Analytic formulae for the Feynman propagator in coordinate space

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April 17, 2020

Abstract

In these notes, we provide an explicit calculation of the Feynman propagator of scalar field theory in coordinate space in four spacetime dimensions. Two different methods of the computation are provided. The methods employed demonstrate that the integral representation of the Feynman propagator cannot be interpreted as an ordinary function but rather as a generalized function (or more precisely a tempered distribution).

We begin with the integral representation of the free-field Feynman propagator of scalar field theory in four spacetime dimensions,

$$
\begin{aligned}
\Delta_F(x) &= \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx} \frac{1}{p^2 - m^2 + i\epsilon} \\
&= \frac{1}{(2\pi)^4} \int d^3\boldsymbol{p} \int_{-\infty}^{\infty} dp_0\, e^{-ip_0x_0 + i\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}} \frac{1}{p_0^2 - \vec{\boldsymbol{p}}^{\,2} - m^2 + i\epsilon}\,. \qquad (1)
\end{aligned}
$$

where p and x are four vectors, $px \equiv p{\cdot}x$ and ϵ is a real positive infinitesimal quantity.

Consider the integral

$$
\mathcal{I} = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} dp_0\, \frac{e^{-ip_0x_0}}{p_0^2 - \vec{\boldsymbol{p}}^{\,2} - m^2 + i\epsilon}\,. \qquad (2)
$$

First, we consider the case of $x_0 > 0$. In the limit of $\epsilon \to 0$, the integrand has poles at $p_0 = p_\pm$, where

$$
p_\pm \equiv \pm\sqrt{\vec{\boldsymbol{p}}^{\,2} + m^2}\,.
$$

To evaluate $\mathcal{I}$, we shall close the contour in the lower half of the complex p_0-plane since $e^{-ip_0x_0}$ is exponentially small along the semicircle at infinity when $x_0 > 0$. Only the pole p_+ lies inside the closed contour. Hence by the residue theorem of complex analysis,

$$
\mathcal{I} = -\frac{\pi i \exp\left\{-ix_0\sqrt{\vec{\boldsymbol{p}}^{\,2} + m^2}\right\}}{\sqrt{\vec{\boldsymbol{p}}^{\,2} + m^2}}\,, \qquad \text{for } x_0 > 0, \qquad (3)
$$

where the minus sign is due to the fact that the closed contour is clockwise.

Second, we consider the case of $x_0 < 0$. In this case, we shall evaluate $\mathcal{I}$ by closing the contour in the upper half of the complex p_0-plane so that $e^{-ip_0x_0}$ is exponentially small along the semicircle at infinity when $x_0 < 0$. Only the pole p_- lies inside the closed contour. Hence by the residue theorem of complex analysis,

$$\mathcal{I} = -\frac{\pi i \exp\left\{ix_0\sqrt{\vec{\boldsymbol{p}}^{\,2}+m^2}\right\}}{\sqrt{\vec{\boldsymbol{p}}^{\,2}+m^2}}\,, \qquad \text{for } x_0 < 0. \tag{4}$$

In this case, the closed contour is counterclockwise and the minus sign arises due to the fact that $p_- - p_+$ is negative.

Without loss of generality, we may choose the z-axis to lie along the vector $\vec{\boldsymbol{x}}$, in which case $e^{i\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}} = e^{ipr\cos\theta}$, where $p \equiv |\vec{\boldsymbol{p}}|$ and $r \equiv |\vec{\boldsymbol{x}}|$. Hence, it follows that

$$\int d\cos\theta\, d\phi\, e^{ipr\cos\theta} = \frac{2\pi}{ipr}\left(e^{ipr} - e^{-ipr}\right) = \frac{4\pi\sin(pr)}{pr}\,. \tag{5}$$

Collecting all of our results above, if follows that

$$i\Delta_F(x_0;\vec{\boldsymbol{x}}) = \frac{1}{4\pi^2 r}\int_0^\infty \frac{p\,\sin(pr)\exp\left\{-i|x_0|\sqrt{p^2+m^2}\right\}}{\sqrt{p^2+m^2}}\,dp\,. \tag{6}$$

Note that the integral above is not convergent due to the oscillatory behavior of the integrand as $p \to \infty$. This is not surprising since $\Delta_F(x)$ is not an ordinary function. In fact, $\Delta_F(x)$ is a tempered distribution, which is an example of a generalized function. Thus, one must regard the integral representation given in eq. (6) in the same way as the integral representation of a delta function given in eq. (A.1).

To perform the integral exhibited in eq. (6), we consult I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* [1], henceforth denoted as G&R. We note the following formula 3.914 no. 9 on p. 495 of G&R which states that[1]

$$\frac{1}{r}\int_0^\infty \frac{p\exp(-z\sqrt{p^2+m^2}\,)}{\sqrt{p^2+m^2}}\sin(pr)\,dp = \frac{m}{\sqrt{r^2+z^2}}\,K_1\left(m\sqrt{r^2+z^2}\right), \quad \text{for } \operatorname{Re} m > 0,\ \operatorname{Re} z > 0. \tag{7}$$

