

Generalized Functions for Physics

Howard E. Haber

Santa Cruz Institute for Particle Physics

University of California, Santa Cruz, CA 95064, USA

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Abstract

In these notes, we provide a practical introduction to generalized functions that are particularly useful in many mathematical physics applications. We focus primarily on identifying the most common generalized functions employed in physics and methods for manipulating them.

1 Examples of Generalized Functions

1.1 The Heavyside step function and the Dirac delta function

The Heavyside step function is defined as,

$$\Theta(k) = \begin{cases} 1, & \text{if } k > 0, \\ 0, & \text{if } k < 0. \end{cases} \quad (1)$$

Although the value of $\Theta(k)$ is not defined at $k = 0$, we shall nevertheless demand that¹

$$\Theta(k) + \Theta(-k) = 1, \quad (2)$$

should be satisfied for all real values of k , *including* the origin, $k = 0$. The Heavyside step function is related to the Dirac delta function by differentiation,

$$\delta(k) = \frac{d\Theta(k)}{dk}. \quad (3)$$

The delta function $\delta(x)$ is not a function at all; instead it is a generalized function that only makes formal sense when first multiplied by a function $f(x)$ that is smooth and non-singular in a neighborhood of the origin, and then integrated over a range of x that may or may not include the origin. We shall also assume that $f(x) \rightarrow 0$ sufficiently fast as $x \rightarrow \pm\infty$ in order that integrals evaluated over the entire real line are convergent. It then follows that surface terms at $x = \pm\infty$, which arise when integrating by parts, vanish. The allowed set of functions $f(x)$ forms the space of test functions. In the case of the delta function, we have

$$\int_{-\infty}^{\infty} \delta(x)f(x) dx = \int_{-\infty}^{\infty} \frac{d\Theta}{dx}f(x) dx = f(x)\Theta(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \Theta(x)\frac{df}{dx} dx = -f(x) \Big|_0^{\infty} = f(0).$$

By a similar computation, one can verify that the generalized function, $x\delta(x) = 0$.

¹In eq. (2), we allow for the possibility that $\Theta(0^+) \neq \Theta(0^-)$, which is equivalent to the statement that $\lim_{\epsilon \rightarrow 0} [\Theta(\epsilon) - \Theta(-\epsilon)] \neq 0$, in which case, $\Theta(0)$ remains undetermined. Thus, it is more precise to rewrite eq. (2) as $\lim_{\epsilon \rightarrow 0^+} [\Theta(k + \epsilon) + \Theta(-k - \epsilon)] = 1$. Indeed, when $\Theta(z)$ is regarded as a generalized function, the specification of the value of $\Theta(z)$ at the origin has no significance (e.g., see p. 63 of Ref. [1]).

The integral representation of the Heavyside step function is given by,

$$\Theta(k) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dx \frac{e^{ikx}}{x - i\varepsilon}, \quad (4)$$

where ε is a positive infinitesimal real number. An explicit derivation of eq. (4) is given in Appendix A. However, it turns out to be simpler to compute the integral on the right hand side of eq. (4) and show that it is equal to $\Theta(k)$.

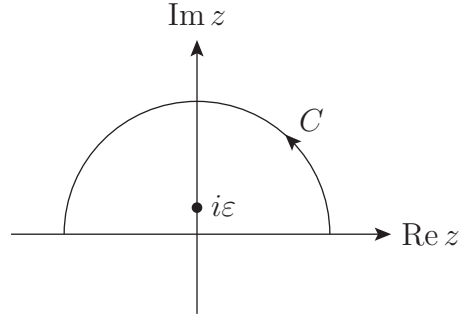
Consider the integral,

$$I(k, \varepsilon) \equiv \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dx \frac{e^{ikx}}{x - i\varepsilon},$$

where $\varepsilon > 0$ by considering a semicircular contour in the complex z plane. Two cases will now be treated.

Case 1: $k > 0$. Then it follows that

$$I(k, \varepsilon) = \frac{1}{2\pi i} \int_C dz \frac{e^{ikz}}{z - i\varepsilon}$$



where C is the closed contour shown above, and the radius of the contour is taken to infinity. Note that because $k > 0$, the integrand is exponentially damped along the semicircular part of the contour C and thus the contribution to the integral along the semicircular arc goes to zero as the radius of the semicircle is taken to infinity.

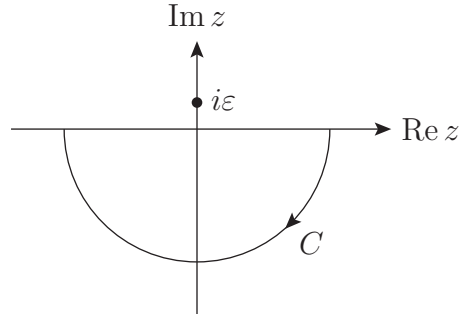
Inside the counterclockwise contour C there exists a simple pole at $z = i\varepsilon$ (since by assumption, $\varepsilon > 0$). Thus, by the residue theorem of complex analysis,

$$I(k, \varepsilon) = 2\pi i \frac{1}{2\pi i} \text{Res} \left(\frac{e^{ikz}}{z - i\varepsilon} \right) = e^{-k\varepsilon},$$

where $\text{Res}f(z) = \lim_{z \rightarrow z_0} (z - z_0)f(z)$ is the residue due to a simple pole at $z = z_0$.

Case 2: $k < 0$. Then it follows that

$$I(k, \varepsilon) = \frac{1}{2\pi i} \int_C dz \frac{e^{ikz}}{z - i\varepsilon}$$



where the contour C is now closed in the lower half plane. Since in this case $k < 0$, the integrand is again exponentially damped along the semicircular part of the contour C and thus the contribution to the integral along the semicircular arc goes to zero as the radius of the semicircle is taken to infinity. But, now the pole lies outside the closed contour C . Hence, by Cauchy's Theorem of complex analysis, it follows that $I(k, \varepsilon) = 0$ for $k < 0$.

Taking the limit as $\varepsilon \rightarrow 0$, we conclude that

$$\lim_{\varepsilon \rightarrow 0} I(k, \varepsilon) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dx \frac{e^{ikx}}{x - i\varepsilon} = \begin{cases} 1, & \text{if } k > 0, \\ 0, & \text{if } k < 0. \end{cases}$$

That is,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dx \frac{e^{ikx}}{x - i\varepsilon} = \Theta(k). \quad (5)$$

The integral representation of the Dirac delta function is given by,

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk. \quad (6)$$

Strictly speaking, the above integral does not converge since the integrand oscillates at infinity with a constant amplitude. Nevertheless, this integral can be interpreted in *the sense of distributions*.² There are two different ways to provide meaning to the integral representation of the delta function. First, one can insert a convergence factor, $e^{-\epsilon k^2}$, to render the integral convergent. Then,

$$\begin{aligned} \delta(x) &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - \epsilon k^2} dk = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi} e^{-x^2/(4\epsilon)} \int_{-\infty}^{\infty} \exp \left[-\epsilon \left(k - \frac{ix}{2\epsilon} \right)^2 \right] dk \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\sqrt{\pi\epsilon}} e^{-x^2/(4\epsilon)}, \end{aligned} \quad (7)$$

which has the property that $\delta(x) = 0$ for any $x \neq 0$ and the area under the delta function is unity,

$$\int_{-\infty}^{\infty} \delta(x) dx = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\sqrt{\pi\epsilon}} \int_{-\infty}^{\infty} e^{-x^2/(4\epsilon)} dx = 1. \quad (8)$$

One can also employ an alternative method to identify the integral representation of the delta function as follows. Consider the generalized function,

$$\mathcal{J}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk = \frac{1}{2\pi} \int_{-\infty}^0 e^{ikx} dk + \frac{1}{2\pi} \int_0^{\infty} e^{ikx} dk = \frac{1}{\pi} \int_0^{\infty} \cos(kx) dk, \quad (9)$$

after performing a variable change, $k \rightarrow -k$, in the second integral above. As expected, the integral of $\cos(kx)$ is not convergent. Nevertheless, one can employ the well known conditionally convergent integral,³

$$\int_0^{\infty} \frac{\sin(kx)}{k} dk = \frac{1}{2}\pi \operatorname{sgn}(x), \quad (10)$$

where $\operatorname{sgn}(x)$ is the sign of the real number x ,⁴

$$\operatorname{sgn}(x) = \Theta(x) - \Theta(-x) = \begin{cases} +1, & \text{for } x > 0, \\ -1, & \text{for } x < 0, \end{cases} \quad (11)$$

²Here, the term *distributions* is being employed as a synonym for generalized functions.

³Eq. (10) is well-known and is often obtained using the residue theorem of complex analysis by integrating e^{ikx}/k over a suitably chosen closed contour.

