The eigenvalues of the quadratic Casimir operator and second-order indices of a simple Lie algebra

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Abstract

In these notes, we demonstrate how to compute the eigenvalue of the quadratic Casimir operator and the second-order index for an irreducible representation of a simple Lie algebra. Explicit results for the fundamental and adjoint representations of $\mathfrak{su}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$ are given. The relation of these results to the dual Coxeter number is clarified. Finally, the dependence on the normalization of the Lie algebra generators is discussed.

I. Introduction

The reader is assumed to be familiar with Dynkin's techniques for analyzing the simple Lie algebras. These methods will be briefly summarized below. The material in these notes and further details can be found in refs. [1–18].

The generators of a Lie group G [which constitute a basis for the corresponding Lie algebra $\mathfrak{g}$] satisfy the commutation relations

$$[\boldsymbol{T_a}\,,\,\boldsymbol{T_b}] = f_{ab}^{c}\,\boldsymbol{T_c}\,, \qquad\qquad a,b,c = 1,2,\ldots,d_G\,, \tag{1}$$

where d_G is the dimension of the Lie group G, and there is an implicit sum over repeated indices. In eq. (1), we employ the mathematics convention in which the $\boldsymbol{T_a}$ are anti-hermitian generators and the f_{ab}^{c} are real structure constants for a compact real Lie algebra. The Killing form is defined in terms of a symmetric metric tensor,

$$g_{ab} = f_{ac}^{d} f_{bd}^{c}\,. \tag{2}$$

The inverse of g_{ab} will be denoted by g^{ab}; that is,

$$g_{ab}g^{bc} = \delta_a^c\,.$$

The adjoint representation consist of $d_G \times d_G$ matrices that represent the $\boldsymbol{T_a}$. These matrices, which we denote henceforth by $\boldsymbol{F_a}$, are defined by:

$$(\boldsymbol{F_a})_b{}^c = -f_{ab}^{c}\,, \tag{3}$$

where b and c label the row and column indices of the $\boldsymbol{F_a}$. Eq. (2) can then be rewritten as:

$$g_{ab} = \mathrm{Tr}(\boldsymbol{F_a}\boldsymbol{F_b})\,. \tag{4}$$

The quadratic Casimir operator, C_2, is defined by

$$C_2 \equiv g^{ab}\boldsymbol{T_a}\boldsymbol{T_b}\,. \tag{5}$$

It is easy to prove that

$$[C_2\,,\,\boldsymbol{T_a}] = 0\,, \qquad\qquad a = 1, 2, \ldots, d_G\,.$$

For a given representation of the Lie algebra $\mathfrak{g}$, the generators are represented by $d_R \times d_R$ matrices $\boldsymbol{R_a}$. By Schur's lemma, any operator that commutes with all the generators of $\mathfrak{g}$ in an *irreducible* representation must be a multiple of the identity operator. Thus, we shall write:

$$C_2(R) = g^{ab}\boldsymbol{R_a}\boldsymbol{R_b} = c_R\mathbb{1}\,, \tag{6}$$

where $\mathbb{1}$ is the $d_R \times d_R$ identity matrix, and c_R is a number that depends only on the representation R. The goal of this note is to compute c_R for any irreducible representation of a simple Lie group. In fact, we can immediately prove the following theorem.

Theorem 1: For the adjoint representation (denoted by $R = A$) of a simple Lie group, $c_A = 1$.

Proof: Using the explicit form for the adjoint representation generators given in eq. (3),

$$C_2(A)_c{}^e \equiv g^{ab}(\boldsymbol{F_a})_c{}^d(\boldsymbol{F_b})_d{}^e = g^{ab}f^d_{ac}f^e_{bd} = c_A\delta^e_c\,.$$

Multiplying both sides of the above equation by δ^c_e and summing over c and e,

$$d_G c_A = g^{ab}f^d_{ac}f^c_{bd} = g^{ab}g_{ab} = d_G\,,$$

and we immediately obtain $c_A = 1$.

II. Root vectors

We choose to work in the Cartan-Weyl basis of $\mathfrak{g}$, where the generators consist of $\{H_j\,,\,E_{\boldsymbol{\alpha}}\}$, which satisfy:

$$[H_j\,,\,H_k] = 0\,, \tag{7}$$

$$[H_j\,,\,E_{\boldsymbol{\alpha}}] = \alpha_j E_{\boldsymbol{\alpha}}\,, \tag{8}$$

$$[E_{\boldsymbol{\alpha}}\,,\,E_{-\boldsymbol{\alpha}}] = \alpha^j H_j\,, \qquad \alpha^j \equiv g^{jk}\alpha_k\,, \tag{9}$$

$$[E_{\boldsymbol{\alpha}}\,,\,E_{\boldsymbol{\beta}}] = \begin{cases} N_{\boldsymbol{\alpha\beta}}E_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\,, & \text{if } \boldsymbol{\alpha}+\boldsymbol{\beta} \text{ is a root and } \boldsymbol{\alpha}+\boldsymbol{\beta} \neq 0\,, \\ 0\,, & \text{if } \boldsymbol{\alpha}+\boldsymbol{\beta} \text{ is not a root}\,. \end{cases} \tag{10}$$

Here, $j = 1, 2, \ldots, \ell$ defines the *rank* ℓ of the Lie algebra (and corresponds to the maximal number of commuting generators), and the root-vectors are real ℓ-dimensional vectors $\boldsymbol{\alpha} = (\alpha_1\,,\,\alpha_2\,,\,\ldots\,,\,\alpha_\ell) \neq \mathbf{0}$ whose components are defined by eq. (8). The set of root vectors is denoted by Δ. Note that $\boldsymbol{\alpha} \in \Delta$ implies that $-\boldsymbol{\alpha} \in \Delta$ and $k\boldsymbol{\alpha} \notin \Delta$ if $k \neq \pm 1$.

The $\ell \times \ell$ block of the positive definite metric tensor is given by:

$$g_{ij} = \sum_{\boldsymbol{\alpha}\in\Delta} \alpha_i \alpha_j \,, \tag{11}$$

and the off-diagonal blocks, $g_{\boldsymbol{\alpha},j} = g_{j,\boldsymbol{\alpha}} = 0$. The inverse of g_{ij} (denoted below by g^{ij}) can be used to define inner products of two vectors that live in the ℓ-dimensional root vector space,

$$(\boldsymbol{\alpha}\,,\,\boldsymbol{\beta}) = g^{jk}\alpha_j\beta_k \,. \tag{12}$$

Note that eqs. (11) and (12) yield the following general formula,

$$(\boldsymbol{\beta}\,,\,\boldsymbol{\gamma}) = \sum_{\boldsymbol{\alpha}\in\Delta} (\boldsymbol{\alpha},\boldsymbol{\beta})(\boldsymbol{\alpha},\boldsymbol{\gamma})\,, \qquad \text{for } \boldsymbol{\beta},\,\boldsymbol{\gamma}\in\Delta. \tag{13}$$

It is convenient to choose the normalization of the generators $E_{\boldsymbol{\alpha}}$ of the Cartan-Weyl basis[1] such that $g_{\boldsymbol{\alpha},-\boldsymbol{\alpha}} = 1$. In this convention, one can show that:

$$|N_{\boldsymbol{\alpha}\boldsymbol{\beta}}|^2 = \tfrac{1}{2}(\boldsymbol{\alpha}\,,\,\boldsymbol{\alpha})q(p+1)\,, \qquad\qquad N_{-\boldsymbol{\alpha},-\boldsymbol{\beta}} = -N^*_{\boldsymbol{\alpha},\boldsymbol{\beta}}\,,$$

where the integers non-negative p and q are determined by the requirement that $\boldsymbol{\beta} + k\boldsymbol{\alpha}$ is a root vector for every integer k that satisfies $-p \leq k \leq q$. In particular,

$$p - q = \frac{2(\boldsymbol{\beta}\,,\,\boldsymbol{\alpha})}{(\boldsymbol{\alpha}\,,\,\boldsymbol{\alpha})}\,. \tag{14}$$

Conventionally, one chooses the phases of the $E_{\boldsymbol{\alpha}}$ such that the $N_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ are real.