Since $z = i|x_0|$ in eq. (6), one cannot immediately employ eq. (7) to evaluate the integral of interest. However, one can define a generalized function that is represented by eq. (6) by replacing $|x_0| \to |x_0| - i\epsilon$, where ϵ is a positive infinitesimal quantity [which is unrelated to the ϵ that appears in eqs. (1) and (2)]. Hence we set $z = i(|x_0| - i\epsilon)$ in eq. (7), which satisfies the condition that $\operatorname{Re} z > 0$. Indeed, this ensures the necessary damping of the integrand as $p \to \infty$ in order to guarantee that eq. (7) is convergent. Thus, we shall assign the following result to the otherwise divergent integral,

$$\frac{1}{r}\int_0^\infty \frac{y\exp(-i|x_0|\sqrt{p^2+m^2}\,)}{\sqrt{p^2+m^2}}\sin(py)\,dp = \lim_{\epsilon\to 0^+}\frac{m}{\sqrt{r^2-x_0^2+i\epsilon}}\,K_1\left(a\sqrt{r^2-x_0^2+i\epsilon}\right). \tag{8}$$

We can identify $r^2 - x_0^2 = -x^\mu x_\mu = -x^2$. Hence, it follows that

$$\frac{1}{r}\int_0^\infty \frac{y\exp(-i|x_0|\sqrt{p^2+m^2}\,)}{\sqrt{p^2+m^2}}\sin(pr)\,dp = \frac{m}{\sqrt{-x^2+i\epsilon}}\,K_1\left(m\sqrt{-x^2+i\epsilon}\right), \tag{9}$$

[1]The constraints on the parameters m and z were omitted by G&R, but they can be found in the corresponding reference cited by G&R. See formula (36) on p. 75 of Ref. [2].

where the $\epsilon \to 0^+$ limit is henceforth implicitly assumed. Consequently, eq. (6) yields,

$$i\Delta_F(x) = \frac{m}{4\pi^2}\,\frac{K_1\big(m\sqrt{-x^2+i\epsilon}\big)}{\sqrt{-x^2+i\epsilon}}\,. \tag{10}$$

Note that in the case of $x^2 > 0$, one must be careful in interpreting both the square root and the Bessel function K_1 of an imaginary argument. Here, I shall employ eq. (5.7.6) of Ref. [3],

$$K_\nu(z) = -\tfrac{1}{2}i\pi e^{-i\pi\nu/2}H_\nu^{(2)}(ze^{-i\pi/2})\,, \qquad \text{for } -\tfrac{1}{2}\pi < \arg z < \pi. \tag{11}$$

One can apply eq. (11) to eq. (10) with $z \equiv m\sqrt{-x^2+i\epsilon}$ in the cases of $x^2 > 0$ and $x^2 < 0$, respectively, since in either case the condition on $\arg z$ is satisfied.

We now compute,

$$\sqrt{-x^2+i\epsilon} = \sqrt{x^2\,e^{i\pi}+i\epsilon} = \sqrt{(x^2-i\epsilon)e^{i\pi}} = e^{i\pi/2}\sqrt{x^2-i\epsilon}\,. \tag{12}$$

In the case of $x^2 < 0$, note that $x^2 - i\epsilon$ lies just below the branch cut that runs along the negative real axis, which implies that $\arg\sqrt{x^2-i\epsilon} \simeq -\frac{1}{2}\pi$. In contrast, $-x^2+i\epsilon$ lies just above the branch cut in the case of $x^2 > 0$, which implies that $\arg\sqrt{-x^2+i\epsilon} \simeq \frac{1}{2}\pi$. Thus, in both cases, it follows that the last step of eq. (12) is valid.

Hence, independently of the sign of x^2,

$$\frac{K_1\big(m\sqrt{-x^2+i\epsilon}\big)}{\sqrt{-x^2+i\epsilon}} = \frac{-\frac{1}{2}i\pi e^{-i\pi/2}H_1^{(2)}(m\sqrt{x^2-i\epsilon})}{e^{i\pi/2}\sqrt{x^2-i\epsilon}} = \tfrac{1}{2}i\pi\frac{H_1^{(2)}(m\sqrt{x^2-i\epsilon})}{\sqrt{x^2-i\epsilon}}\,. \tag{13}$$

It then follows that an equivalent form of eq. (10) is given by,

$$i\Delta_F(x) = \frac{im}{8\pi}\,\frac{H_1^{(2)}(m\sqrt{x^2-i\epsilon})}{\sqrt{x^2-i\epsilon}}\,. \tag{14}$$

Admittedly, eq. (10) is more convenient in the case of $x^2 < 0$, whereas eq. (14) is more convenient in the case of $x^2 > 0$. Hence, one can replace eqs. (10) and (14) with the more convenient expression,[2]

$$\boxed{i\Delta_F(x) = \frac{m}{4\pi^2}\left[\frac{K_1\big(m\sqrt{-x^2+i\epsilon}\big)}{\sqrt{-x^2+i\epsilon}}\,\Theta(-x^2) + \frac{i\pi H_1^{(2)}\big(m\sqrt{x^2-i\epsilon}\big)}{2\sqrt{x^2-i\epsilon}}\,\Theta(x^2)\right]\,,} \tag{15}$$

where we have employed the step function, $\Theta(z) = 1$ for $z > 0$ and $\Theta(z) = 0$ for $z < 0$, subject to the condition that $\Theta(z) + \Theta(-z) = 1$.[3] Indeed, the form of $\Delta_F(x)$ is consistent with our previous assertion that $\Delta_F(x)$ is a generalized function.

An alternative derivation of eq. (15) is provided in Appendix A.

[2] A different technique for evaluating the integrals of this section is presented in Ref. [4]. In this work, the authors also demonstrate that both eqs. (10) and (14) are separately valid, independently of the sign of x^2.