⁴Some books define $\operatorname{sgn}(0) = 0$, in which case, eq. (10) would be valid at $x = 0$. However, when $\operatorname{sgn}(x)$ is regarded as a generalized function, the specification of the value at the origin has no significance (cf. footnote 1).

which satisfies

$$\frac{d}{dx} \operatorname{sgn}(x) = \frac{d}{dx} [\Theta(x) - \Theta(-x)] = 2\delta(x), \quad (12)$$

in light of eq. (3). As a result, we can identify $\mathcal{J}(x)$ given by eq. (9) as the derivative of a conditionally convergent integral,

$$\mathcal{J}(x) = \frac{\partial}{\partial x} \frac{1}{\pi} \int_0^\infty \frac{\sin(kx)}{k} dk = \frac{1}{2} \frac{d}{dx} \operatorname{sgn}(x) = \delta(x), \quad (13)$$

in agreement with the integral representation of the delta function [cf. eqs. (6) and (9)].

1.2 Additional examples of generalized functions

The generalized function, $P(1/x)$, can be defined by the following equation,

$$P \int_{-\infty}^{\infty} \frac{f(x)}{x} dx = \int_0^\infty \frac{f(x) - f(-x)}{x} dx, \quad (14)$$

where $f(x)$ is regular in a neighborhood of the real axis and vanishes as $|x| \rightarrow \infty$. It is shown in Appendix B that eq. (14) is equivalent to the definition of the Cauchy principal value,

$$P \int_{-\infty}^{\infty} \frac{f(x) dx}{x} \equiv \lim_{\delta \rightarrow 0^+} \left\{ \int_{-\infty}^{-\delta} \frac{f(x) dx}{x} + \int_{\delta}^{\infty} \frac{f(x) dx}{x} \right\}. \quad (15)$$

where $\delta \rightarrow 0^+$ indicates that the limit is taken for positive values of δ .

There is an alternate definition of $P(1/x)$ that is also quite useful. Noting that

$$\frac{d}{dx} \ln|x| = \frac{1}{x}, \quad \text{for } x \neq 0, \quad (16)$$

one can extend this result to $x = 0$ by treating $d \ln|x|/dx$ as a generalized function. For any well-behaved test function $f(x)$ that vanishes sufficiently fast as $x \rightarrow \pm\infty$, it follows from an integration by parts that

$$\int_{-\infty}^{\infty} f(x) \frac{d}{dx} \ln|x| dx = - \int_{-\infty}^{\infty} \ln|x| f'(x) dx = - \lim_{\delta \rightarrow 0^+} \int_{|x| \geq \delta} \ln|x| f'(x) dx. \quad (17)$$

where $f'(x) \equiv df/dx$, and the boundary terms vanish due to the behavior of $f(x)$ at $\pm\infty$. Note that the limiting process above is smooth, since the integral above exists for all values of $\delta \geq 0$. To complete the calculation, we integrate by parts once more to obtain,

$$\int_{-\infty}^{\infty} f(x) \frac{d}{dx} \ln|x| dx = \lim_{\delta \rightarrow 0^+} \left[\int_{|x| \geq \delta} \frac{f(x)}{x} dx - [f(\delta) - f(-\delta)] \ln \delta \right]. \quad (18)$$

However, $[f(\delta) - f(-\delta)] \ln \delta = \mathcal{O}(\delta \ln \delta)$ which vanishes as $\delta \rightarrow 0$. Thus, we end up with

$$\int_{-\infty}^{\infty} f(x) \frac{d}{dx} \ln|x| dx = \lim_{\delta \rightarrow 0^+} \int_{|x| \geq \delta} \frac{f(x)}{x} dx = P \int_{-\infty}^{\infty} \frac{f(x)}{x} dx, \quad (19)$$

in light of eq. (15). Hence, we can identify the generalized function,

$$P \frac{1}{x} = \frac{d}{dx} \ln|x|, \quad (20)$$

which now attains meaning at $x = 0$ via eq. (19). An alternative derivation of eq. (20) is given in Appendix B.

In Ref. [2], two additional functions, x_+^{-1} and x_-^{-1} are defined via

$$x_+^{-1} = \Theta(x) \frac{1}{x} = \begin{cases} x^{-1}, & \text{for } x > 0, \\ 0, & \text{for } x < 0. \end{cases} \quad (21)$$

$$x_-^{-1} = -\Theta(-x) \frac{1}{x} = \begin{cases} 0, & \text{for } x > 0, \\ |x|^{-1}, & \text{for } x < 0. \end{cases} \quad (22)$$

As in the case of $P(1/x)$, we would like to extend the definition of x_+^{-1} and x_-^{-1} such that they yield finite results when integrated against a test function over the real axis. The corresponding generalized functions are defined by,⁵

$$\frac{1}{x_+} = \lim_{\mu \rightarrow 0} \left\{ \Theta(x) \frac{1}{x^{1-\mu}} - \frac{1}{\mu} \delta(x) \right\}, \quad (23)$$

$$\frac{1}{x_-} = \lim_{\mu \rightarrow 0} \left\{ \Theta(-x) \frac{1}{(-x)^{1-\mu}} - \frac{1}{\mu} \delta(x) \right\}, \quad (24)$$

where μ is a real and positive infinitesimal. It then follows that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{f(x)}{x_+} dx &= \lim_{\mu \rightarrow 0} \left\{ \int_0^{\infty} \frac{f(x)}{x^{1-\mu}} dx - \frac{1}{\mu} \int_{-\infty}^{\infty} \delta(x) f(x) dx \right\} \\ &= \lim_{\mu \rightarrow 0} \left\{ \int_0^1 \frac{f(x)}{x^{1-\mu}} dx - \frac{1}{\mu} f(0) \right\} + \int_1^{\infty} \frac{f(x)}{x} dx \\ &= \int_0^1 \frac{f(x) - f(0)}{x} dx + \int_1^{\infty} \frac{f(x)}{x} dx + \lim_{\mu \rightarrow 0} \left\{ f(0) \int_0^1 \frac{dx}{x^{1-\mu}} dx - \frac{1}{\mu} f(0) \right\}. \end{aligned} \quad (25)$$

In the last step above, we wrote $f(x) = f(x) - f(0) + f(0)$ and took the $\mu \rightarrow 0$ limit in the first term, which is allowed since the corresponding integral converges for any smooth function $f(x)$ that vanishes sufficiently fast as $x \rightarrow \pm\infty$. Finally, the last two terms in eq. (25) cancel exactly, and we end up with a well-defined and finite result,

$$\int_{-\infty}^{\infty} \frac{f(x)}{x_+} dx = \int_0^{\infty} \frac{f(x) - f(0)\Theta(1-x)}{x} dx = \int_0^1 \frac{f(x) - f(0)}{x} dx + \int_1^{\infty} \frac{f(x)}{x} dx. \quad (26)$$

A similar computation yields,

$$\int_{-\infty}^{\infty} \frac{f(x)}{x_-} dx = \int_0^{\infty} \frac{f(-x) - f(0)\Theta(1-x)}{x} dx = \int_0^1 \frac{f(-x) - f(0)}{x} dx + \int_1^{\infty} \frac{f(-x)}{x} dx. \quad (27)$$

⁵The conditions, $xx_+^{-1} = \Theta(x)$ and $xx_-^{-1} = -\Theta(-x)$, do not yield unique generalized functions. For example, $x[x_+^{-1} + C\delta(x)] = \Theta(x)$ for any constant C . The definitions given in eqs. (23) and (24) are taken from Ref. [1] and are motivated by the desire that the integrals given in eqs. (26) and (27) should be well-defined and finite. Some books write $\text{Pf}(1/x_{\pm})$ in eqs. (23) and (24) to distinguish these generalized functions from the functions defined in eqs. (21) and (22), where Pf stands for *pseudofunction* [3–5]. However, we choose to follow Refs. [1,2] in omitting the symbol Pf when employing the generalized functions x_{\pm}^{-1} in these notes.