One can introduce an ordering of the root vectors by defining $\boldsymbol{\alpha} > \boldsymbol{\beta}$ if the first non-zero component of $\boldsymbol{\alpha} - \boldsymbol{\beta}$ with respect to some fixed basis is positive. The roots can be divided up into two sets: the set of positive roots, denoted by Δ_+, and the set of negative roots, denoted by Δ_-. Note that the quadratic Casimir operator can be written in terms of the Cartan-Weyl basis as:

$$C_2 = \sum_{j=1}^{\ell} g^{ij}H_iH_j + \sum_{\boldsymbol{\alpha}\in\Delta_+} \left(E_{\boldsymbol{\alpha}}E_{-\boldsymbol{\alpha}} + E_{-\boldsymbol{\alpha}}E_{\boldsymbol{\alpha}}\right)\,. \tag{15}$$

Finally, we define the *simple* roots to be a positive root that cannot be expressed as a sum of two other positive roots. One can prove that there are precisely ℓ positive roots in a Lie algebra of rank ℓ. The set of simple roots is denoted by Π.

[1]More generally, eq. (9) is given by $[E_{\boldsymbol{\alpha}}\,,\,E_{-\boldsymbol{\alpha}}] = g_{\boldsymbol{\alpha},-\boldsymbol{\alpha}}\,g^{ij}\alpha_jH_i$. Since the normalization of the $E_{\boldsymbol{\alpha}}$ is not fixed by eq. (8), we are free to rescale the $E_{\boldsymbol{\alpha}}$ and $E_{-\boldsymbol{\alpha}}$ separately such that $g_{\boldsymbol{\alpha},-\boldsymbol{\alpha}} = 1$.

Theorem 2: If $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in \Pi$ and $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, then $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is not a root, and

$$(\boldsymbol{\alpha}\,,\,\boldsymbol{\beta}) \leq 0\,. \tag{16}$$

Proof: If $\boldsymbol{\alpha} - \boldsymbol{\beta} \in \Delta_+$, then $\boldsymbol{\alpha} = (\boldsymbol{\alpha} - \boldsymbol{\beta}) + \boldsymbol{\beta}$ shows that $\boldsymbol{\alpha}$ is the sum of two positive roots, which is impossible as $\boldsymbol{\alpha} \in \Pi$. Likewise, if $\boldsymbol{\beta} - \boldsymbol{\alpha} \in \Delta_+$, then $\boldsymbol{\beta} = (\boldsymbol{\beta} - \boldsymbol{\alpha}) + \boldsymbol{\alpha}$ shows that $\boldsymbol{\beta}$ is the sum of two positive roots, which is impossible as $\boldsymbol{\beta} \in \Pi$. Since $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, it follows that $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is not a root. This implies that $p = 0$ in eq. (14), and it follows that $(\boldsymbol{\alpha}\,,\,\boldsymbol{\beta}) \leq 0$.

It is convenient to introduce the $\ell \times \ell$ Cartan matrix A_{ij}, which is defined by,[2]

$$A_{ij} \equiv \frac{2(\boldsymbol{\alpha}_i\,,\,\boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_i\,,\,\boldsymbol{\alpha}_i)}\,, \tag{17}$$

where i and j label the simple roots. Note that $A_{ii} = 2$ and $A_{ij} \leq 0$ for $i \neq j$. There is a one-to-one correspondence between the possible Cartan matrices and the Dynkin diagrams that characterize the possible simple Lie groups.

Given the Cartan matrix, one can compute the inner product of any two simple roots as follows. First we note the following result obtained in refs. [4, 9],

$$(\boldsymbol{\alpha}_i\,,\,\boldsymbol{\alpha}_i) = \left[\tfrac{1}{2}\sum_{\boldsymbol{\beta}\in\Delta^+}\left\{\sum_{j=1}^{\ell} k_j^{\beta} A_{ij}\right\}^2\right]^{-1}\,, \qquad \boldsymbol{\alpha}_i \in \Pi\,. \tag{18}$$

where A_{ij} [defined in eq. (17)] depends on $\boldsymbol{\alpha}_i$, the positive root $\boldsymbol{\beta}$ has been expressed in terms of the simple roots via $\boldsymbol{\beta} = \sum_{j=1}^{\ell} k_j^{\beta}\boldsymbol{\alpha}_j$ and the k_j^{β} are nonnegative integers. It then follows from eq. (17) that $(\boldsymbol{\alpha}_i\,,\,\boldsymbol{\alpha}_j) = \frac{1}{2}A_{ij}(\boldsymbol{\alpha}_i\,,\,\boldsymbol{\alpha}_i)$.

We next introduce the Weyl reflection, which acts on a root vector as follows:

$$S_i(\boldsymbol{\alpha}) \equiv \boldsymbol{\alpha} - \frac{2(\boldsymbol{\alpha}\,,\,\boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i\,,\,\boldsymbol{\alpha}_i)}\boldsymbol{\alpha}_i\,, \qquad \boldsymbol{\alpha} \in \Delta \quad \text{and} \quad \boldsymbol{\alpha}_i \in \Pi\,.$$

Three immediate properties of S_i are:

$$S_i(\boldsymbol{\alpha}_i) = -\boldsymbol{\alpha}_i\,, \tag{19}$$

$$\big(S_i(\boldsymbol{\alpha})\,,\,\boldsymbol{\beta}\big) = \big(\boldsymbol{\alpha}\,,\,S_i(\boldsymbol{\beta})\big)\,, \tag{20}$$

$$\big(S_i(\boldsymbol{\alpha})\,,\,S_i(\boldsymbol{\alpha})\big) = (\boldsymbol{\alpha},\boldsymbol{\alpha})\,. \tag{21}$$

Additional properties of the Weyl reflection are summarized by the following theorem.

Theorem 3: If $\boldsymbol{\alpha} \in \Delta_+$ and $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}_i$ (for some simple root $\boldsymbol{\alpha}_i \in \Pi$), then $S_i(\boldsymbol{\alpha}) > 0$. Moreover, if $S_i(\boldsymbol{\alpha}) = S_i(\boldsymbol{\beta})$, then $\boldsymbol{\alpha} = \boldsymbol{\beta}$.

Proof: [6] Any positive root $\boldsymbol{\alpha} \in \Delta_+$ can be written as

$$\boldsymbol{\alpha} = k_i\boldsymbol{\alpha}_i + \sum_{j\neq i} k_j\boldsymbol{\alpha}_j\,, \qquad k_i \text{ and } k_j \text{ are nonnegative integers.}$$

[2]Some books define the expression given in eq. (17) to be the transpose of the Cartan matrix.