[3] One need not specify the values of $\Theta(0^+)$ and $\Theta(0^-)$. Indeed, when $\Theta(z)$ is regarded as a generalized function, the specification of the value of $\Theta(z)$ at the origin has no significance (e.g., see p. 63 of Ref. [5]).

It is instructive to examine the leading singularities of the Feynman propagator near the light cone, This is most easily done by employing the expansions for $H_1^{(2)}(z))$ and for $K_1(z)$ which are either given or can be deduced from the results on pp. 927–928 of G&R,

$$\tfrac{1}{2}i\pi H_1^{(2)}(z) = -\frac{1}{z} + \frac{z}{2}\left[\ln\left(\frac{z}{2}\right) + \gamma - \frac{1}{2} + \frac{i\pi}{2}\right] + \mathcal{O}(z^3)\,, \tag{16}$$

$$K_1(z) = \frac{1}{z} + \frac{z}{2}\left[\ln\left(\frac{z}{2}\right) + \gamma - \frac{1}{2}\right] + \mathcal{O}(z^3)\,. \tag{17}$$

Since $H_1^{(2)}(z)$ and $K_1(z)$ possess branch cuts along the negative real axis, eqs. (16) and (17) are valid for $|\arg z| < \pi$.

To obtain the leading singularities of $i\Delta_F(x)$, one can either insert the expansion given in eq. (17) into eq. (10) or the expansion given in eq. (16) into eq. (14) to obtain,

$$i\Delta_F(x) \sim \begin{cases} \dfrac{1}{4\pi^2(-x^2+i\epsilon)} + \dfrac{m^2}{8\pi^2}\left[\ln\left(\tfrac{1}{2}m\sqrt{-x^2+i\epsilon}\right) + \gamma - \tfrac{1}{2}\right]\,, & \text{as } x^2 \to 0^-, \\ -\dfrac{1}{4\pi^2(x^2-i\epsilon)} + \dfrac{m^2}{8\pi^2}\left[\ln\left(\tfrac{1}{2}m\sqrt{x^2-i\epsilon}\right) + \gamma - \tfrac{1}{2} + \tfrac{1}{2}i\pi\right]\,, & \text{as } x^2 \to 0^+. \end{cases} \tag{18}$$

In light of eq. (12), we see that the two limiting cases above are analytic continuations of each other. Indeed, it is a simple matter to check that if $x^2 > 0$ then

$$\begin{aligned} \tfrac{1}{2}i\pi + \lim_{\epsilon\to 0}\ln\sqrt{x^2-i\epsilon} &= \tfrac{1}{2}\left[i\pi + \lim_{\epsilon\to 0}\ln(x^2-i\epsilon)\right] = \tfrac{1}{2}\left[i\pi + \ln|x^2| - i\pi\Theta(-x^2)\right] \\ &= \ln\sqrt{|x^2|} + \tfrac{1}{2}i\pi\left[1-\Theta(-x^2)\right] = \ln\sqrt{|x^2|} + \tfrac{1}{2}i\pi\,\Theta(x^2)\,, \end{aligned} \tag{19}$$

where we have employed the identity, $\Theta(x^2) + \Theta(-x^2) = 1$. The same end result is obtained if $x^2 < 0$, since

$$\lim_{\epsilon\to 0}\ln\sqrt{-x^2+i\epsilon} = \tfrac{1}{2}\lim_{\epsilon\to 0}\ln(-x^2+i\epsilon) = \tfrac{1}{2}\left[\ln|x^2| + i\pi\Theta(x^2)\right] = \ln\sqrt{|x^2|} + \tfrac{1}{2}i\pi\,\Theta(x^2)\,. \tag{20}$$

Thus, we may combine both limits in eq. (18) into a single equation,

$$i\Delta_F(x) \sim -\frac{1}{4\pi^2(x^2-i\epsilon)} + \frac{m^2}{8\pi^2}\left[\ln\left(\tfrac{1}{2}m\sqrt{|x^2|}\right) + \gamma - \tfrac{1}{2} + \tfrac{1}{2}i\pi\,\Theta(x^2)\right]\,, \qquad \text{as } x^2 \to 0. \tag{21}$$

Finally, we can make use of the Sokhotski-Plemelj formula (see, e.g., p. 27 of Ref. [6]),

$$\frac{1}{z \pm i\epsilon} = \mathrm{P}\,\frac{1}{z} \mp i\pi\delta(z)\,, \tag{22}$$

where P is the Cauchy principal value prescription, which is employed when evaluating the integral of the product of a generalized function, $\mathrm{P}(1/x)$, and a smooth test function according to the following rule,

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)}{x}\,dx \equiv \lim_{\delta\to 0^+}\left\{\int_{-\infty}^{-\delta}\frac{f(x)}{x}\,dx + \int_{\delta}^{\infty}\frac{f(x)}{x}\,dx\right\}\,. \tag{23}$$

Hence, eqs. (21) and (22) yield,

$$\boxed{i\Delta_F(x) = -\frac{i}{4\pi}\,\delta(x^2) - \frac{1}{4\pi^2}\,\mathrm{P}\,\frac{1}{x^2} + \frac{m^2}{8\pi^2}\left[\ln\left(\frac{m\sqrt{|x^2|}}{2}\right) + \gamma - \frac{1}{2} + \frac{i\pi}{2}\Theta(x^2) + \mathcal{O}(m^2x^2)\right]}\,, \tag{24}$$

where the terms of $\mathcal{O}(m^2x^2)$ vanish on the light cone.