It is easy to verify that the generalized functions $1/x_{\pm}$ satisfy,

$$\text{P}\frac{1}{x} = \frac{1}{x_+} - \frac{1}{x_-}. \quad (28)$$

Note that eq. (28) is true for $x \neq 0$ in light of the definitions of x_{\pm} given by eqs. (21) and (22). In addition, one can check that subtracting eq. (27) from eq. (26) yields eq. (14). One can also use eq. (28) to obtain another definition of $\text{P}(1/x)$ by employing eqs. (23) and (24),

$$\text{P}\frac{1}{x} = \lim_{\mu \rightarrow 0} \left\{ \Theta(x) \frac{1}{x^{1-\mu}} - \Theta(-x) \frac{1}{(-x)^{1-\mu}} \right\}. \quad (29)$$

In light of $|x| = -x$ for $x < 0$, it follows that

$$\Theta(x) \frac{1}{x^{1-\mu}} - \Theta(-x) \frac{1}{(-x)^{1-\mu}} = [\Theta(x) - \Theta(-x)] \frac{1}{|x|^{1-\mu}} = \frac{\text{sgn}(x)}{|x|^{1-\mu}}, \quad (30)$$

where $\text{sgn}(x)$ is the sign function defined in eq. (11). Hence, eq. (29) yields

$$\text{P}\frac{1}{x} = \lim_{\mu \rightarrow 0} \frac{\text{sgn}(x)}{|x|^{1-\mu}}. \quad (31)$$

It is a simple exercise to verify that multiplying the right hand side of eq. (31) by $f(x)$ and integrating over the real line yields eq. (14),

$$\begin{aligned} \text{P} \int \frac{f(x)}{x} dx &= \lim_{\mu \rightarrow 0} \int_{-\infty}^{\infty} \frac{f(x)}{|x|^{1-\mu}} \text{sgn}(x) dx = \lim_{\mu \rightarrow 0} \left\{ \int_0^{\infty} \frac{f(x)}{x^{1-\mu}} dx - \int_{-\infty}^0 \frac{f(x)}{x^{1-\mu}} dx \right\} \\ &= \lim_{\mu \rightarrow 0} \int_0^{\infty} \frac{f(x) - f(-x)}{x^{1-\mu}} dx = \int_0^{\infty} \frac{f(x) - f(-x)}{x} dx, \end{aligned} \quad (32)$$

after changing the integration variable $x \rightarrow -x$ in the second integral in the penultimate step above. In the final step, we can set $\mu = 0$ since the resulting integral is well-defined and finite, under the assumption that $f(x)$ is a smooth function that vanishes as $x \rightarrow \pm\infty$.

Next, we consider the function,

$$\frac{1}{|x|} = [\Theta(x) - \Theta(-x)] \frac{1}{x} = \begin{cases} x^{-1}, & \text{for } x > 0, \\ -x^{-1}, & \text{for } x < 0. \end{cases} \quad (33)$$

We again propose to extend the definition of $|x|^{-1}$ such that it yields a finite result when integrated against a test function over the real axis. The corresponding generalized function, denoted by $\text{Pf}(1/|x|)$, is defined by,⁶

$$\text{Pf} \frac{1}{|x|} = \lim_{\mu \rightarrow 0} \left\{ \frac{1}{|x|^{1-\mu}} - \frac{2}{\mu} \delta(x) \right\}. \quad (34)$$

⁶A generalized function that coincides with $1/|x|$ for $x \neq 0$ not unique; see footnote 5. One particularly convenient choice is $\text{Pf}(1/|x|)$, which in the notation of Refs. [1,2] is denoted as $1/|x|$. In contrast to footnote 5, we prefer to keep the Pf (pseudofunction) symbol in this case [3–5].

In order to see the relation between $\text{Pf}(1/|x|)$ and the generalized functions x_{\pm}^{-1} , we employ eq. (2) to obtain the identity,

$$\frac{1}{|x|^{1-\mu}} = \frac{1}{|x|^{1-\mu}} [\Theta(x) + \Theta(-x)] = \Theta(x) \frac{1}{x^{1-\mu}} + \Theta(-x) \frac{1}{(-x)^{1-\mu}}. \quad (35)$$

Hence, one can re-express eq. (34) as,

$$\text{Pf} \frac{1}{|x|} = \lim_{\mu \rightarrow 0} \left\{ \Theta(x) \frac{1}{x^{1-\mu}} + \Theta(-x) \frac{1}{(-x)^{1-\mu}} - \frac{2}{\mu} \delta(x) \right\}. \quad (36)$$

Comparing with eqs. (23) and (24), we see that

$$\text{Pf} \frac{1}{|x|} = \frac{1}{x_+} + \frac{1}{x_-}. \quad (37)$$

Note that eq. (37) is clearly true for $x \neq 0$ in light of the initial definitions of x_{\pm}^{-1} and $|x|^{-1}$ given by eqs. (21), (22) and (33).

It is convenient to rewrite eq. (27) by changing the integration variable $x \rightarrow -x$ and writing $|x| = -x$ for $x < 0$, which yields

$$\int_{-\infty}^{\infty} \frac{f(x)}{x_-} dx = - \int_{-\infty}^0 \frac{f(x) - f(0)\Theta(1+x)}{x} dx = \int_{-\infty}^{-1} \frac{f(x)}{|x|} dx + \int_{-1}^0 \frac{f(x) - f(0)}{|x|} dx. \quad (38)$$

Using eqs. (26) and (38), it then follows that

$$\int_{-\infty}^{\infty} f(x) \text{Pf} \frac{1}{|x|} dx \equiv \int_{-\infty}^{-1} \frac{f(x)}{|x|} dx + \int_{-1}^1 \frac{f(x) - f(0)}{|x|} dx + \int_1^{\infty} \frac{f(x)}{|x|} dx, \quad (39)$$

which is a well-defined and finite result, assuming $f(x)$ is smooth and vanishes sufficiently fast as $x \rightarrow \pm\infty$.

A few more generalized functions are of interest. First, $\lim_{\varepsilon \rightarrow 0} 1/(x \pm i\varepsilon)$ is related to other known generalized functions via the Sokhotski-Plemelj formula (which is derived in Section 2),

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{x \pm i\varepsilon} = \text{P} \frac{1}{x} \mp i\pi \delta(x), \quad (40)$$

where $\varepsilon > 0$ is an infinitesimal real quantity. Perhaps less well known is the related formula,⁷

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{x \pm i\varepsilon} \ln(x \pm i\varepsilon) = \text{P} \frac{1}{x} \ln|x| \mp i\pi \frac{1}{x_-} + \frac{1}{2}\pi^2 \delta(x), \quad (41)$$

which as in the case of eq. (40) consists of two separate equations—one where the upper signs are employed and one where the lower signs are employed. In eq. (41), x is a real variable and the complex logarithm is given by its principal value,

$$\lim_{\varepsilon \rightarrow 0} \ln(x \pm i\varepsilon) = \ln|x| \pm i\pi \Theta(-x), \quad (42)$$

on the cut complex plane, where the branch cut runs along the negative real axis.

⁷The symbol for the generalized function, $(x \pm i\varepsilon)^{-1} \ln(x \pm i\varepsilon)$, should be considered as a single (unified) symbol. In particular, one should not split this symbol into the product of two separate generalized functions. That is, one cannot derive eq. (41) by simply multiplying eqs. (40) and (42).

To derive eq. (41), we first take the square of eq. (42),

$$\lim_{\varepsilon \rightarrow 0} \ln^2(x \pm i\varepsilon) = \ln^2|x| - \pi^2\Theta(-x) \pm 2\pi i\Theta(-x) \ln|x|, \quad (43)$$

where we have used the fact that $[\Theta(-x)]^2 = \Theta(-x)$. In order to derive eq. (41), we shall take the derivative of eq. (43) and divide by two. The derivative of the first term on the right hand side of eq. (43) is

$$\frac{d}{dx} \ln^2|x| = 2 \ln|x| \frac{d}{dx} \ln|x| = 2 \ln|x| \text{P} \frac{1}{x}, \quad (44)$$

where we have used eq. (20). The derivative of the second term on the right hand side of eq. (43) is easily obtained after noting that

$$\frac{d}{dx} \Theta(-x) = -\frac{d}{dx} \Theta(x) = -\delta(x).$$

The derivative of the third term on the right hand side of eq. (43) requires a little more effort. Under the assumption that the test function $f(x)$ is smooth and vanishes sufficiently fast as $x \rightarrow \pm\infty$, we evaluate (with the help of integration by parts),

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \frac{d}{dx} [\Theta(-x) \ln|x|] dx &= - \int_{-\infty}^0 \frac{df}{dx} \ln(-x) dx = - \lim_{\delta \rightarrow 0} \int_{-\infty}^{-\delta} \frac{df}{dx} \ln(-x) dx \\ &= - \lim_{\delta \rightarrow 0} \left\{ \ln(-x) f(x) \Big|_{-\infty}^{-\delta} - \int_{-\infty}^{-\delta} \frac{f(x)}{x} dx \right\} \\ &= - \lim_{\delta \rightarrow 0} \left\{ \ln \delta f(\delta) - \int_{-\infty}^{-\delta} \frac{f(x)}{x} dx \right\} \\ &= - \lim_{\delta \rightarrow 0} \left\{ f(0) \int_{-1}^{-\delta} \frac{dx}{x} - \int_{-\infty}^{-\delta} \frac{f(x)}{x} dx \right\} \\ &= \int_{-\infty}^0 \frac{f(x) - f(0)\Theta(1+x)}{x} dx \\ &= - \int_{-\infty}^{\infty} \frac{f(x)}{x_-} dx. \end{aligned} \quad (45)$$

In the derivation above, we noted that $f(\delta) = f(0) + \delta f'(0) + \mathcal{O}(\delta^2)$ and made use of $\lim_{\delta \rightarrow 0} \delta \ln \delta = 0$. In the final step, we employed eq. (38). It then follows that

$$\frac{d}{dx} [\Theta(-x) \ln|x|] = -\frac{1}{x_-}. \quad (46)$$

A similar analysis (see, e.g., p. 25 of Ref. [2]) yields,

$$\frac{d}{dx} [\Theta(x) \ln|x|] = \frac{1}{x_+}. \quad (47)$$

We now add eqs. (46) and (47) and employ eq. (28) and $\Theta(x) + \Theta(-x) = 1$ to obtain,

$$\frac{d}{dx} \ln |x| = \text{P} \frac{1}{x}, \quad (48)$$

in agreement with eq. (20). Likewise, if we subtract eq. (46) from eq. (47) and employ eqs. (11) and (37), we obtain,

$$\frac{d}{dx} [\text{sgn}(x) \ln |x|] = \text{Pf} \frac{1}{|x|}. \quad (49)$$

In some books, eqs. (46)–(49) are employed as the definitions of the corresponding generalized functions, x_-^{-1} , x_+^{-1} , $\text{P}(1/x)$, and $\text{Pf}(1/|x|)$, respectively.