Then,

$$S_i(\boldsymbol{\alpha}) = \boldsymbol{\alpha} - \frac{2(\boldsymbol{\alpha}\,,\,\boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i\,,\,\boldsymbol{\alpha}_i)}\boldsymbol{\alpha}_i = -\boldsymbol{\alpha}_i\left[k_i + \sum_{j\neq i} k_j A_{ij}\right] + \sum_{j\neq i} k_j \boldsymbol{\alpha}_j\,.$$

Since $S_i(\boldsymbol{\alpha}) \in \Delta$, it follows that $S_i(\boldsymbol{\alpha})$ is either positive or negative. If $S_i(\boldsymbol{\alpha}) < 0$, then we must have $k_j = 0$ for $j \neq i$, in which case $\boldsymbol{\alpha} = \boldsymbol{\alpha}_i$ (i.e. $k_i = 1$) and we recover eq. (19). Hence if $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}_i$, it then follows that $S_i(\boldsymbol{\alpha}) > 0$. If $S_i(\boldsymbol{\alpha}) = S_i(\boldsymbol{\beta})$, then $\boldsymbol{\alpha} - \boldsymbol{\beta} = \kappa\boldsymbol{\alpha}_i$, where

$$\kappa = \frac{2(\boldsymbol{\alpha} - \boldsymbol{\beta}\,,\,\boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i\,,\,\boldsymbol{\alpha}_i)}\,.$$

Inserting $\boldsymbol{\alpha} - \boldsymbol{\beta} = \kappa\boldsymbol{\alpha}_i$ into the expression above yields $\kappa\boldsymbol{\alpha}_i = 2\kappa\boldsymbol{\alpha}_i$, and we conclude that $\kappa = 0$ or $\boldsymbol{\alpha} = \boldsymbol{\beta}$.

One consequence of Theorem 3 is that S_i maps the set of positive roots excluding $\boldsymbol{\alpha}_i$ into itself, where the map is one-to-one and onto. Thus, if we define the Weyl vector $\boldsymbol{\delta}$ to be half the sum of the positive roots,

$$\boldsymbol{\delta} \equiv \tfrac{1}{2} \sum_{\boldsymbol{\alpha}\in\Delta_+} \boldsymbol{\alpha}\,, \tag{22}$$

then using eq. (19),

$$S_i(\boldsymbol{\delta}) = \tfrac{1}{2}S_i\left(\boldsymbol{\alpha}_i + \sum_{j\neq i}\boldsymbol{\alpha}_j\right) = \tfrac{1}{2}\left(-\boldsymbol{\alpha}_i + \sum_{j\neq i}\boldsymbol{\alpha}_j\right) = \boldsymbol{\delta} - \boldsymbol{\alpha}_i\,. \tag{23}$$

Hence, eq. (20) yields,

$$(S_i(\boldsymbol{\delta})\,,\,\boldsymbol{\alpha}_i) = (\boldsymbol{\delta}\,,\,S_i(\boldsymbol{\alpha}_i))\;.$$

Using eqs. (19) and (23), it follows that

$$(\boldsymbol{\delta} - \boldsymbol{\alpha}_i\,,\,\boldsymbol{\alpha}_i) = -(\boldsymbol{\delta}\,,\,\boldsymbol{\alpha}_i)\,.$$

Rearranging the above result then yields:

$$\frac{2(\boldsymbol{\delta}\,,\,\boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i\,,\,\boldsymbol{\alpha}_i)} = 1\,, \qquad \text{for } \boldsymbol{\alpha}_i \in \Pi. \tag{24}$$

Finally, we introduce the dual root or co-root of $\boldsymbol{\alpha} \in \Delta$,

$$\boldsymbol{\alpha}^\vee \equiv \frac{2\boldsymbol{\alpha}}{(\boldsymbol{\alpha}\,,\,\boldsymbol{\alpha})}\,. \tag{25}$$

In terms of the dual root, the Cartan matrix can be defined as

$$A_{ij} = (\boldsymbol{\alpha}_i^\vee\,,\,\boldsymbol{\alpha}_j)\,,$$

and the Weyl reflection acts on a root vector as follows:

$$S_i(\boldsymbol{\alpha}) = \boldsymbol{\alpha} - (\boldsymbol{\alpha}\,,\,\boldsymbol{\alpha}_i^\vee)\boldsymbol{\alpha}_i\,.$$

Eq. (24) then can be rewritten as:

$$(\boldsymbol{\delta}\,,\,\boldsymbol{\alpha}_i^\vee) = 1\,, \qquad \text{for } \boldsymbol{\alpha}_i \in \Pi. \tag{26}$$

III. Irreducible representations and weights

In a unitary representation of a simple Lie algebra, the representation matrices of the Cartan-Weyl generators satisfy $H_j^\dagger = H_j$ and $E_{\boldsymbol{\alpha}}^\dagger = E_{-\boldsymbol{\alpha}}$. To construct a particular representation of $\mathfrak{g}$, one determines the basis vectors of the representation space, denoted collectively by $|\boldsymbol{m}\rangle$. These vectors are chosen to be the simultaneous eigenvectors of the commuting Hermitian generators H^j,

$$H_j\,|\boldsymbol{m}\rangle = m_j\,|\boldsymbol{m}\rangle\ .$$

The components of the real ℓ-dimensional vector $\boldsymbol{m} = (m_1\,,\,m_2\,,\,\ldots\,,\,m_\ell)$ are the corresponding eigenvalues of H_j. The ℓ-dimensional vector space in which the $\boldsymbol{m}$ reside is called the vector space of weight vectors. One can formulate a ordering of vectors of the weight space by introducing the rule that $\boldsymbol{m} > \boldsymbol{n}$ if the first non-zero component of $\boldsymbol{m}-\boldsymbol{n}$ is positive. An important theorem in Lie algebra representation theory states that for a given irreducible representation, the *highest* weight $|\boldsymbol{M}\rangle$ is non-degenerate and uniquely fixes the representation. Moreover,

$$E_{\boldsymbol{\alpha}}\,|\boldsymbol{M}\rangle = 0\,, \qquad \text{for all}\ \ \boldsymbol{\alpha}\in\Delta_+\,. \tag{27}$$

Given a conventional ordered list, $\{\boldsymbol{\alpha}_1\,,\,\boldsymbol{\alpha}_2\,,\,\ldots\,,\,\boldsymbol{\alpha}_\ell\}$, of the simple roots of $\mathfrak{g}$, one can define the following quantities,

$$n_i \equiv \frac{2(\boldsymbol{M}\,,\,\boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i\,,\,\boldsymbol{\alpha}_i)} = (\boldsymbol{M}\,,\,\boldsymbol{\alpha}_i^\vee)\,, \qquad\qquad i = 1,2,\ldots,\ell\,. \tag{28}$$

One can then prove that the n_i are non-negative integers. Thus, an irreducible representation can be identified by the ordered list $(n_1\,,\,n_2\,,\,\ldots\,,\,n_\ell)$, where the n_i are called the *Dynkin labels* of the irreducible representation. Since $\boldsymbol{M}$ is a vector that lives in an ℓ-dimensional space, it can be expanded in terms of the root vectors,

$$\boldsymbol{M} = \sum_{k=1}^{\ell} p_k \boldsymbol{\alpha}_k\,. \tag{29}$$

Inserting this expansion into eq. (28) and using eq. (17) yields

$$n_j = \sum_{k=1}^{\ell} A_{jk} p_k\,, \tag{30}$$

where the p_k are real and rational. Inverting this result gives:

$$p_k = \sum_{j=1}^{\ell} (A^{-1})_{kj} n_j\,. \tag{31}$$

IV. A general formula for c_R and the second-order index $I_2(R)$

In a representation R,

$$C_2(R)\,|\boldsymbol{M}\rangle = c_R\,|\boldsymbol{M}\rangle\ .$$

To compute c_R, we employ eq. (15) to obtain:

$$\begin{aligned}
C_2(R)\,|\boldsymbol{M}\rangle &= (\boldsymbol{M},\boldsymbol{M})\,|\boldsymbol{M}\rangle + \sum_{\boldsymbol{\alpha}\in\Delta_+} (E_{\boldsymbol{\alpha}}E_{-\boldsymbol{\alpha}} + E_{-\boldsymbol{\alpha}}E_{\boldsymbol{\alpha}})\,|\boldsymbol{M}\rangle \\
&= (\boldsymbol{M},\boldsymbol{M})\,|\boldsymbol{M}\rangle + \sum_{\boldsymbol{\alpha}\in\Delta_+} [E_{\boldsymbol{\alpha}}\,,\,E_{-\boldsymbol{\alpha}}]\,|\boldsymbol{M}\rangle \\
&= (\boldsymbol{M},\boldsymbol{M})\,|\boldsymbol{M}\rangle + \sum_{\boldsymbol{\alpha}\in\Delta_+} (\boldsymbol{\alpha},\boldsymbol{M})\,|\boldsymbol{M}\rangle\ .
\end{aligned}$$