The limit of eq. (24) as $m \to 0$ is noteworthy,

$$\lim_{m\to 0} i\Delta_F(x) = -\frac{i}{4\pi}\,\delta(x^2) - \frac{1}{4\pi^2}\,\mathrm{P}\,\frac{1}{x^2}\,. \tag{25}$$

In obtaining eq. (25), we have used dimensional analysis to conclude that all terms in eq. (24) that vanish on the light cone must be proportional to a positive power of m^2.

To check the result of eq. (25), it is instructive to perform an exact calculation of $i\Delta_F(x)$ in the case of $m = 0$ by returning to eq. (6)

$$i\Delta_F(x_0;\boldsymbol{\vec{x}})_{m=0} = \frac{1}{4\pi^2 r}\int_0^\infty \sin(pr)e^{-ip|x_0|}\,dp\,. \tag{26}$$

To evaluate this integral, I will make use of the integral representation of the step function (see, e.g., p. 151 of Ref. [6]),

$$\Theta(k) = \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x - i\epsilon}\,dx\,. \tag{27}$$

Multiplying eq. (27) by i and then taking the inverse Fourier transform yields an expression for the generalized function $(x - i\epsilon)^{-1}$,

$$\frac{1}{x - i\epsilon} = i\int_{-\infty}^{\infty}\Theta(k)e^{-ikx}\,dk = i\int_0^\infty e^{-ikx}\,dk\,. \tag{28}$$

Employing eq. (28) in evaluating eq. (26),

$$\begin{aligned} i\Delta_F(x)_{m=0} &= -\frac{i}{8\pi^2 r}\int_0^\infty \left[e^{-ip(|x_0|-r)} - e^{-ip(|x_0|+r)}\right]dp \\ &= -\frac{1}{8\pi^2 r}\left[\frac{1}{|x_0| - r - i\epsilon} - \frac{1}{|x_0| + r - i\epsilon}\right] = -\frac{1}{4\pi^2\big(x_0^2 - r^2 - i\epsilon\big)} \\ &= -\frac{1}{4\pi^2(x^2 - i\epsilon)}\,, \end{aligned} \tag{29}$$

where we have identified $x^2 = x_0^2 - r^2$. Thus we have recovered the $m \to 0$ limit of eq. (24). Equivalently, one can again employ the Sokhotski-Plemelj formula [cf. eq. (22)] to obtain

$$i\Delta_F(x)_{m=0} = -\frac{1}{4\pi^2}\left[\mathrm{P}\,\frac{1}{x^2} + i\pi\delta(x^2)\right] = -\frac{i}{4\pi}\delta(x^2) - \frac{1}{4\pi^2}\,\mathrm{P}\,\frac{1}{x^2}\,, \tag{30}$$

thereby confirming the result of eq. (25).

Appendix A: Alternative methods for evaluating $\boldsymbol{\Delta_F(x)}$

Our method for identifying the explicit form for the generalized function $\Delta_F(x)$ was to insert a convergence factor in the integrand of eq. (6), $\exp\bigl[-\epsilon\sqrt{p^2+m^2}\bigr]$, and then take $\epsilon \to 0^+$ at the end of the calculation. Indeed, this method is often employed to interpret the integral representation of the delta function,

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk = \lim_{\epsilon\to 0^+}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\epsilon k^2}\,dk\,, \tag{A.1}$$

after inserting the convergence factor $e^{-\epsilon k^2}$.

An alternative method is to rewrite the integral given in eq. (6) as the derivative of a conditionally convergent integral, which then can be computed explicitly. We can also employ this alternative method to identify the integral representation of the delta function as follows. Consider the generalized function,

$$\mathcal{J}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{0} e^{ikx}\,dk + \frac{1}{2\pi}\int_{0}^{\infty} e^{ikx}\,dk = \frac{1}{\pi}\int_0^\infty \cos(kx)\,dk\,, \tag{A.2}$$

after performing a variable change, $k \to -k$, in the second integral above. Note that the integral representation of $\mathcal{J}(x)$ is not convergent due to the oscillatory behavior of the integrand as $k \to \infty$ [as in the case of eq. (6)]. Nevertheless, one can employ the well known conditionally convergent integral,

$$\int_0^\infty \frac{\sin(kx)}{k}\,dk = \tfrac{1}{2}\pi\,\text{sgn}(x)\,, \tag{A.3}$$

where $\text{sgn}(x)$ is the sign of the real number x.[4] Noting that

$$\text{sgn}(x) = 2\Theta(x) - 1\,, \tag{A.4}$$

it follows that

$$\int_0^\infty \frac{\sin(kx)}{k}\,dk = \pi\bigl[\Theta(x) - \tfrac{1}{2}\bigr]\,. \tag{A.5}$$

Using eq. (A.2), we can identify $\mathcal{J}(x)$ as the derivative of a conditionally convergent integral,

$$\mathcal{J}(x) = \frac{\partial}{\partial x}\frac{1}{\pi}\int_0^\infty \frac{\sin(kx)}{k}\,dk = \frac{d}{dx}\Theta(x) = \delta(x)\,, \tag{A.6}$$

in agreement with the well-known integral representation of the delta function.