Finally, taking the the derivative of eq. (43) and dividing by two, we end up with

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{x \pm i\varepsilon} \ln(x \pm i\varepsilon) = \text{P} \frac{1}{x} \ln |x| + \frac{1}{2} \pi^2 \delta(x) \mp \pi i \frac{1}{x_-}. \quad (50)$$

Thus, eq. (41) has been established.⁸

As application of eqs. (40) and (41), consider the Fourier transform of $\ln |x|$, which is exhibited in eq. (78) of Section 3. However, this formula differs from the following one that is found in Ref. [2],⁹

$$\int_{-\infty}^{\infty} \ln |x| e^{ikx} dx = i \left\{ \left[-\gamma + \frac{1}{2} i\pi - \ln(k + i\varepsilon) \right] \frac{1}{k + i\varepsilon} + \left[\gamma + \frac{1}{2} i\pi + \ln(k + i\varepsilon) \right] \frac{1}{k - i\varepsilon} \right\}, \quad (51)$$

where γ is the Euler-Mascheroni constant. In eq. (51) and in what follows, we shall always assume the $\varepsilon \rightarrow 0$ limit without explicitly indicating the limit symbol.

We can simplify eq. (51) as follows. First, we write

$$\begin{aligned} \int_{-\infty}^{\infty} \ln |x| e^{ikx} dx &= -i\gamma \left(\frac{1}{k + i\varepsilon} - \frac{1}{k - i\varepsilon} \right) - \frac{\pi}{2} \left(\frac{1}{k + i\varepsilon} + \frac{1}{k - i\varepsilon} \right) \\ &\quad - i \left\{ \frac{1}{k + i\varepsilon} \ln(k + i\varepsilon) - \frac{1}{k - i\varepsilon} \ln(k - i\varepsilon) \right\}. \end{aligned} \quad (52)$$

Using eqs. (40) and (41), we end up with

$$\int_{-\infty}^{\infty} \ln |x| e^{ikx} dx = -2\pi\gamma\delta(k) - \pi \left(\text{P} \frac{1}{k} + 2 \frac{1}{k_-} \right). \quad (53)$$

Finally, employing eqs. (28) and (37) yields

$$\text{P} \frac{1}{k} + 2 \frac{1}{k_-} = \text{Pf} \frac{1}{|k|}. \quad (54)$$

⁸An alternative proof of eq. (41) can be found on pp. 96–98 of Ref. [2].

⁹Eq. (51) is also derived on pp. 160–161 of Ref. [3]. However, one must correct a typographical error that appears in eq. (6.4.57) of this work, where $-i(u - i0)^{-1}$ should be replaced by $+i(u - i0)^{-1}$. On pp. 153–154 of Ref. [3], one can also find a derivation of eq. (55) below, although again one must correct a typographical error in eq. (6.4.33d) of this work, where $1/u$ should be replaced by $1/|u|$.

Inserting eq. (54) into eq. (53), we end up with

$$\int_{-\infty}^{\infty} \ln|x| e^{ikx} dx = -\pi \left[\text{Pf} \frac{1}{|k|} + 2\gamma\delta(k) \right], \quad (55)$$

which reproduces the result of eq. (78).

2 The Sokhotski-Plemelj Formula

The Sokhotski-Plemelj formula is a relation between the following generalized functions,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{x \pm i\epsilon} = \text{P} \frac{1}{x} \mp i\pi\delta(x), \quad (56)$$

where $\epsilon > 0$ is an infinitesimal real quantity. This identity formally makes sense only when first multiplied by a function $f(x)$ that is smooth and non-singular in a neighborhood of the origin and vanishes sufficiently fast as $x \rightarrow \pm\infty$ to ensure convergence when integrated along the real line. Moreover, all surface terms at $\pm\infty$ that arise when integrating by parts are assumed to vanish.

To establish eq. (56), we shall prove that

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{f(x) dx}{x \pm i\epsilon} = \text{P} \int_{-\infty}^{\infty} \frac{f(x) dx}{x} \mp i\pi f(0), \quad (57)$$

where the Cauchy principal value integral is defined in eq. (15), under the assumption that $f(x)$ is regular in a neighborhood of the real axis and vanishes as $|x| \rightarrow \infty$.

In this section, I will provide three different derivations of eq. (57). The first derivation is a mathematically non-rigorous proof of eq. (57), which should at least provide some insight into the origin of this result. A more rigorous derivation starts with a contour integral in the complex plane,

$$\int_C \frac{f(z) dz}{z}.$$

By defining C appropriately, we will obtain two different expressions for this integral. Setting the two resulting expressions equal yields eq. (57) with the upper sign. Complex conjugating this result yields eq. (57) with the lower sign. Finally, an elegant third proof makes direct use of the theory of distributions. Finally, a useful check is to consider the Fourier transform of eq. (56), which is presented in Section 3.

Eq. (56) can be generalized as follows,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{x - x_0 \pm i\epsilon} = \text{P} \frac{1}{x - x_0} \mp i\pi\delta(x - x_0), \quad (58)$$

where

$$\text{P} \int_{-\infty}^{\infty} \frac{f(x) dx}{x - x_0} \equiv \lim_{\delta \rightarrow 0^+} \left\{ \int_{-\infty}^{x_0 - \delta} \frac{f(x) dx}{x - x_0} + \int_{x_0 + \delta}^{\infty} \frac{f(x) dx}{x - x_0} \right\}. \quad (59)$$

Note that eq. (57) and its generalization via eqs. (58) and (59) involve integration along the real axis. These ideas generalize further to the so-called Cauchy type integrals as shown in Appendix C, and yield the Plemelj formulae of complex variables theory.

2.1 A non-rigorous derivation of the Sokhotski-Plemelj formula

We begin with the identity,

$$\frac{1}{x \pm i\epsilon} = \frac{x \mp i\epsilon}{x^2 + \epsilon^2},$$

where ϵ is a positive infinitesimal quantity. Thus, for any smooth function that is non-singular in a neighborhood of the origin,

$$\int_{-\infty}^{\infty} \frac{f(x) dx}{x \pm i\epsilon} = \int_{-\infty}^{\infty} \frac{xf(x) dx}{x^2 + \epsilon^2} \mp i\epsilon \int_{-\infty}^{\infty} \frac{f(x) dx}{x^2 + \epsilon^2}. \quad (60)$$

The first integral on the right had side of eq. (60),

$$\int_{-\infty}^{\infty} \frac{xf(x) dx}{x^2 + \epsilon^2} = \int_{-\infty}^{-\delta} \frac{xf(x) dx}{x^2 + \epsilon^2} + \int_{\delta}^{\infty} \frac{xf(x) dx}{x^2 + \epsilon^2} + \int_{-\delta}^{\delta} \frac{xf(x) dx}{x^2 + \epsilon^2}. \quad (61)$$

In the first two integrals on the right hand side of eq. (61), it is safe to take the limit $\epsilon \rightarrow 0$. In the third integral on the right hand side of eq. (61), if δ is small enough, then we can approximate $f(x) \simeq f(0)$ for values of $|x| < \delta$. Hence, eq. (61) yields,

$$\int_{-\infty}^{\infty} \frac{xf(x) dx}{x^2 + \epsilon^2} = \lim_{\delta \rightarrow 0} \left\{ \int_{-\infty}^{-\delta} \frac{f(x) dx}{x} + \int_{\delta}^{\infty} \frac{f(x) dx}{x} \right\} + f(0) \int_{-\delta}^{\delta} \frac{x dx}{x^2 + \epsilon^2}. \quad (62)$$

However,

$$\int_{-\delta}^{\delta} \frac{x dx}{x^2 + \epsilon^2} = 0,$$

since the integrand is an odd function of x that is being integrated symmetrically about the origin. Hence, eq. (62) yields

$$\int_{-\infty}^{\infty} \frac{xf(x) dx}{x^2 + \epsilon^2} = \text{P} \int_{-\infty}^{\infty} \frac{f(x) dx}{x}, \quad (63)$$

where P denotes the Cauchy principal value prescription [cf. eq. (15)].