In terms of $\boldsymbol{\delta} \equiv \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\Delta_+}\boldsymbol{\alpha}$, which is defined in eq. (22), we can write:

$$\boxed{C_2(R)\,|\boldsymbol{M}\rangle = (\boldsymbol{M}\ ,\ \boldsymbol{M}+2\boldsymbol{\delta})\,|\boldsymbol{M}\rangle} \tag{32}$$

That is,

$$c_R = (\boldsymbol{M}\ ,\ \boldsymbol{M}+2\boldsymbol{\delta})\,. \tag{33}$$

Using eq. (29),

$$\begin{aligned}
(\boldsymbol{M}\ ,\ \boldsymbol{M}+2\boldsymbol{\delta})\,|\boldsymbol{M}\rangle) &= \left(\sum_{k=1}^{\ell} p_k\boldsymbol{\alpha}_k\ ,\ \boldsymbol{M}+2\boldsymbol{\delta}\right) \\
&= \tfrac{1}{2}\sum_{k=1}^{\ell} p_k\left[(\boldsymbol{\alpha}_k\ ,\ \boldsymbol{\alpha}_k)\,(n_k+2)\right]\ .
\end{aligned}$$

after making use of eq. (24). Finally, inserting eq. (31) for p_k,

$$\boxed{c_R = \tfrac{1}{2}\sum_{j=1}^{\ell}\sum_{k=1}^{\ell}(\boldsymbol{\alpha}_k\ ,\ \boldsymbol{\alpha}_k)(n_k+2)(A^{-1})_{kj}n_j} \tag{34}$$

Eq. (34) is our basic result, which has also been obtained in ref. [17]. This is sometimes rewritten in terms of the symmetrized Cartan matrix, which is defined by [7]:

$$G_{ij} \equiv \frac{2}{(\boldsymbol{\alpha}_j\ ,\ \boldsymbol{\alpha}_j)}A_{ij} = \frac{4(\boldsymbol{\alpha}_i\ ,\ \boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_i\ ,\ \boldsymbol{\alpha}_i)(\boldsymbol{\alpha}_j\ ,\ \boldsymbol{\alpha}_j)} = (\boldsymbol{\alpha}_i^\vee\ ,\ \boldsymbol{\alpha}_j^\vee)\,. \tag{35}$$

The inverse of the symmetrized Cartan matrix, which we shall denote by G^{ij} is therefore given by:

$$G^{ij} = \tfrac{1}{2}(\boldsymbol{\alpha}_i\ ,\ \boldsymbol{\alpha}_i)A^{-1}_{ij}\,. \tag{36}$$

One can immediately check that $G_{ij}G^{jk} = \delta_i^k$ as required. Hence, eq. (34) can be rewritten as [19]:

$$c_R = \sum_{j=1}^{\ell}\sum_{k=1}^{\ell}(n_k+2)G^{kj}n_j\,. \tag{37}$$

For any irreducible representation,

$$\mathrm{Tr}(\boldsymbol{R_a}\boldsymbol{R_b}) = I_2(R)g_{ab}\,, \tag{38}$$

where $I_2(R)$ is called the second-order index of the representation R. By virtue of eq. (4), the second-order index of the adjoint representation is $I_2(A) = 1$. For an arbitrary irreducible representation R, taking the trace of eq. (6) yields:

$$\boxed{c_R = \frac{I_2(R)d_G}{d_R}} \tag{39}$$

where d_G is the dimension of the Lie algebra (which is also equal to the number of generators) and d_R is the dimension of the representation. For the adjoint representation ($R = A$), we have $d_R = d_G$, in which case we obtain the expected result,

$$c_A = I_2(A) = 1\,. \tag{40}$$

V. The quadratic Casimir operator and second-order index for irreducible representations of $\mathfrak{su}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$

We begin by listing the inverse Cartan matrices for $\mathfrak{su}(\ell+1)$, $\mathfrak{so}(2\ell)$, $\mathfrak{so}(2\ell+1)$ and $\mathfrak{sp}(\ell)$, where ℓ is the rank of the corresponding Lie algebras [4].

$\mathfrak{su}(\ell+1)\ \ (\ell \geq 1)$:

$$A^{-1} = \frac{1}{\ell+1}\begin{pmatrix} \ell & \ell-1 & \ell-2 & \ell-3 & \cdots & 3 & 2 & 1 \\ \ell-1 & 2(\ell-1) & 2(\ell-2) & 2(\ell-3) & \cdots & 6 & 4 & 2 \\ \ell-2 & 2(\ell-2) & 2(\ell-2) & 3(\ell-3) & \cdots & 9 & 6 & 3 \\ \ell-3 & 2(\ell-3) & 3(\ell-3) & 4(\ell-3) & \cdots & 12 & 8 & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 3 & 6 & 9 & 12 & \cdots & 3(\ell-2) & 2(\ell-2) & \ell-2 \\ 2 & 4 & 6 & 8 & \cdots & 2(\ell-2) & 2(\ell-1) & \ell-1 \\ 1 & 2 & 3 & 4 & \cdots & \ell-2 & \ell-1 & \ell \end{pmatrix},$$

$$\mathfrak{so}(2\ell+1)\ \ (\ell \geq 4): \qquad A^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & \cdots & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & \cdots & 4 & 4 & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & \ell-2 & \ell-2 & \frac{1}{2}(\ell-2) \\ 1 & 2 & 3 & 4 & \cdots & \ell-2 & \ell-1 & \frac{1}{2}(\ell-1) \\ 1 & 2 & 3 & 4 & \cdots & \ell-2 & \ell-1 & \frac{1}{2}\ell \end{pmatrix},$$

$$\mathfrak{so}(2\ell)\ \ (\ell \geq 4): \qquad A^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & 2 & \cdots & 2 & 1 & 1 \\ 1 & 2 & 3 & 3 & \cdots & 3 & \frac{3}{2} & \frac{3}{2} \\ 1 & 2 & 3 & 4 & \cdots & 4 & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & \ell-2 & \frac{1}{2}(\ell-2) & \frac{1}{2}(\ell-2) \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \cdots & \frac{1}{2}(\ell-2) & \frac{1}{4}\ell & \frac{1}{4}(\ell-2) \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \cdots & \frac{1}{2}(\ell-2) & \frac{1}{4}(\ell-2) & \frac{1}{4}\ell \end{pmatrix},$$

$$\mathfrak{sp}(\ell)\ \ (\ell \geq 4): \qquad A^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & \cdots & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & \cdots & 4 & 4 & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & \ell-2 & \ell-2 & \ell-2 \\ 1 & 2 & 3 & 4 & \cdots & \ell-2 & \ell-1 & \ell-1 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \cdots & \frac{1}{2}(\ell-2) & \frac{1}{2}(\ell-1) & \frac{1}{2}\ell \end{pmatrix}.$$