Let us employ this alternative strategy in evaluating eq. (6). To perform the integral exhibited in eq. (6), we observe the following two formulae, 3.876 nos. 1 and 2 on p. 486 of G&R,

$$\int_0^\infty \frac{\sin(|x_0|\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}}\cos(pr)\,dp = \tfrac{1}{2}\pi J_0\bigl(m\sqrt{x_0^2-r^2}\bigr)\Theta(x_0^2-r^2), \tag{A.7}$$

[4] Some books define $\text{sgn}(0)=0$, in which case, eq. (A.3) would be valid at $x=0$. However, when $\text{sgn}(x)$ is regarded as a generalized function, the specification of the value at the origin has no significance (cf. footnote 3).

$$\int_0^\infty \frac{\cos(|x_0|\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}}\cos(pr)\,dp = -\tfrac{1}{2}\pi Y_0\big(m\sqrt{x_0^2-r^2}\big)\Theta(x_0^2-r^2)$$

$$+K_0\big(m\sqrt{r^2-x_0^2}\big)\Theta(r^2-x_0^2), \qquad \text{(A.8)}$$

which satisfy the conditions specified by G&R since m, r and $|x_0|$ are all positive. Combining the two integrals above yields,

$$\int_0^\infty \frac{\exp(-i|x_0|\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}}\cos(pr)\,dy = -\tfrac{1}{2}i\pi\big[J_0\big(m\sqrt{x_0^2-r^2}\big) - iY_0\big(m\sqrt{x_0^2-r^2}\big)\big]\Theta(x_0^2-r^2)$$

$$+K_0\big(m\sqrt{r^2-x_0^2}\big)\Theta(r^2-x_0^2)\,. \qquad \text{(A.9)}$$

It then follows that

$$\frac{1}{r}\int_0^\infty \frac{p\exp(-i|x_0|\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}}\sin(pr)\,dp = -\frac{1}{r}\frac{\partial}{\partial r}\int_0^\infty \frac{\exp(-i|x_0|\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}}\cos(pr)\,dp$$

$$= 2\frac{d}{dx^2}\Big[K_0\big(m\sqrt{-x^2}\big)\,\Theta(-x^2) - \tfrac{1}{2}\pi Y_0\big(m\sqrt{x^2}\big)\Theta(x^2)\Big] - i\pi\frac{d}{dx^2}\Big[J_0\big(m\sqrt{x^2}\big)\Theta(x^2)\Big]$$

$$= 2\frac{d}{dx^2}\Big[K_0\big(m\sqrt{-x^2}\big)\,\Theta(-x^2) - \tfrac{1}{2}i\pi H_0^{(2)}\big(m\sqrt{x^2}\big)\Theta(x^2)\Big]\,, \qquad \text{(A.10)}$$

after introducing the Hankel function of the second kind, $H_0^{(2)}(z) \equiv J_z(z) - iY_1(z)$ and identifying the square of the position four-vector, $x^2 = x_0^2 - r^2$.

Thus, we focus on the quantity,

$$F(x^2) \equiv K_0\big(m\sqrt{-x^2}\big)\,\Theta(-x^2) - \tfrac{1}{2}i\pi H_0^{(2)}\big(m\sqrt{x^2}\big)\Theta(x^2)\,. \qquad \text{(A.11)}$$

In order to compute dF/dx^2, we must pay attention to the behavior of $F(x^2)$ in the vicinity of $x^2 = 0$. To facilitate this analysis, we shall employ the small argument expansions of the Bessel functions, which can be deduced from results given on pp. 927–928 of G&R,

$$K_0(z) = -\ln\left(\frac{z}{2}\right) - \gamma + \mathcal{O}(z^2)\,, \qquad \text{(A.12)}$$

$$\tfrac{1}{2}i\pi H_0^{(2)}(z) = \tfrac{1}{2}i\pi + \ln\left(\frac{z}{2}\right) + \gamma + \mathcal{O}(z^2)\,, \qquad \text{(A.13)}$$

where γ is Euler's constant. It is therefore convenient to define,

$$\widetilde{K}_0(z) = K_0(z) + \ln\left(\frac{z}{2}\right)\,, \qquad \text{(A.14)}$$

$$\widetilde{H}_0^{(2)}(z) = H_0^{(2)}(z) + \frac{2i}{\pi}\ln\left(\frac{z}{2}\right)\,, \qquad \text{(A.15)}$$

each of which has a finite limit as $z \to 0$. Hence, we can rewrite eq. (A.11) as,

$$F(x^2) = \widetilde{K}_0\big(m\sqrt{-x^2}\big)\,\Theta(-x^2) - \tfrac{1}{2}i\pi\widetilde{H}_0^{(2)}\big(m\sqrt{x^2}\big)\Theta(x^2)$$

$$-\ln\big(\tfrac{1}{2}m\sqrt{-x^2}\big)\Theta(-x^2) - \ln\big(\tfrac{1}{2}m\sqrt{x^2}\big)\Theta(x^2)\,. \qquad \text{(A.16)}$$

Simplifying the second line of eq. (A.16) by employing the relation,

$$f(x^2)\Theta(x^2) + f(-x^2)\Theta(-x^2) = f(|x^2|)\,, \tag{A.17}$$

it follows that

$$F(x^2) = \widetilde{K}_0\big(m\sqrt{-x^2}\big)\,\Theta(-x^2) - \tfrac{1}{2}i\pi \widetilde{H}_0^{(2)}\big(m\sqrt{x^2}\big)\Theta(x^2) - \ln\big(\tfrac{1}{2}m|x^2|^{1/2}\big)\,. \tag{A.18}$$