Next, we consider the second integral on the right hand side of eq. (60). Since ϵ is an infinitesimal quantity, the only significant contribution from

$$\epsilon \int_{-\infty}^{\infty} \frac{f(x) dx}{x^2 + \epsilon^2}$$

can come from the integration region where $x \simeq 0$, where the integrand behaves like ϵ^{-2} . Thus, we can again approximate $f(x) \simeq f(0)$, in which case we obtain

$$\epsilon \int_{-\infty}^{\infty} \frac{f(x) dx}{x^2 + \epsilon^2} \simeq \epsilon f(0) \int_{-\infty}^{\infty} \frac{dx}{x^2 + \epsilon^2} = \pi f(0), \quad (64)$$

where we have made use of

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + \epsilon^2} = \frac{1}{\epsilon} \tan^{-1}(x/\epsilon) \Big|_{-\infty}^{\infty} = \frac{\pi}{\epsilon}.$$

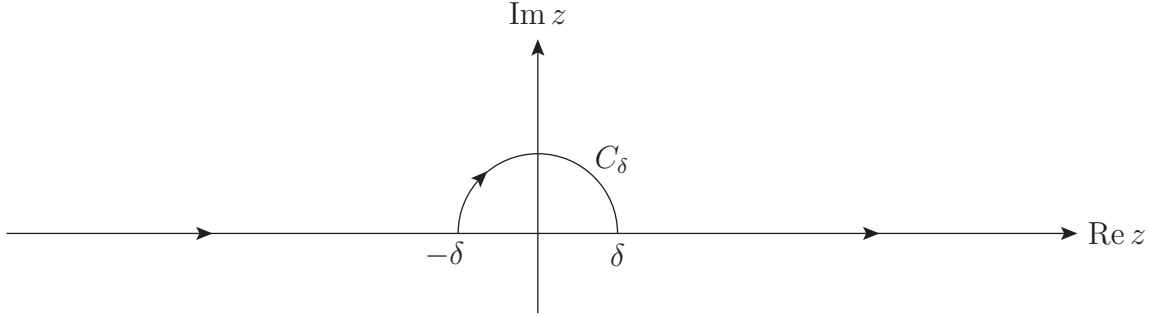
Using the results of eqs. (63) and (64), we see that eq. (60) yields,

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{f(x) dx}{x \pm i\epsilon} = \text{P} \int_{-\infty}^{\infty} \frac{f(x) dx}{x} \mp i\pi f(0), \quad (65)$$

which establishes eq. (57).

2.2 A more rigorous derivation of the Sokhotski-Plemelj formula

We consider the following path of integration in the complex plane, denoted by C , shown below.



That is, C is the contour along the real axis from $-\infty$ to $-\delta$, followed by a semicircular path C_δ (of radius δ), followed by the contour along the real axis from δ to ∞ . The infinitesimal quantity δ is defined to be a real positive quantity. Then

$$\int_C \frac{f(x)}{x} dx = \text{P} \int_{-\infty}^{\infty} \frac{f(x)}{x} dx + \int_{C_\delta} \frac{f(x)}{x} dx, \quad (66)$$

where the principal value integral is defined in eq. (15). In the limit of $\delta \rightarrow 0$, we can approximate $f(x) \simeq f(0)$ in the last integral on the right hand side of eq. (66). Noting that the contour C_δ can be parameterized as $x = \delta e^{i\theta}$ for $0 \leq \theta \leq \pi$, we end up with

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} \frac{f(x)}{x} dx = f(0) \lim_{\delta \rightarrow 0} \int_\pi^0 \frac{i\delta e^{i\theta}}{\delta e^{i\theta}} d\theta = -i\pi f(0).$$

Hence,

$$\int_C \frac{f(x)}{x} dx = \text{P} \int_{-\infty}^{\infty} \frac{f(x)}{x} dx - i\pi f(0). \quad (67)$$

We can also evaluate the left hand side of eq. (67) by deforming the contour C to a contour C' that consists of a straight line that runs from $-\infty + i\varepsilon$ to $\infty + i\varepsilon$, where ε is a positive infinitesimal (of the same order of magnitude as δ). Assuming that $f(x)$ has no singularities in an infinitesimal neighborhood around the real axis, we are free to deform the contour C into C' without changing the value of the integral. It follows that

$$\int_C \frac{f(x)}{x} dx = \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{f(x)}{x} dx = \int_{-\infty}^{\infty} \frac{f(y+i\varepsilon)}{y+i\varepsilon} dy, \quad (68)$$

where in the last step we have made a change of the integration variable.

Since ε is infinitesimal, we can approximate $f(y+i\varepsilon) \simeq f(y)$.¹⁰ Thus, after relabeling the integration variable y as x , eq. (68) yields

$$\int_C \frac{f(x)}{x} dx = \int_{-\infty}^{\infty} \frac{f(x)}{x+i\varepsilon} dx. \quad (69)$$

¹⁰More precisely, we can expand $f(y+i\varepsilon)$ in a Taylor series about $\varepsilon = 0$ to obtain $f(y+i\varepsilon) = f(y) + \mathcal{O}(\varepsilon)$. At the end of the calculation, we may take $\varepsilon \rightarrow 0$, in which case the $\mathcal{O}(\varepsilon)$ terms vanish.

Inserting this result back into eq. (67) yields

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{f(x)}{x + i\varepsilon} dx = \text{P} \int_{-\infty}^{\infty} \frac{f(x)}{x} - i\pi f(0). \quad (70)$$

Eq. (70) is also valid if $f(x)$ is replaced by $f^*(x)$. We can then take the complex conjugate of the resulting equation. The end result is¹¹

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{f(x)}{x \pm i\varepsilon} dx = \text{P} \int_{-\infty}^{\infty} \frac{f(x)}{x} \mp i\pi f(0),$$

in agreement with eq. (65).

2.3 An elegant derivation of the Sokhotski-Plemelj formula

We can employ $\text{P}(1/x) = d \ln |x|/dx$ obtained in eq. (20) to provide a very elegant derivation of eq. (56). We begin with the definition of the principal value of the complex logarithm,

$$\ln z = \ln |z| + i \arg z,$$

where $\arg z$ is the principal value of the argument (or phase) of the complex number z , with the convention that $-\pi < \arg z \leq \pi$. In particular, for real x and a positive infinitesimal ϵ ,

$$\lim_{\epsilon \rightarrow 0} \ln(x \pm i\epsilon) = \ln |x| \pm i\pi \Theta(-x), \quad (71)$$

as previously noted in eq. (42).

Differentiating eq. (71) with respect to x and employing eq. (20) and

$$\frac{d}{dx} \Theta(-x) = -\frac{d}{dx} \Theta(x) = -\delta(x), \quad (72)$$

immediately yields the Sokhotski-Plemelj formula,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{x \pm i\epsilon} = \text{P} \frac{1}{x} \mp i\pi \delta(x). \quad (73)$$

In this derivation, we have made use of the fact that the derivative of the principal value of the complex logarithm is $d \ln z/dz = 1/z$ for all complex values of z that do not lie on the branch cut along the negative real axis.

3 Fourier transforms of distributions

The generalized functions specified in eqs. (20) and (56), which we repeat below,

$$\frac{d}{dx} \ln |x| = \text{P} \frac{1}{x}, \quad (74)$$

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{x \pm i\varepsilon} = \text{P} \frac{1}{x} \mp i\pi \delta(x), \quad (75)$$

¹¹Alternatively, we can repeat the above derivation where the contour C_δ is replaced by a semicircle of radius δ in the lower half complex plane, which yields eq. (67) with $-i$ replaced by i . Finally, after deforming the contour of integration to a new contour that consists of a straight line that runs from $-\infty - i\varepsilon$ to $\infty - i\varepsilon$, one obtains eq. (69) with i replaced by $-i$.

are only meaningful when multiplied by a test function $f(x)$ and integrated over a region of the real line that may or may not include the origin. Here, we shall require that the test functions are infinitely differentiable and vanish at $\pm\infty$ faster than any inverse power of x .¹² Clearly, e^{ikx} does not satisfy this requirement for a test function. Nevertheless, one can define Fourier transforms of generalized functions by using the well known property of the Fourier transform,

$$\int_{-\infty}^{\infty} \tilde{f}(k)g(k) dk = \int_{-\infty}^{\infty} f(k)\tilde{g}(k) dx, \quad (76)$$

where

$$\tilde{f}(k) \equiv \int_{-\infty}^{\infty} f(x)e^{ikx} dx.$$

If $f(x)$ is a generalized function and $g(x)$ is a test function, then it follows that $\tilde{g}(x)$ exists and is well defined. The Fourier transform of $f(x)$, denoted by $\tilde{f}(k)$, is *defined* via eq. (76).