For the cases of $\ell = 2$ and $\ell = 3$, we have:

$$\mathfrak{sp}(2): \quad A^{-1} = \begin{pmatrix} 1 & 1 \\ \tfrac{1}{2} & 1 \end{pmatrix}, \qquad \mathfrak{so}(4): \quad A^{-1} = \begin{pmatrix} \tfrac{1}{2} & 0 \\ 0 & \tfrac{1}{2} \end{pmatrix}, \qquad \mathfrak{so}(5): \quad A^{-1} = \begin{pmatrix} 1 & \tfrac{1}{2} \\ 1 & 1 \end{pmatrix},$$

$$\mathfrak{sp}(3): \quad A^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ \tfrac{1}{2} & 1 & \tfrac{3}{2} \end{pmatrix}, \quad \mathfrak{so}(6): \quad A^{-1} = \begin{pmatrix} 1 & \tfrac{1}{2} & \tfrac{1}{2} \\ \tfrac{1}{2} & \tfrac{3}{4} & \tfrac{1}{4} \\ \tfrac{1}{2} & \tfrac{1}{4} & \tfrac{3}{4} \end{pmatrix}, \quad \mathfrak{so}(7): \quad A^{-1} = \begin{pmatrix} 1 & 1 & \tfrac{1}{2} \\ 1 & 2 & 1 \\ 1 & 2 & \tfrac{3}{2} \end{pmatrix}.$$

We also need the length of each simple root. In the Cartan-Weyl basis introduced above, the length of each simple root is fixed according to eq. (18). These can be evaluated explicitly, and the final results are given by [4]:

$$\mathfrak{su}(\ell+1): \quad (\boldsymbol{\alpha}_k\,,\,\boldsymbol{\alpha}_k) = \frac{1}{\ell+1}\,, \qquad k = 1, 2, \ldots, \ell\,, \tag{41}$$

$$\mathfrak{so}(2\ell+1): \quad (\boldsymbol{\alpha}_k\,,\,\boldsymbol{\alpha}_k) = \begin{cases} \dfrac{1}{2\ell-1}\,, & \text{for } \; k = 1, 2, \ldots, \ell-1\,, \\ \dfrac{1}{2(2\ell-1)}\,, & \text{for } \; k = \ell\,, \end{cases} \tag{42}$$

$$\mathfrak{so}(2\ell): \quad (\boldsymbol{\alpha}_k\,,\,\boldsymbol{\alpha}_k) = \frac{1}{2(\ell-1)}\,, \qquad \text{for } k = 1, 2, \ldots, \ell\,, \quad (\ell \neq 1)\,, \tag{43}$$

$$\mathfrak{sp}(\ell): \quad (\boldsymbol{\alpha}_k\,,\,\boldsymbol{\alpha}_k) = \begin{cases} \dfrac{1}{2(\ell+1)}\,, & \text{for } \; k = 1, 2, \ldots, \ell-1\,, \\ \dfrac{1}{\ell+1}\,, & \text{for } \; k = \ell\,, \end{cases} \tag{44}$$

Finally, we need to identify specific irreducible representations of $\mathfrak{su}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$ We define the fundamental (or defining) representation of the corresponding groups to be the n-dimensional matrix representation that defines the groups $\mathrm{SU}(n)$ and $\mathrm{SO}(n)$, respectively, and the $2n$-dimensional representation that defines the group $\mathrm{Sp}(n)$.[3] In terms of the Dynkin labels, $\boldsymbol{n} \equiv (n_1, n_2, \ldots, n_\ell)$, the fundamental representations are given by:

$$\boldsymbol{n} = (1, 0, 0, \ldots, 0)\,,$$

for $\mathfrak{su}(\ell+1)$ (for $\ell \geq 1$), $\mathfrak{so}(2\ell+1)$ (for $\ell \geq 2$), $\mathfrak{so}(2\ell)$ (for $\ell \geq 3$), and $\mathfrak{sp}(\ell)$ (for $\ell \geq 1$). For the case of $\mathfrak{so}(3)$, $\boldsymbol{n} = 2$ for the fundamental three-dimensional representation (since $\boldsymbol{n} = 1$ is the two-dimensional spinor representation). For the case of $\mathfrak{so}(4)$, $\boldsymbol{n} = (1, 1)$ for

[3]The reader is warned that what we call $\mathrm{Sp}(n)$ is often called $\mathrm{Sp}(2n)$ in the literature.

the fundamental four-dimensional representation (since $\boldsymbol{n}=(1,0)$ and $\boldsymbol{n}=(0,1)$ are two inequivalent two-dimensional spinor representations).

Eq. (34) then yields:

$$\mathfrak{su}(\ell+1)\;:\quad c_F=\frac{1}{2(\ell+1)}\left[(A^{-1})_{11}+2\sum_{k=1}^{\ell}(A^{-1})_{k1}\right]=\frac{\ell(\ell+2)}{2(\ell+1)^2}\,,\qquad \ell\geq 1\,,$$

$$\mathfrak{so}(2\ell+1)\;:\quad c_F=\frac{1}{2(2\ell-1)}\left[(A^{-1})_{11}+2\sum_{k=1}^{\ell-1}(A^{-1})_{k1}+(A^{-1})_{\ell 1}\right]=\frac{\ell}{2\ell-1}\,,\qquad \ell\geq 2\,,$$

$$\mathfrak{so}(2\ell)\;:\quad c_F=\frac{1}{4(\ell-1)}\left[(A^{-1})_{11}+2\sum_{k=1}^{\ell}(A^{-1})_{k1}\right]=\frac{2\ell-1}{4(\ell-1)}\,,\qquad \ell\geq 3\,,$$

$$\mathfrak{sp}(\ell)\;:\quad c_F=\frac{1}{4(\ell+1)}\left[(A^{-1})_{11}+2\sum_{k=1}^{\ell-1}(A^{-1})_{k1}+4(A^{-1})_{\ell 1}\right]=\frac{2\ell+1}{4(\ell+1)}\,,\qquad \ell\geq 1\,.$$

The above results can be rewritten as:

$$\mathfrak{su}(n)\;:\quad c_F=\frac{n^2-1}{2n^2}\,,\qquad (n\geq 2)\,,\tag{45}$$

$$\mathfrak{so}(n)\;:\quad c_F=\frac{n-1}{2(n-2)}\,,\qquad (n\geq 5)\,,\tag{46}$$

$$\mathfrak{sp}(n)\;:\quad c_F=\frac{2n+1}{4(n+1)}\,,\qquad (n\geq 5)\,.\tag{47}$$

We note that the dimensions of the fundamental representations (d_F) and the adjoint representations (d_G) [the latter is equal to the number of generators] of the simple classical Lie algebras are given by:

$$\mathfrak{su}(n)\;:\quad d_F=n\,,\qquad d_G=n^2-1\,,\tag{48}$$

$$\mathfrak{so}(n)\;:\quad d_F=n\,,\qquad d_G=\tfrac{1}{2}n(n-1)\,,\tag{49}$$

$$\mathfrak{sp}(n)\;:\quad d_F=2n\,,\qquad d_G=n(2n+1)\,.\tag{50}$$

Using eq. (39), one obtains the second-order index of the fundamental representation:

$$\mathfrak{su}(n)\;:\quad I_2(F)=\frac{1}{2n}\,,\qquad (n\geq 2)\,,\tag{51}$$

$$\mathfrak{so}(n)\;:\quad I_2(F)=\frac{1}{n-2}\,,\qquad (n\geq 5)\,,\tag{52}$$

$$\mathfrak{sp}(n)\;:\quad I_2(F)=\frac{1}{2(n+1)}\,,\qquad (n\geq 1)\,.\tag{53}$$

We now examine the adjoint representation and check that Theorem 1 is satisfied. The Dynkin labels of the adjoint representation are given by:

$$\mathfrak{su}(n): \quad \boldsymbol{n} = (1\,,0\,,0\,,\ldots\,,0\,,0\,,1)\,, \qquad (n \geq 3)\,, \tag{54}$$

$$\mathfrak{so}(n): \quad \boldsymbol{n} = (0\,,1\,,0\,,0\,,\ldots\,,0\,,0)\,, \qquad (n \geq 5)\,. \tag{55}$$

$$\mathfrak{sp}(n): \quad \boldsymbol{n} = (2\,,0\,,0\,,0\,,\ldots\,,0\,,0)\,, \qquad (n \geq 1)\,. \tag{56}$$