We can now differentiate $F(x^2)$ with respect to x^2. Noting that,

$$\frac{d}{dz}\widetilde{K}_0(z) = -\widetilde{K}_1(z) \equiv -K_1(z) + \frac{1}{z}\,, \tag{A.19}$$

$$\frac{d}{dz}\widetilde{H}_0^{(2)}(z) = -\widetilde{H}_1^{(2)}(z) \equiv -H_1^{(2)}(z) + \frac{2i}{\pi z}\,, \tag{A.20}$$

where we have defined $\widetilde{K}_1(z)$ and $\widetilde{H}_1^{(2)}(z)$ such that the leading singular pieces of $K_1(z)$ and $H_1^{(2)}(z)$ as $z \to 0$ are removed, it follows that,

$$\begin{aligned}\frac{d}{dx^2}F(x^2) = \frac{m}{2}\left[\frac{\widetilde{K}_1(m\sqrt{-x^2})}{\sqrt{-x^2}}\,\Theta(-x^2) + \frac{i\pi\widetilde{H}_1^{(2)}(m\sqrt{x^2})}{2\sqrt{x^2}}\,\Theta(x^2)\right] - \frac{1}{2}\frac{d}{dx^2}\ln|x^2| \\ -\big[\widetilde{K}_0\big(m\sqrt{-x^2}\big) + \tfrac{1}{2}i\pi\widetilde{H}_0^{(2)}\big(m\sqrt{x^2}\big)\big]\delta(x^2)\,,\end{aligned} \tag{A.21}$$

after employing $\delta(x^2) = d\Theta(x^2)/dx^2$ and noting that $\delta(x^2) = \delta(-x^2)$. The second line of eq. (A.21) is evaluated by employing $f(x^2)\delta(x^2) = f(0)\delta(x^2)$, where $f(x^2)$ is a smooth function. Hence,

$$-\big[\widetilde{K}_0\big(m\sqrt{-x^2}\big) + \tfrac{1}{2}i\pi\widetilde{H}_0^{(2)}\big(m\sqrt{x^2}\big)\big]\delta(x^2) = -\big[\widetilde{K}_0(0) + \tfrac{1}{2}i\pi\widetilde{H}_0^{(2)}(0)\big]\delta(x^2) = -\tfrac{1}{2}i\pi\delta(x^2)\,, \tag{A.22}$$

in light of eqs. (A.12) and (A.13).

Finally, we make use of the following result that is derived in Appendix B,

$$\frac{d}{dx^2}\ln|x^2| = \mathrm{P}\,\frac{1}{x^2}\,, \tag{A.23}$$

where the symbol P stands for the principal value prescription. Hence, after using eqs. (A.21)–(A.23) the end result is,

$$\frac{d}{dx^2}F(x^2) = \frac{m}{2}\left[\frac{\widetilde{K}_1(m\sqrt{-x^2})}{\sqrt{-x^2}}\,\Theta(-x^2) + \frac{i\pi\widetilde{H}_1^{(2)}(m\sqrt{x^2})}{2\sqrt{x^2}}\,\Theta(x^2)\right] - \frac{1}{2}\left[\mathrm{P}\,\frac{1}{x^2} + i\pi\delta(x^2)\right]. \tag{A.24}$$

Consequently, eq. (A.10) yields,

$$\frac{1}{r}\int_0^\infty \frac{y\exp(-i|x_0|\sqrt{p^2+m^2}\,)}{\sqrt{p^2+m^2}}\sin(py)\,dp = -\mathrm{P}\,\frac{1}{x^2} - i\pi\,\delta(x^2) \tag{A.25}$$

$$+m\left[\frac{\widetilde{K}_1(m\sqrt{-x^2})}{\sqrt{-x^2}}\,\Theta(-x^2) + \frac{i\pi\widetilde{H}_1^{(2)}(m\sqrt{x^2})}{2\sqrt{x^2}}\,\Theta(x^2)\right]. \tag{A.26}$$

Hence, it follows from eqs. (6) and (A.26) that

$$\boxed{i\Delta_F(x) = -\frac{i}{4\pi}\,\delta(x^2) + \frac{m}{4\pi^2}\,\mathrm{P}\left\{\frac{K_1\big(m\sqrt{-x^2}\big)}{\sqrt{-x^2}}\,\Theta(-x^2) + \frac{i\pi H_1^{(2)}\big(m\sqrt{x^2}\big)}{2\sqrt{x^2}}\,\Theta(x^2)\right\}\,,} \tag{A.27}$$

after re-expressing $\widetilde{K}_1$ and $\widetilde{H}_1^{(2)}$ in terms of K_1 and $H_1^{(2)}$, respectively, and making use of the identity, $\Theta(x^2) + \Theta(-x^2) = 1$. The principal value prescription affects only those terms in eq. (A.27) inside the braces that behave as $1/x^2$ as $x^2 \to 0$.

Inserting the expansions given by eqs. (16) and (17) into eq. (A.27) and making use of eq. (A.17), it follows that

$$i\Delta_F(x) = -\frac{i}{4\pi}\,\delta(x^2) - \frac{1}{4\pi^2}\,\mathrm{P}\,\frac{1}{x^2} + \frac{m^2}{8\pi^2}\left[\ln\left(\frac{m\sqrt{|x^2|}}{2}\right) + \gamma - \frac{1}{2} + \frac{i\pi}{2}\Theta(x^2) + \mathcal{O}(m^2x^2)\right]\,, \tag{A.28}$$

in agreement with eq. (24). Note that the principal value prescription is not needed for the logarithmic term in eq. (24), since the integral of $\ln(\tfrac{1}{2}m\sqrt{|x^2|})$ multiplied by a well behaved test function, performed over an integration range that includes the point $x^2 = 0$, is convergent.