One can now check the validity of eqs. (74) and (75) by computing their Fourier transforms. To compute the Fourier transform of eq. (74), we make use of the property of Fourier transforms that

$$\int_{-\infty}^{\infty} \frac{df(x)}{dx} e^{ikx} dx = -ik\tilde{f}(k).$$

Hence,

$$\int_{-\infty}^{\infty} \frac{d}{dx} \ln|x| e^{ikx} dx = -ik \int_{-\infty}^{\infty} \ln|x| e^{ikx} dx. \quad (77)$$

The calculation of the right-hand side of eq. (77) is rather involved, since it only exists in the sense of distributions. One can show that¹³

$$\int_{-\infty}^{\infty} \ln|x| e^{ikx} dx = -\pi \left[\text{Pf} \frac{1}{|k|} + 2\gamma\delta(k) \right], \quad (78)$$

where γ is the Euler-Mascheroni constant, and the generalized function $\text{Pf}(1/|k|)$ was defined in eq. (34) [see also eq. (39)].

Inserting the result of eq. (51) into eq. (77) and using $k\delta(k) = 0$ and¹⁴

$$k \left(\text{Pf} \frac{1}{|k|} \right) = \frac{k}{|k|} = \text{sgn}(k), \quad (79)$$

the end result is given by,

$$\int_{-\infty}^{\infty} \frac{d}{dx} \ln|x| e^{ikx} dx = i\pi \text{sgn}(k). \quad (80)$$

¹²Generalized functions that are defined by employing test functions with these characteristics are called tempered distributions.

¹³For example, see pp. 153–154 and pp. 160–161 of Ref. [3]. There are two typographical errors on these pages. In eq. (6.4.33d), $1/u$ should be $1/|u|$ and in the last term in eq. (6.4.57), $-i(u-i0)^{-1}$ should be $+i(u-i0)^{-1}$. Eq. (51) is a consequence of the corrected eq. (6.4.57).

¹⁴When we multiply $\text{Pf}(1/|k|)$ by k , the singularity at $k=0$ is canceled and the prescription indicated by eq. (39) is no longer required. Noting that $k/|k|$ is equal to sign of k for $k \neq 0$, we end up with eq. (79).

Next, we consider

$$\text{P} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x} dx = \text{P} \int_{-\infty}^{\infty} \frac{\cos(kx)}{x} dx + i\text{P} \int_{-\infty}^{\infty} \frac{\sin(kx)}{x} dx. \quad (81)$$

Since $\cos(kx)/x$ is an odd function of x (i.e., it changes sign under $x \rightarrow -x$), it immediately follows from the definition of the Cauchy principle value that

$$\text{P} \int_{-\infty}^{\infty} \frac{\cos(kx)}{x} dx = 0. \quad (82)$$

Moreover, we observe that $\lim_{x \rightarrow 0} \sin(kx)/x = k$; that is, $\sin(kx)/x$ is regular at $x = 0$. Thus,

$$\text{P} \int_{-\infty}^{\infty} \frac{\sin(kx)}{x} dx = \int_{-\infty}^{\infty} \frac{\sin(kx)}{x} dx = \text{sgn}(k) \int_{-\infty}^{\infty} \frac{\sin y}{y} dy = \pi \text{sgn}(k). \quad (83)$$

Note that the P symbol has no effect on the integral given by eq. (83), since the integrand is regular at $x = 0$. The factor of $\text{sgn}(k)$ arises after changing the integration variable, $y = kx$. When $k < 0$, the integration limits must be reversed, which then leads to the extra minus sign. Inserting eqs. (82) and (83) into eq. (81) then yields,

$$\text{P} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x} dx = i\pi \text{sgn}(k). \quad (84)$$

In light of eqs. (80) and (84), we have verified that the Fourier transform of eq. (74) is satisfied.

Likewise, we can verify that the Fourier transform of eq. (75) is satisfied. The following result is required,

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x \pm i\varepsilon} dx = \mp 2\pi i \Theta(\mp k), \quad (85)$$

which follows from the results of Section 1.1 (see also Appendix A). Then, using eqs. (2), (11) and (84), it follows that the Fourier transform of eq. (75) is

$$\mp 2\pi i \Theta(\mp k) = i\pi [\Theta(k) - \Theta(-k)] \mp i\pi [\Theta(k) + \Theta(-k)]. \quad (86)$$

It is a simple matter to verify that eq. (86) is satisfied for either choice of sign.

Since the Fourier transform of a generalized function and its inverse Fourier transform are unique, one can conclude that if the Fourier transforms of eqs. (74) and (75) are satisfied, then eqs. (74) and (75) are valid identities. Thus, the validity of eq. (86) provides a fourth independent derivation of the Sokhotski-Plemelj formula.

4 The Riemann-Lebesgue Lemma

The Riemann Lebesgue Lemma is one of the most important results of Fourier analysis and asymptotic analysis. It has many physics applications, especially in studies of wave phenomena. In this short note, I will provide a simple proof of the Riemann-Lebesgue lemma which will be adequate for most cases that arise in physical applications.

The simplest form of the Riemann-Lebesgue lemma states that for a function $f(x)$ for which the integral

$$\int_a^b |f(x)| dx < \infty, \quad (87)$$

where a and b are real numbers, we have

$$\lim_{k \rightarrow \infty} \int_a^b f(x) e^{ikx} dx = 0. \quad (88)$$

Sometimes, the result of eq. (88) appears in the form,

$$\lim_{k \rightarrow \infty} e^{ikx} = 0. \quad (89)$$

Of course eq. (89) makes no sense when interpreted as a standard limit in mathematical analysis. However, if one interprets the limit of eq. (89) in the *sense of distributions*, i.e. by treating e^{ikx} as a generalized function, then eq. (89) can be assigned a useful meaning [6].

If we further assume that $f(x)$ has certain “nice” properties [e.g., a sufficient (but not necessary) condition is that $f(x)$ is continuously differentiable for $a \leq x \leq b$], then it follows that

$$\int_a^b f(x) e^{ikx} dx = \mathcal{O}\left(\frac{1}{k}\right), \quad \text{as } k \rightarrow \infty. \quad (90)$$

Moreover, eqs. (88) and (90) continue to hold if $a \rightarrow -\infty$ and/or $b \rightarrow \infty$, assuming that eq. (87) holds over the infinite interval.

We will present a proof of eq. (88) under the assumption that $f(x)$ is continuous over the closed interval $a \leq x \leq b$. The origin of eq. (88) in this case is not too difficult to understand. In the limit of $k \rightarrow \infty$, the factor e^{ikx} oscillates faster and faster such that $f(x) e^{ikx}$ averages out to zero over any finite region of x inside the interval.

If we can assume that $f(x)$ is N -times differentiable in the region $a \leq x \leq b$, one can derive eq. (90) simply by a repeated integration by parts. Namely,

$$\begin{aligned} \int_a^b f(x) e^{ikx} dx &= \frac{f(x)}{ik} e^{ikx} \Big|_a^b - \frac{1}{ik} \int_a^b f'(x) e^{ikx} dx \\ &= \frac{e^{ikb} f(b) - e^{ika} f(a)}{ik} - \frac{f'(x)}{(ik)^2} e^{ikx} \Big|_a^b + \frac{1}{(ik)^2} \int_a^b f''(x) e^{ikx} dx \\ &= \sum_{n=0}^{N-1} (-1)^n \frac{e^{ikb} f^{(n)}(b) - e^{ika} f^{(n)}(a)}{(ik)^{n+1}} + \mathcal{O}\left(\frac{1}{k^{N+1}}\right), \end{aligned} \quad (91)$$

where $f^{(0)}(x) \equiv f(x)$ and $f^{(n)}(x) \equiv d^n f/dx^n$. For example $f'(b)$ is equal to the first derivative of $f(x)$ evaluated at $x = b$, etc. Taking $N = 1$ in eq. (91), we see that the leading term that survives is of $\mathcal{O}(1/k)$ as asserted by eq. (90).

More generally, we can prove eq. (88) without the assumption $f(x)$ is differentiable in the interval. We do this by writing the integral

$$\mathcal{I}(k) = \int_a^b f(x) e^{ikx} dx,$$

in two different but equivalent ways:¹⁵

$$\mathcal{I}(k) = \int_a^{a+\pi/k} f(x) e^{ikx} dx + \int_{a+\pi/k}^b f(x) e^{ikx} dx \quad (92)$$

and

$$\mathcal{I}(k) = \int_a^{b-\pi/k} f(x) e^{ikx} dx + \int_{b-\pi/k}^b f(x) e^{ikx} dx. \quad (93)$$

By a change of variables, $x' = x - \pi/k$, it is straightforward to verify that

$$\int_{a+\pi/k}^b f(x) e^{ikx} dx = - \int_a^{b-\pi/k} f\left(x + \frac{\pi}{k}\right) e^{ikx} dx, \quad (94)$$

after using $e^{-i\pi} = -1$ and dropping the primes from the x in the second integral. Thus, writing \mathcal{I} as one half the sum of eqs. (92) and (93), and employing eq. (94), it follows that

$$\begin{aligned} \mathcal{I}(k) = \frac{1}{2} \int_a^{a+\pi/k} f(x) e^{ikx} dx + \frac{1}{2} \int_{b-\pi/k}^b f(x) e^{ikx} dx \\ + \frac{1}{2} \int_a^{b-\pi/k} \left[f(x) - f\left(x + \frac{\pi}{k}\right) \right] e^{ikx} dx. \end{aligned} \quad (95)$$

We now take the limit of $k \rightarrow \infty$. The mean value theorem of for integrals states that if $f(x)$ is continuous and bounded over a closed interval, $a \leq x \leq b$, then

$$\int_a^b f(x) dx = f(c)(b-a),$$

for some real number c that lies in the interval $a \leq c \leq b$. Applying this to the first two integrals in eq. (95), we immediately conclude that

$$\int_a^{a+\pi/k} f(t) e^{ikx} dx = \mathcal{O}\left(\frac{1}{k}\right), \quad \int_{b-\pi/k}^b f(k) e^{ikx} dx = \mathcal{O}\left(\frac{1}{k}\right),$$

which vanish in the limit of $k \rightarrow \infty$.