For $\mathfrak{su}(2)$, the adjoint representation is given by $\boldsymbol{n} = 2$. For $\mathfrak{so}(n)$, the adjoint representation corresponds to the antisymmetric part of the Kronecker product of $n \otimes n$. However, the cases of $n \leq 6$ must be treated separately, as $\boldsymbol{n} = (0, 1)$ is a spinor representation of $\mathfrak{so}(4)$ and of $\mathfrak{so}(5)$, whereas $\boldsymbol{n} = (0, 1, 0)$ is a spinor representation of $\mathfrak{so}(6)$.[4] For $\mathfrak{so}(3)$, the fundamental and adjoint representations coincide and correspond to $\boldsymbol{n} = 2$. For $\mathfrak{so}(4)$, which is semisimple, the adjoint representation is not irreducible. For $\mathfrak{so}(5)$, the adjoint representation is given by $\boldsymbol{n} = (0, 2)$. For $\mathfrak{so}(6)$, the adjoint representation is given by $\boldsymbol{n} = (0, 1, 1)$.

We now evaluate the quadratic Casimir operator using eq. (34).

$$\mathfrak{su}(\ell+1): \quad c_A = \frac{1}{2(\ell+1)}\left[(A^{-1})_{11} + (A^{-1})_{\ell 1} + (A^{-1})_{1\ell} + (A^{-1})_{\ell\ell} + 2\sum_{k=1}^{\ell}[(A^{-1})_{k1} + (A^{-1})_{k\ell}]\right]$$
$$= 1\,, \qquad \ell \geq 2\,,$$

$$\mathfrak{so}(2\ell+1): \quad c_A = \frac{1}{2(2\ell-1)}\left[(A^{-1})_{22} + 2\sum_{k=1}^{\ell-1}(A^{-1})_{k2} + (A^{-1})_{\ell 2}\right] = 1\,, \qquad \ell \geq 3\,.$$

$$\mathfrak{so}(2\ell): \quad c_A = \frac{1}{4(\ell-1)}\left[(A^{-1})_{22} + 2\sum_{k=1}^{\ell}(A^{-1})_{k2}\right] = 1\,, \qquad \ell \geq 4\,.$$

$$\mathfrak{sp}(\ell): \quad c_A = \frac{1}{\ell+1}\left[(A^{-1})_{11} + \sum_{k=1}^{\ell-1}(A^{-1})_{k1} + 2(A^{-1})_{\ell 1}\right] = 1\,, \qquad \ell \geq 1\,.$$

I have also checked that the cases of $\mathfrak{su}(2)$, $\mathfrak{so}(3)$, $\mathfrak{so}(5)$ and $\mathfrak{so}(6)$ yield $c_A = 1$.

VI. The dual Coxeter number

We now introduce the maximal weight of the adjoint representation, denoted by $\boldsymbol{\theta}$, which also coincides with the highest positive root [7]. It is well known that for a simple Lie algebra, there are at most two roots of different length, called long roots and short roots, respectively.[5] Furthermore, one can prove that $(\boldsymbol{\theta}, \boldsymbol{\theta}) \geq (\boldsymbol{\alpha}, \boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \Delta$,

[4] In general, for n odd there is one fundamental irreducible spinor representation of $\mathfrak{so}(n)$ given by $\boldsymbol{n} = (0\,,0\,,\ldots\,,0\,,1)$. For n even there are two fundamental irreducible spinor representations of $\mathfrak{so}(n)$ given by $\boldsymbol{n} = (0\,,0\,,\ldots\,,0\,,1\,,0)$ and $\boldsymbol{n} = (0\,,0\,,\ldots\,,0\,,0\,,1)$.

[5] Roots are conventionally called long in cases where all roots are of the same length.

which implies that $\boldsymbol{\theta}$ must be a long root. One can expand $\boldsymbol{\theta}$ in terms of the simple roots with positive integer coefficients,

$$\boldsymbol{\theta} = \sum_{k=1}^{\ell} a_k \boldsymbol{\alpha}_k . \tag{57}$$

The *Coxeter number* of a simple Lie algebra is defined as [7]:

$$h \equiv 1 + \sum_{k=1}^{\ell} a_k \,.$$

Likewise, one can expand $\boldsymbol{\theta}^\vee \equiv 2\boldsymbol{\theta}/(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})$ in terms of the dual roots,

$$\boldsymbol{\theta}^\vee = \sum_{k=1}^{\ell} a_k^\vee \boldsymbol{\alpha}_k^\vee \,, \tag{58}$$

where

$$a_k^\vee = \frac{(\boldsymbol{\alpha}_k\,,\boldsymbol{\alpha}_k)}{(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})} a_k \,, \tag{59}$$

after using eqs. (25) and (57). The *dual Coxeter number* is then defined as [7]:

$$g \equiv 1 + \sum_{k=1}^{\ell} a_k^\vee \,. \tag{60}$$

For a simply laced Lie algebra (defined as a simple Lie algebra whose roots are all of equal length), we have $g = h$.

The dual Coxeter number can be related to the eigenvalue of the quadratic Casimir operator in the adjoint representation, c_A, as follows. Taking the inner product of the Weyl vector with $\boldsymbol{\theta}^\vee$ using eqs. (22) and (58), and employing eq. (26), it follows that

$$(\boldsymbol{\delta}, \boldsymbol{\theta}^\vee) = \sum_{k=1}^{\ell} a_k^\vee (\boldsymbol{\delta}, \boldsymbol{\alpha}_k^\vee) = \sum_{k=1}^{\ell} a_k^\vee \,. \tag{61}$$

Since $\boldsymbol{\theta}$ is the maximal weight of the adjoint representation, eqs. (60) and (61) yield,

$$g = 1 + (\boldsymbol{\delta}, \boldsymbol{\theta}^\vee) = 1 + \frac{(\boldsymbol{\delta}, 2\boldsymbol{\theta})}{(\boldsymbol{\theta}, \boldsymbol{\theta})} = \frac{(\boldsymbol{\theta} + 2\boldsymbol{\delta}, \boldsymbol{\theta})}{(\boldsymbol{\theta}, \boldsymbol{\theta})} = \frac{c_A}{(\boldsymbol{\theta}, \boldsymbol{\theta})} \,, \tag{62}$$

after using eq. (33) and the symmetry property of the inner product. Since $c_A = 1$, we conclude that:

$$g = \frac{1}{(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})} = \begin{cases} n\,, & \text{for } \mathfrak{su}(n)\,, & (n \geq 2)\,, \\ 2\,, & \text{for } \mathfrak{so}(3)\,, & \\ n-2\,, & \text{for } \mathfrak{so}(n)\,, & (n \geq 4)\,, \\ n+1\,, & \text{for } \mathfrak{sp}(n)\,, & (n \geq 1)\,, \end{cases} \tag{63}$$

after using eqs. (41)–(44) for the length of the long root.