In particular, eqs. (22) and (A.28) imply that the leading singular behavior of $i\Delta_F(x)$ is

$$i\Delta_F(x) = -\frac{i}{4\pi}\,\delta(x^2) - \frac{1}{4\pi^2}\,\mathrm{P}\,\frac{1}{x^2} + \ldots = \lim_{\epsilon\to 0^+}\frac{-1}{4\pi^2(x^2 - i\epsilon)} + \ldots\,, \tag{A.29}$$

where $\ldots$ represents subleading terms as $x^2 \to 0$, and the $\epsilon \to 0$ limit is taken only after integrating the product of $\Delta_F(x)$ and a well-behaved test function. Consequently, eq. (A.27) is equivalent to

$$i\Delta_F(x) = \frac{m}{4\pi^2}\left[\frac{K_1\big(m\sqrt{-x^2+i\epsilon}\big)}{\sqrt{-x^2+i\epsilon}}\,\Theta(-x^2) + \frac{i\pi H_1^{(2)}\big(m\sqrt{x^2-i\epsilon}\big)}{2\sqrt{x^2-i\epsilon}}\,\Theta(x^2)\right]\,, \tag{A.30}$$

in agreement with eq. (15).

Yet another alternative method for evaluating the integral given by eq. (6)

This method is based on pp. 68–73 of Ref. [7] (see also Section 2.3 of Ref. [8]). Denoting $E_p \equiv \sqrt{p^2+m^2}$, we can rewrite eq. (6) as,

$$\begin{aligned}\Delta_F(x_0;\vec{\boldsymbol{x}}) &= \frac{-i}{4\pi^2 r}\int_0^\infty \frac{p\sin(pr)}{E_p}e^{-i|x_0|E_p}\,dp = \frac{-1}{8\pi^2 r}\int_0^\infty \frac{p(e^{ipr}-e^{-ipr})}{E_p}e^{-i|x_0|E_p}\,dp \\ &= \frac{-1}{8\pi^2 r}\int_{-\infty}^\infty \frac{p}{E_p}e^{ipr}e^{-i|x_0|E_p}\,dp = \frac{i}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^\infty \frac{dp}{E_p}e^{ipr}e^{-i|x_0|E_p}\,.\end{aligned} \tag{A.31}$$

Introducing the rapidity ζ,

$$E_p = m\cosh\zeta\,, \qquad\qquad p = m\sinh\zeta\,, \tag{A.32}$$

it follows that $dp = E_p\,d\zeta$. Hence,

$$i\Delta_F(x_0;\vec{\boldsymbol{x}}) = -\frac{1}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^\infty d\zeta\,\exp\big[-im\big(|x_0|\cosh\zeta - r\sinh\zeta\big)\big]\,. \tag{A.33}$$

We consider two cases.

Case 1: $x^2 \equiv x_0^2 - r^2 > 0$ (or equivalently, $|x_0| > r$)

It is convenient to define a new variable η such that the condition $x^2 \equiv x_0^2 - r^2$ is satisfied,

$$|x_0| = \sqrt{x^2}\,\cosh\eta\,, \qquad r = \sqrt{x^2}\,\sinh\eta\,. \tag{A.34}$$

It then follows that $|x_0|\cosh\zeta - r\sinh\zeta = \sqrt{x^2}\,\cosh(\zeta-\eta)$. Hence,

$$\begin{aligned} i\Delta_F(x) &= -\frac{1}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} d\zeta\ \exp\bigl[-im\sqrt{x^2}\,\cosh(\zeta-\eta)\bigr] \\ &= -\frac{1}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} d\zeta\ \exp\bigl[-im\sqrt{x^2}\,\cosh\zeta\bigr] \\ &= \frac{1}{2\pi^2}\frac{d}{dx^2}\int_0^{\infty} d\zeta\ \bigl[\cos(m\sqrt{x^2}\cosh\zeta) - i\sin(m\sqrt{x^2}\cosh\zeta)\bigr]\,, \end{aligned} \tag{A.35}$$

after making use of the symmetry of the integrand under $\zeta \to -\zeta$ to express the final result as an integral from 0 to ∞.

We now make use of G&R, formulae 3.714 nos. 2 and 3:

$$\int_0^\infty \sin(z\cosh x)dx = \tfrac{1}{2}\pi J_0(z)\,, \qquad \int_0^\infty \cos(z\cosh x)dx = -\tfrac{1}{2}\pi Y_0(z)\,, \qquad \text{for } \operatorname{Re} z > 0\,. \tag{A.36}$$

Introducing $H_0^{(2)}(z) \equiv J_0(z) - iY_0(z)$, we end up with

$$i\Delta_F(x) = -\frac{i}{4\pi}\frac{d}{dx^2}H_0^{(2)}(m\sqrt{x^2})\,, \qquad \text{for } x^2 > 0\,. \tag{A.37}$$

Case 2: $x^2 \equiv x_0^2 - r^2 < 0$ (or equivalently, $|x_0| < r$)

It is again convenient to define a variable η such that the condition $x^2 \equiv x_0^2 - r^2$ is satisfied,

$$|x_0| = \sqrt{-x^2}\,\sinh\eta\,, \qquad r = \sqrt{-x^2}\,\cosh\eta\,. \tag{A.38}$$

It then follows that $|x_0|\cosh\zeta - r\sinh\zeta = -\sqrt{-x^2}\,\sinh(\zeta-\eta)$. Hence,

$$\begin{aligned} i\Delta_F(x) &= -\frac{1}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} d\zeta\ \exp\bigl[im\sqrt{-x^2}\,\sinh(\zeta-\eta)\bigr] \\ &= -\frac{1}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} d\zeta\ \exp\bigl[im\sqrt{-x^2}\,\sinh\zeta\bigr] \\ &= \frac{1}{2\pi^2}\frac{d}{dx^2}\int_0^{\infty} d\zeta\ \cos(m\sqrt{-x^2}\sinh\zeta)\,. \end{aligned} \tag{A.39}$$

Note that in the second line of eq. (A.39), after expressing the exponential function in the integrand as a linear combination of a sine and cosine function, the integrand of the term proportional to the sine function, which is an odd function of ζ, integrates to zero. Thus, we can again make use of the symmetry of the remaining term of integrand under $\zeta \to -\zeta$ to express the final result as an integral from 0 to ∞.

