Im}\,\mathrm{Li}_2(x) = -\pi\ln(x)\Theta(x-1)$ for real $x > 1$, where the Heavyside step function is defined such that $\Theta(x - 1) = 1$ for $x > 1$ and $\Theta(x - 1) = 0$ for $x < 1$.[2]

The Clausen function is defined as

$$\mathrm{Cl}_2(\theta) = -\int_0^\theta \ln\big|2\sin\big(\tfrac{1}{2}\phi\big)\big|\, d\phi\,, \tag{A.3}$$

for $\theta \in \mathbb{R}$. A useful convergent series representation of the Clausen function is

$$\mathrm{Cl}_2(\theta) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}\,. \tag{A.4}$$

The Clausen function is an odd function of θ that satisfies the following periodic properties:

$$\mathrm{Cl}_2(2n\pi \pm \theta) = \mathrm{Cl}_2(\pm\theta) = \pm\,\mathrm{Cl}_2(\theta)\,, \quad \text{for all integer values of } n, \tag{A.5}$$

and $\mathrm{Cl}_2(\pi + \theta) = -\,\mathrm{Cl}_2(\pi - \theta)$. Note that $\mathrm{Cl}_2(n\pi) = 0$ for any integer n, and $\mathrm{Cl}_2(\frac{1}{2}\pi) = G$, where G is Catalan's constant,

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \simeq 0.91596559\,. \tag{A.6}$$

[1] In light of eq. (A.2), $\mathrm{Li}_2(1) = \zeta(2) = \frac{1}{6}\pi^2$.

[2] It is now more common in the literature to define $\mathrm{Li}_2(x) = \lim_{\varepsilon\to 0}\mathrm{Li}_2(x + i\varepsilon)$ for real $x > 1$, although this choice of conventions affects none of the results presented in these notes.

References

[1] S.M. Barr and A. Zee, "Electric Dipole Moment of the Electron and of the Neutron," Phys. Rev. Lett. **65**, 21 (1990) [Erratum: ibid. **65**, 2920 (1990)].

[2] D. Chang, W.S. Hou and W.Y. Keung, "Two loop contributions of flavor changing neutral Higgs bosons to $\mu \to e\gamma$," Phys. Rev. D **48**, 217 (1993) [arXiv:hep-ph/9302267].

[3] H.E. Haber and D. Wyler, "Radiative Neutralino Decay," Nucl. Phys. B **323**, 267 (1989).

[4] Leonard Lewin, *Polylogarithms and Associated Functions* (Elsevier North Holland, New York, 1981).