It is instructive to rederive eq. (62) as follows. In light of eqs. (59) and (60),

$$g = 1 + \frac{1}{(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})}\sum_{k=1}^{\ell}(\boldsymbol{\alpha}_k\,,\,\boldsymbol{\alpha}_k)a_k\,. \tag{64}$$

The Dynkin labels for $\boldsymbol{\theta}$,

$$n_j^\theta \equiv \frac{2(\boldsymbol{\theta}\,,\,\boldsymbol{\alpha}_j)}{(\boldsymbol{\alpha}_j\,,\,\boldsymbol{\alpha}_j)}\,,$$

are given explicitly in eqs. (54)–(56) for $\mathfrak{su}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$, respectively. Following eqs. (30) and (31), we can write

$$n_k^\theta = \sum_{j=1}^{\ell} A_{kj}a_j\,, \qquad\qquad a_k = \sum_{j=1}^{\ell}(A^{-1})_{kj}n_j^\theta\,. \tag{65}$$

It follows that:

$$\begin{aligned}(\boldsymbol{\theta}\,,\,\boldsymbol{\theta}) &= \sum_{j=1}^{\ell}\sum_{k=1}^{\ell} a_j a_k(\boldsymbol{\alpha}_k\,,\,\boldsymbol{\alpha}_j) = \tfrac{1}{2}\sum_{j=1}^{\ell}\sum_{k=1}^{\ell} a_j a_k(\boldsymbol{\alpha}_k\,,\,\boldsymbol{\alpha}_k)A_{kj} \\ &= \tfrac{1}{2}\sum_{k=1}^{\ell} a_k(\boldsymbol{\alpha}_k\,,\,\boldsymbol{\alpha}_k)n_k^\theta = \tfrac{1}{2}\sum_{j=1}^{\ell}\sum_{k=1}^{\ell}(\boldsymbol{\alpha}_k\,,\,\boldsymbol{\alpha}_k)n_k^\theta(A^{-1})_{kj}n_j^\theta\,.\end{aligned}$$

Using eqs. (64) and (65), the dual Coxeter number can be rewritten as:

$$g = 1 + \frac{1}{(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})}\sum_{j=1}^{\ell}\sum_{k=1}^{\ell}(\boldsymbol{\alpha}_k\,,\,\boldsymbol{\alpha}_k)(A^{-1})_{kj}n_j^\theta\,.$$

Hence, eq. (34) yields,

$$c_A = \tfrac{1}{2}\sum_{j=1}^{\ell}\sum_{k=1}^{\ell}(\boldsymbol{\alpha}_k\,,\,\boldsymbol{\alpha}_k)(n_k^\theta + 2)(A^{-1})_{kj}n_j^\theta = (\boldsymbol{\theta}\,,\,\boldsymbol{\theta}) + (g-1)(\boldsymbol{\theta}\,,\,\boldsymbol{\theta}) = g(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})\,,$$

in agreement with the result of eq. (62).

The second-order index and the eigenvalue of the Casimir operator in the fundamental representation are related to the dual Coxeter number by using eqs. (51)–(53),

$$\mathfrak{su}(n)\,: \qquad I_2(F) = \frac{1}{2g}\,, \qquad\qquad (n \geq 2)\,, \tag{66}$$

$$\mathfrak{so}(n)\,: \qquad I_2(F) = \frac{1}{g}\,, \qquad\qquad (n \geq 5)\,, \tag{67}$$

$$\mathfrak{sp}(n)\,: \qquad I_2(F) = \frac{1}{2g}\,, \qquad\qquad (n \geq 1)\,, \tag{68}$$

and eq. (39) then yields,

$$c_F = \frac{g_G I_2(F)}{d_F}\,.$$

For completeness, we provide an explicit computation of $(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})$. Multiplying eq. (11) by g^{ij} and summing over i and j yields [20],

$$\sum_{\boldsymbol{\alpha}\in\Delta}(\boldsymbol{\alpha}\,,\,\boldsymbol{\alpha}) = \ell\,, \tag{69}$$

where ℓ is the rank of the group. This result can be used to compute $(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})$ as follows. All roots of a simply laced Lie algebra are of equal length. By definition, the maximal root $\boldsymbol{\theta}$ is regarded as a long root. Since there are $d_G-\ell$ non-zero roots, it follows from eq. (69) that $\ell=(d_G-\ell)(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})$, or

$$(\boldsymbol{\theta}\,,\,\boldsymbol{\theta}) = \frac{\ell}{d_G-\ell}\,,\qquad \text{for}\;\; \mathfrak{g}=\mathfrak{su}(\ell+1)\;\;[\ell\geq 1]\;\;\text{and}\;\;\mathfrak{so}(2\ell)\;\;[\ell\geq 2]\,.$$

For $\mathfrak{so}(2\ell+1)$ $[\ell\geq 2]$, there are $\ell-1$ long roots and one short root. We use Weyl reflections to generate the remaining roots, which results in $(\ell-1)(d_G-\ell)/\ell$ long roots and $(d_G-\ell)/\ell$ short roots. For $\mathfrak{sp}(\ell)$ $[\ell\geq 1]$, there is one long root and $\ell-1$ short roots. We use Weyl reflections to generate the remaining roots, which results in $(d_G-\ell)/\ell$ long roots and $(\ell-1)(d_G-\ell)/\ell$ short roots. Hence, for $\mathfrak{so}(2\ell+1)$ $[\ell\geq 2]$, eq. (69) yields:

$$\ell = \left[\frac{(\ell-1)(d_G-\ell)}{\ell}+\frac{d_G-\ell}{2\ell}\right](\boldsymbol{\theta}\,,\,\boldsymbol{\theta}) = \frac{(2\ell-1)(d_G-\ell)}{2\ell}(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})\,,$$

and for $\mathfrak{sp}(\ell)$ $[\ell\geq 1]$, eq. (69) yields:

$$\ell = \left[\frac{d_G-\ell}{\ell}+\frac{(\ell-1)(d_G-\ell)}{2\ell}\right](\boldsymbol{\theta}\,,\,\boldsymbol{\theta}) = \frac{(\ell+1)(d_G-\ell)}{2\ell}(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})\,,$$

after using the known fact that for $\mathfrak{so}(2\ell+1)$ $[\ell\geq 2]$ and $\mathfrak{sp}(\ell)$ $[\ell\geq 1]$, the length of the short roots is half of the length of the long roots (the latter includes the $\boldsymbol{\theta}$). Therefore,

$$(\boldsymbol{\theta}\,,\,\boldsymbol{\theta}) = \frac{2\ell^2}{d_G-\ell}\times\begin{cases}\dfrac{1}{2\ell-1}\,, & \text{for}\;\;\mathfrak{g}=\mathfrak{so}(2\ell+1)\,, \qquad (\ell\geq 2)\,,\\[2ex] \dfrac{1}{\ell+1}\,, & \text{for}\;\;\mathfrak{g}=\mathfrak{sp}(\ell)\,, \qquad (\ell\geq 1)\,.\end{cases}$$

Using eqs. (48)–(50), we end up with:

$$(\boldsymbol{\theta}\,,\,\boldsymbol{\theta}) = \begin{cases}\dfrac{1}{n}\,, & \text{for}\;\mathfrak{su}(n)\,, \qquad (n\geq 2)\,,\\[2ex] \dfrac{1}{2}\,, & \text{for}\;\mathfrak{so}(3)\,,\\[2ex] \dfrac{1}{n-2}\,, & \text{for}\;\mathfrak{so}(n)\,, \qquad (n\geq 4)\,,\\[2ex] \dfrac{1}{n+1}\,, & \text{for}\;\mathfrak{sp}(n)\,, \qquad (n\geq 1)\,,\end{cases}$$

in agreement with eq. (63).

Additional properties of the Coxeter number and the dual Coxeter number can be found in ref. [21].

VII. The Strange Formula

For completeness, I shall record a formula first obtained by Freudenthal and de Vries [22]. This is a formula for the length of the Weyl vector [cf. eq. (22)],

$$(\boldsymbol{\delta}, \boldsymbol{\delta}) = \tfrac{1}{24} d_G \,, \tag{70}$$

where d_G is the dimension of the Lie algebra. This remarkable formula can be verified explicitly for all the simple Lie algebras. Elementary proofs of eq. (70), known as the *strange formula*, can be found in refs. [23–25].