Employing G&R, formulae 3.714 no. 1:

$$\int_0^\infty \cos(z\sinh x)dx = K_0(z)\,, \qquad \text{for } \operatorname{Re} z > 0\,, \tag{A.40}$$

it then follows that

$$i\Delta_F(x) = \frac{1}{2\pi^2}\,\frac{d}{dx^2}K_0(m\sqrt{-x^2})\,, \qquad \text{for } x^2 < 0. \tag{A.41}$$

Combining the results of eqs. (A.37) and (A.41),

$$i\Delta_F(x) = \frac{1}{2\pi^2}\,\frac{d}{dx^2}\left[K_0(m\sqrt{-x^2})\,\Theta(-x^2) - \tfrac{1}{2}i\pi H_0^{(2)}(m\sqrt{x^2})\,\Theta(x^2)\right], \tag{A.42}$$

which reproduces eq. (A.10). The computation of the derivative with respect to x^2 then follows the same steps previously employed in deriving the final result given in eq. (A.27).

Appendix B: Proof of $d\ln|x|/dx = \mathrm{P}(1/x)$

This proof is taken from pp. 25–26 of Ref. [9] (see also p. 83 of Ref. [6]). Consider $\ln|x|$ as a generalized function. Noting that

$$\frac{d}{dx}\ln|x| = \frac{1}{x}\,, \qquad \text{for } x \neq 0\,, \tag{B.1}$$

one can extend this result to $x = 0$ by treating $d\ln|x|/dx$ as a generalized function. For any well-behaved test function $f(x)$ that vanishes sufficiently fast as $x \to \pm\infty$, it follows from an integration by parts that

$$\int_{-\infty}^\infty f(x)\frac{d}{dx}\ln|x|\,dx = -\int_{-\infty}^\infty \ln|x|f'(x)\,dx = -\lim_{\delta\to 0^+}\int_{|x|\geq\delta}\ln|x|f'(x)\,dx\,. \tag{B.2}$$

where $f'(x) \equiv df/dx$, and the boundary terms vanish due to the behavior of $f(x)$ at $\pm\infty$. Note that the limiting process above is smooth, since the integral above exists for all values of $\delta \geq 0$. To complete the calculation, we integrate by parts once more to obtain,

$$\int_{-\infty}^\infty f(x)\frac{d}{dx}\ln|x|\,dx = \lim_{\delta\to 0^+}\left[\int_{|x|\geq\delta}\frac{f(x)}{x}\,dx - \big[f(\delta) - f(-\delta)\big]\ln\delta\right]. \tag{B.3}$$

However, $\big[f(\delta) - f(-\delta)\big]\ln\delta = \mathcal{O}(\delta\ln\delta)$ which vanishes as $\delta \to 0$. Thus, we end up with

$$\int_{-\infty}^\infty f(x)\frac{d}{dx}\ln|x|\,dx = \lim_{\delta\to 0^+}\int_{|x|\geq\delta}\frac{f(x)}{x}\,dx\,. \tag{B.4}$$

We recognize the right hand side of eq. (B.4) as the Cauchy principal value prescription [cf. eq. (23)]. Hence, we can identify the generalized function,

$$\frac{d}{dx}\ln|x| = \mathrm{P}\,\frac{1}{x}\,, \tag{B.5}$$

which is meaningful at $x = 0$ via eq. (B.4).

References

[1] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Eighth Edition, edited by Daniel Zwillinger and Victor Moll (Academic Press, Waltham, MA, 2015).

[2] *Table of Integral Transforms*, Volume 1, edited by A. Erdélyi (McGraw-Hill, New York, 1954).

[3] N.N. Lebedev, *Special Functions and Their Applications* (Dover Publications, Mineola, New York, 1972),

[4] H.-H. Zhang, K.-X. Feng, S.-W. Qiu, A. Zhao and X.-S. Li, "On analytic formulas of Feynman propagators in position space," Chin. Phys. C **34**, 1576 (2010) [arXiv:0811.1261 [math-ph]].

[5] D.S. Jones, *The theory of generalised functions* (Cambridge University Press, Cambridge, UK, 1982).

[6] Ram P. Kanwal, *Generalized Functions—Theory and Application*, Third Edition (Birkhäuser, Boston, MA, 2004).

[7] Walter Greiner and Joachim Reinhardt, *Quantum Electrodynamics*, Fourth Edition (Springer-Verlag, Berlin, Germany, 2009).

[8] Günter Scharf, *Finite Quantum Electrodynamics—The Causal Approach*, Third Edition (Dover Publications, Inc., Mineola, New York, 2014).

[9] I.M. Gel'fand and G.E. Shilov, *Generalized Functions—Properties and Operations*, Volume 1 (Academic Press, Inc., New York, 1964).