Finally, under the assumption that $f(x)$ is continuous at all points in the closed interval $a \leq x \leq b$, it follows that

$$\lim_{k \rightarrow \infty} \int_a^{b-\pi/k} \left[f(x) - f\left(x + \frac{\pi}{k}\right) \right] e^{ikx} dx = 0. \quad (96)$$

This is true because a function that is continuous at all points in a closed, bounded interval is *uniformly* continuous over the interval.¹⁶ Hence, one can make the integrand in eq. (96) arbitrarily small by choosing k sufficiently large. The limit of eq. (96) is thus established, and the Riemann-Lebesgue lemma stated in eq. (88) is proven.

¹⁵This proof is taken from pp. 39–40 of Ref. [7].

¹⁶For further details, see e.g. pp. 228–229 of Ref. [8].

Note that the argument above does *not* necessarily imply that

$$\int_a^{b-\pi/k} \left[f(x) - f\left(x + \frac{\pi}{k}\right) \right] e^{ikx} = \mathcal{O}\left(\frac{1}{k}\right), \quad (97)$$

as $k \rightarrow \infty$. However, if the function $f(x)$ is continuously differentiable in the interval, then we can employ the mean value theorem for differentiable functions, which states that

$$f(b) - f(a) = f'(c)(b - a), \quad \text{for some } c \text{ between } a \text{ and } b.$$

It follows that

$$f(x) - f\left(x + \frac{\pi}{k}\right) = -\frac{\pi}{k} f'(x + c), \quad \text{for } 0 \leq c \leq \frac{\pi}{k}.$$

Hence, in this case eq. (97) does hold, in which case eq. (90) is satisfied.¹⁷

The extension to cases where $a \rightarrow -\infty$ and/or $b \rightarrow \infty$ is straightforward. For example, suppose that

$$\int_a^\infty |f(x)| dx < \infty. \quad (98)$$

Then, noting that one can write

$$\int_a^\infty f(x) e^{ikx} dx = \int_a^b f(x) e^{ikx} dx + \epsilon,$$

where

$$\epsilon < \int_b^\infty |f(x)| dx,$$

it follows [in light of eq. (98)] that it is always possible to make ϵ arbitrarily small by a suitable (finite) choice of b . Hence, we can use eq. (88) to conclude that

$$\lim_{k \rightarrow \infty} \int_a^\infty f(x) e^{ikx} dx = 0. \quad (99)$$

Finally, it should be noted that taking the real and imaginary parts of eq. (88) under the assumption that $f(x)$ is a real valued function yields,

$$\lim_{k \rightarrow \infty} \int_a^b f(x) \sin kx dx = 0, \quad (100)$$

$$\lim_{k \rightarrow \infty} \int_a^b f(x) \cos kx dx = 0, \quad (101)$$

with a similar extension to cases where $a \rightarrow -\infty$ and/or $b \rightarrow \infty$. That is,

$$\lim_{k \rightarrow \infty} \cos kx = \lim_{k \rightarrow \infty} \sin kx = 0, \quad (102)$$

where the limit is interpreted in the sense of distributions [cf. discussion below eq. (89)]. If we further assume that $f(x)$ has certain “nice” properties [cf. comments above eq. (90)], then the two integrals given in eqs. (100) and (101) behave as $\mathcal{O}(1/k)$ as $k \rightarrow \infty$.

¹⁷We also obtain eq. (97) under slightly weaker conditions in which the function $f(x)$ satisfies the so-called *Lipschitz condition*, $|f(x) - f(y)| \leq M|x - y|$ for all x and y in the interval for some positive finite bound M . Indeed a Lipschitz continuous function is uniformly continuous (although the converse is not necessarily true). A Lipschitz continuous function need not be differentiable. On the other hand a differentiable function whose derivative is bounded on the interval satisfies the Lipschitz condition. Thus, to establish eq. (97) is sufficient to require that $f(k)$ is Lipschitz continuous in the interval.

By consulting a table of Fourier transforms [9], one can see many examples of functions that satisfy eq. (87). In all cases, you will find that the corresponding Fourier transform satisfies eq. (88). It is interesting to look for cases that satisfy eq. (88) and not eq. (90). For example,

$$\int_0^\infty x^{\nu-1} e^{-ax} e^{ikx} dx = \frac{\Gamma(\nu)}{(a - ik)^\nu}, \quad \text{for } a > 0 \text{ and } \text{Re } \nu > 0.$$

Indeed, eqs. (87) and (88) are satisfied for all $\text{Re } \nu > 0$, whereas eq. (90) is only satisfied when $\text{Re } \nu \geq 1$.

Finally, we provide the following interesting example that contradicts eq. (100). Choosing $f(x) = \sin(x^2)$ and consulting a table of integrals (see, e.g., I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products* [10], henceforth denoted as G&R), we note the following result on formula 3.691 number 5 on p. 419 of G&R,

$$\int_0^\infty \sin(x^2) \sin kx dx = \frac{1}{2} \sqrt{\pi} \cos\left(\frac{k^2 + \pi}{4}\right),$$

which does *not* vanish as $k \rightarrow \infty$. However, this is not surprising in light of the fact that eq. (98) [with $a = 0$] is not satisfied. Indeed, even though

$$\int_0^\infty \sin(x^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}},$$

is finite, it turns out that

$$\int_0^\infty |\sin(x^2)| dx$$

diverges. Thus, one of the key assumptions underlying the Riemann-Lebesgue lemma does not hold in this case.

Appendix A: An explicit derivation of the Fourier transformation of the Heavyside step function

The goal of this Appendix is to express the step function as a Fourier transform,

$$\Theta(k) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ikx} f(x) dx, \tag{A.1}$$

where the function $f(x)$ is to be determined.¹⁸

The function $f(x)$ is determined by the inverse Fourier transform,

$$f(x) = \int_{-\infty}^\infty e^{-ikx} \Theta(k) dk. \tag{A.2}$$

This integral is not well defined. However it can be re-interpreted in *the sense of distributions*. What this phrase really means is that quantities are treated as generalized functions (also called distributions), which make sense only when integrated against test functions that are smooth, regular, and vanish sufficiently fast at $\pm\infty$.

¹⁸This Appendix is based on a derivation given on p. 151 of Ref. [3].

We can evaluate $f(x)$ using the following trick. Note that

$$1 = \int_{-\infty}^{\infty} e^{-ikx} \delta(k) dk = \int_{-\infty}^{\infty} e^{-ikx} \frac{d\Theta(k)}{dk} dk. \quad (\text{A.3})$$

We now integrate by parts. We can set the surface term to zero by employing

$$\lim_{k \rightarrow \pm\infty} e^{-ikx} = 0, \quad (\text{A.4})$$

where the limit is interpreted in the sense of distributions [cf. eq. (89)]. It then follows that

$$1 = - \int_{-\infty}^{\infty} \Theta(k) \frac{d}{dk} e^{-ikx} dk = ix \int_{-\infty}^{\infty} \Theta(k) e^{-ikx} dk = ix f(x). \quad (\text{A.5})$$

To solve eq. (A.5), let us define $h(x) \equiv if(x)$ and consider the equation

$$xh(x) = 1. \quad (\text{A.6})$$

The solution to this equation for $x \neq 0$ is clearly $h(x) = 1/x$. But, how should we deal with $x = 0$? The answer is again to appeal to generalized functions. That is, eq. (A.6) should be interpreted as

$$\int_{-\infty}^{\infty} xh(x)g(x) dx = \int_{-\infty}^{\infty} g(x) dx, \quad (\text{A.7})$$

for any smooth regular test function $g(x)$ that vanishes sufficiently fast at $\pm\infty$.