VIII. An alternative normalization convention

We highlight two implicit normalization conditions employed in this note. First, $g_{ab} = \mathrm{Tr}(\boldsymbol{F_a F_b})$ defines the Killing metric, which is normalized by a coefficient of 1. Second, the roots are normalized by

$$\sum_{\boldsymbol{\alpha}} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_j = g_{ij} \,.$$

It is convenient to alter these conventions as follows. First, we redefine [7,16]

$$g_{ab} = \frac{1}{g\eta} \mathrm{Tr}(\boldsymbol{F_a F_b}) \,, \tag{71}$$

where g is the dual Coxeter number and η is an additional rescaling factor. In order to be consistent with eq. (11), we shall simultaneously rescale the roots so that

$$\frac{1}{g\eta} \sum_{\boldsymbol{\alpha}} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_j = g_{ij} \,. \tag{72}$$

Multiplying by g^{ij} and summing over i and j yields:

$$\sum_{\boldsymbol{\alpha} \in \Delta} (\boldsymbol{\alpha} \,, \, \boldsymbol{\alpha}) = g\eta\ell \,,$$

which replaces eq. (69) and fixes the length of the root vectors. We can identify:

$$\eta = (\boldsymbol{\theta} \,, \, \boldsymbol{\theta}) \,,$$

since $g\eta = g(\boldsymbol{\theta} \,, \, \boldsymbol{\theta}) = 1$ returns us to our previous conventions [cf. eq. (63)].

This rescaling can be viewed in two equivalent ways. As presented above, it can be viewed simply as a rescaling of the definition of the Killing metric. Note that in this interpretation, the eigenvalue of the Casimir operator and the second-order index are independent of the choice of basis for the generators, since the definitions given by eqs. (5) and (38) are covariant with respect to their indices. That is, rescaling the Lie algebra generators automatically rescales the Killing metric, leaving the eigenvalue of the Casimir operator and the second-order index invariant. However, we can also view eq. (71) as a rescaling of the definition of the Lie algebra generators, with g_{ab} held fixed. In practice, one typically chooses the basis for the Lie algebra generators such that $g_{ab} = \delta_{ab}$, where the coefficient in front of the Kronecker delta is held fixed at 1. In this interpretation, the eigenvalue of the Casimir operator and the second-order index depend on the normalization of the Lie algebra generators.[6] Of course, both interpretations are equally valid.

The symmetrized Cartan matrix defined in eq. (35) also depends on the overall scale of the roots. However, we shall simply redefine it as:

$$G_{ij} = \frac{(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})}{(\boldsymbol{\alpha}_j\,,\,\boldsymbol{\alpha}_j)} A_{ij}\,. \tag{73}$$

Note that eq. (73) is independent of the convention for the normalization of the length of the roots. The inverse of the redefined symmetrized Cartan matrix is given by

$$G^{ij} = \frac{(\boldsymbol{\alpha}_i\,,\,\boldsymbol{\alpha}_i)}{(\boldsymbol{\theta}\,,\,\boldsymbol{\theta})} A^{-1}_{ij}\,.$$

This matrix is called the quadratic form matrix in ref. [7]. The explicit forms of the G^{ij} for the simple Lie groups are given in refs. [3, 7].

As noted above, the eigenvalue of the quadratic Casimir operator and the second-order index are rescaled by ηg, and we shall denote the corresponding rescaled quantities by capital letters,

$$C_R \equiv \eta g c_R\,, \qquad\qquad T_R \equiv \eta g I_2(R)\,. \tag{74}$$

In particular, eq. (40) implies that:

$$C_A = T_A = \eta g\,. \tag{75}$$

If we multiply eq. (37) by ηg, and rescale G^{-1} as indicated above, we obtain [16]:

$$C_R = \eta g c_R = \tfrac{1}{2}\eta \sum_{j=1}^{\ell}\sum_{k=1}^{\ell}(a_i+2)G^{ij}a_j\,, \tag{76}$$

where eq. (63) has been used to convert to the new definition of G^{-1}.

[6] In ref. [17], this viewpoint is described on the top of p. 302 as follows. "If all generators in a given simple Lie algebra are multiplied with a common factor λ, the structure constants f^c_{ab} are multiplied with λ and the Killing form is multiplied with λ^2. For convenience, the inner product in the root space [cf. eq. (12)] is redefined to be Euclidean again, namely, the metric tensor in the root space is δ^{ij} instead of g^{ij}."

Finally, we note that the strange formula of Freudenthal and de Vries [cf. eq. (70)] is also rescaled by ηg,

$$(\boldsymbol{\delta}, \boldsymbol{\delta}) = \tfrac{1}{24}\eta g d_G \,. \tag{77}$$

It is often convenient to choose the squared-length of the longest root to be equal to 2 (see, e.g., Ref. [16]). That is,[7]

$$\eta \equiv (\boldsymbol{\theta}\,, \boldsymbol{\theta}) = 2\,. \tag{78}$$

In this convention, our original definition of the symmetrized Cartan matrix defined in eq. (35) and the rescaled version defined in eq. (73) are of the same form. Consequently, when $\eta = 2$, the form of eqs. (37) and (76) coincide since both c_R and G^{ij} scale in the same way.

As a consequence of eqs. (39), (74) and (75),

$$C_F = \frac{T_F d_G}{d_F}\,, \qquad\qquad C_A = T_A = \frac{T_F}{I_2(F)}\,.$$

It then follows that:

$$\mathfrak{su}(n):\quad C_F = T_F\left(\frac{n^2-1}{n}\right), \qquad C_A = T_A = 2nT_F\,, \qquad (n \geq 2)\,, \tag{79}$$

$$\mathfrak{so}(n):\quad C_F = \tfrac{1}{2}T_F(n-1)\,, \qquad C_A = T_A = T_F(n-2)\,, \qquad (n \geq 5)\,. \tag{80}$$

$$\mathfrak{sp}(n):\quad C_F = \tfrac{1}{2}T_F(2n+1)\,, \qquad C_A = T_A = 2T_F(n+1)\,, \qquad (n \geq 1)\,. \tag{81}$$

Comparing the above results with eqs. (63) and (75), it follows that the normalization of the Lie algebra generators are fixed according to [19]:

$$\mathfrak{su}(n):\qquad T_F = \tfrac{1}{2}\eta\,, \qquad (n \geq 2)\,,$$

$$\mathfrak{so}(n):\qquad T_F = \eta\,, \qquad (n \geq 5)\,,$$

$$\mathfrak{sp}(n):\qquad T_F = \tfrac{1}{2}\eta\,, \qquad (n \geq 1)\,.$$

Of course, the above results are consistent with eqs. (66)–(68), in light of eq. (74).

As noted in eq. (78) and in footnote 7, $\eta = 2$ is the common choice in the mathematics literature. In contrast, $\eta = 1$ is more typically employed in the physics literature, especially in the case of the $\mathfrak{su}(n)$ Lie algebra. Although a universal choice for η is desirable, it is not required. As a result, it is not uncommon to see different conventions for η applied to different simple Lie algebras [19]. For example, the results of eqs. (79)–(81) agree with Table 3 of ref. [5], where $T_F = 1$ has been taken for *all* $\mathfrak{su}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$ generators. This choice requires a different choice of η for $\mathfrak{so}(n)$ as compared to $\mathfrak{su}(n)$ and $\mathfrak{sp}(n)$. It is also common for physicists to choose $T_F = \frac{1}{2}$ for $\mathfrak{su}(n)$ and $\mathfrak{sp}(n)$ and $T_F = 2$ for $\mathfrak{so}(n)$, which again requires a different choice of η for $\mathfrak{so}(n)$ as compared to $\mathfrak{su}(n)$ and $\mathfrak{sp}(n)$.

[7] This convention is common in the mathematics literature. It is motivated by the observation that in this convention, $I_2(R)$ is always an integer.

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