The most general solution to the inhomogeneous equation, $xh(x) = 1$, must be of the form,

$$h(x) = h_p(x) + h_h(x), \quad (\text{A.8})$$

where $h_p(x)$ is a particular solution that satisfies $xh_p(x) = 1$ and $h_h(x)$ is the solution to the homogeneous equation, $xh_h(x) = 0$. I claim that one choice for the particular solution to eq. (A.6) is,

$$h_p(x) = \text{P} \frac{1}{x}, \quad (\text{A.9})$$

where P indicates the Cauchy principal value prescription defined in eq. (15) [see also Appendix B].

Let us check that $h(x) = h_p(x)$ given by eq. (A.9) provides a solution to eq. (A.7). It is sufficient to observe that,

$$\text{P} \int_{-\infty}^{\infty} x \frac{1}{x} g(x) dx = \int_{-\infty}^{\infty} g(x) dx, \quad (\text{A.10})$$

where the P symbol can be dropped on the right hand side of eq. (A.10) since the corresponding integral is now well defined. Hence, it follows that (in the sense of distributions),

$$x \text{P} \frac{1}{x} = 1, \quad (\text{A.11})$$

and eq. (A.9) is verified.

We now turn to the most general solution to the homogeneous equation,

$$xh_h(x) = 0. \quad (\text{A.12})$$

We shall solve eq. (A.12) using a Fourier transform technique. We first write

$$h_h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} q(k) dk. \quad (\text{A.13})$$

Inverting the Fourier transform yields

$$q(k) = \int_{-\infty}^{\infty} e^{-ikx} h_h(x) dx. \quad (\text{A.14})$$

We now take the derivative of $q(k)$ with respect to k to obtain,

$$\frac{dq}{dk} = -i \int_{-\infty}^{\infty} e^{-ikx} x h_h(x) dx = 0. \quad (\text{A.15})$$

where we have used eq. (A.12) in the final step. The most general solution to the differential equation $dq/dk = 0$ is $q(k) = C$, where C is an arbitrary constant.¹⁹ Inserting this solution back into eq. (A.13), we end up with

$$h_h(x) = \frac{C}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk = C \delta(x), \quad (\text{A.16})$$

after employing the integral representation of the delta function [cf. eq. (6)].

It is simple to check the validity of eq. (A.16). In particular,

$$\int_{-\infty}^{\infty} x \delta(x) g(x) dx = x g(x) \Big|_{x=0} = 0, \quad (\text{A.17})$$

where again we have used the fact that $g(x)$ is a smooth regular function. It then follows that

$$x \delta(x) = 0, \quad (\text{A.18})$$

in the sense of distributions. Hence, $xh_h(x) = Cx\delta(x) = 0$ as required. Combining the results of eqs. (A.9) and (A.16), one obtains the most general solution of eq. (A.6),

$$h(x) = h_p(x) + h_h(x) = \text{P} \frac{1}{x} + C \delta(x). \quad (\text{A.19})$$

Returning to eq. (A.5), it follows in light of eq. (A.19) that

$$h(x) = if(x) = i \int_{-\infty}^{\infty} \Theta(k) e^{-ikx} dk = \text{P} \frac{1}{x} + C \delta(x), \quad (\text{A.20})$$

¹⁹This statement is trivial if solutions are restricted to the space of ordinary functions. Nevertheless, $q(k) = C$ is still the unique solution of $dq/dk = 0$ even if the solution space is expanded to include generalized functions. A proof of this assertion can be found on pp. 39–41 of Ref. [2].

where the constant C is still yet to be determined. To fix the constant C we proceed as follows. Replacing $x \rightarrow -x$ in eq. (A.20) yields,

$$-P\frac{1}{x} + C\delta(x) = i \int_{-\infty}^{\infty} \Theta(k)e^{ikx} dk = i \int_{-\infty}^{\infty} \Theta(-k)e^{-ikx} dk, \quad (\text{A.21})$$

after noting that $\delta(-x) = \delta(x)$ and changing the integration variable from k to $-k$. Adding eqs. (A.20) and (A.21) and using eq. (2), we end up with

$$2C\delta(x) = i \int_{-\infty}^{\infty} [\Theta(k) + \Theta(-k)]e^{-ikx} dk = i \int_{-\infty}^{\infty} e^{-ikx} dk. \quad (\text{A.22})$$

Finally, using the integral representation of the delta function [cf. eq. (6)], we conclude that $C = i\pi$. We now insert this value of C into eq. (A.20) and employ the Sokhotski-Plemelj formula [cf. eq. (40)],

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{x - i\varepsilon} = P\frac{1}{x} + i\pi\delta(x). \quad (\text{A.23})$$

The end result is

$$if(x) = i \int_{-\infty}^{\infty} \Theta(k)e^{-ikx} dk = \lim_{\varepsilon \rightarrow 0} \frac{1}{x - i\varepsilon}. \quad (\text{A.24})$$

Returning to eq. (A.1), we conclude that

$$\Theta(k) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x - i\varepsilon} dx, \quad (\text{A.25})$$

which reconfirms the result exhibited in eq. (4).

It is instructive to revisit eq. (A.20) with $C = i\pi$, which yields the noteworthy result,

$$i \int_{-\infty}^{\infty} \Theta(k)e^{-ikx} dk = P\frac{1}{x} + i\pi\delta(x). \quad (\text{A.26})$$

In particular, in light of eq. (1), the complex conjugate of eq. (A.26) yields,

$$\int_0^{\infty} e^{ikx} dk = iP\frac{1}{x} + \pi\delta(x). \quad (\text{A.27})$$

Equivalently, we can employ eq. (A.23) and rewrite eq. (A.27) in the form,

$$\int_0^{\infty} e^{ikx} dk = \lim_{\varepsilon \rightarrow 0} \frac{i}{x + i\varepsilon}. \quad (\text{A.28})$$

Of course, eqs. (A.27) and (A.28) must be interpreted in the sense of distributions (since the integrals do not converge in the usual sense²⁰). Moreover, taking real and imaginary parts of eq. (A.27) yield,

$$\int_0^{\infty} \cos kx dk = \pi\delta(x), \quad (\text{A.29})$$

$$\int_0^{\infty} \sin kx dk = P\frac{1}{x}. \quad (\text{A.30})$$

²⁰Compare this with the integral in eq. (6), which does not converge in the usual sense, but nevertheless provides an integral representation of the delta function in the sense of distributions.

Once again, the integrals of eqs. (A.29) and (A.30) must be interpreted in the sense of distributions. For example, we have already obtained eq. (A.29) in light of eqs. (9) and (13).

One additional consequence of eq. (A.26) can be extracted if we invert the Fourier transform,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \left[\text{P} \frac{1}{x} + i\pi\delta(x) \right] dx = i\Theta(k). \quad (\text{A.31})$$

Using the Cauchy principal value prescription and integrating over the delta function yields

$$\frac{1}{2\pi} \text{P} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x} dx = i \left[\Theta(k) - \frac{1}{2} \right]. \quad (\text{A.32})$$

Using eq. (2), it follows that

$$\Theta(k) - \frac{1}{2} = \Theta(k) - \frac{1}{2} [\Theta(k) + \Theta(-k)] = \frac{1}{2} [\Theta(k) - \Theta(-k)] = \frac{1}{2} \text{sgn}(k). \quad (\text{A.33})$$

Inserting this result into eq. (A.32) yields,

$$\text{P} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x} dx = i\pi \text{sgn}(k). \quad (\text{A.34})$$

Taking the real and imaginary parts of eq. (A.34) yields,

$$\text{P} \int_{-\infty}^{\infty} \frac{\cos kx}{x} dx = 0, \quad (\text{A.35})$$

$$\int_{-\infty}^{\infty} \frac{\sin kx}{x} dx = \pi \text{sgn}(k). \quad (\text{A.36})$$

Note that the vanishing of the integral in eq. (A.35) is due to the fact that the integrand is an odd function of x , which integrates to zero due to the Cauchy principal value prescription. The P symbol is not needed in eq. (A.36) since the corresponding integrand has a finite limit as $k \rightarrow 0$.

As one final check, let us take the derivative of eq. (A.34) with respect to k and employ eq. (12). This yields (once again) the integral representation of the delta function (where the P symbol can be dropped as the resulting integrand is not singular at $x = 0$).

Appendix B: Appendix: The Cauchy Principle Value

In this Appendix, we verify that the definitions of the Cauchy principal value given in eqs. (14), (15) and (20) are all equivalent. We begin with eq. (14), which we repeat here for the convenience of the reader,

$$\text{P} \int_{-\infty}^{\infty} \frac{f(x)}{x} dx = \int_0^{\infty} \frac{f(x) - f(-x)}{x} dx. \quad (\text{B.1})$$

For any positive real number δ , the following equation is an identity,

$$\int_0^{\infty} \frac{f(x) - f(-x)}{x} dx = \int_0^{\delta} \frac{f(x) - f(-x)}{x} dx + \int_{\delta}^{\infty} \frac{f(x)}{x} dx + \int_{-\infty}^{-\delta} \frac{f(x)}{x} dx, \quad (\text{B.2})$$

where the last integral on the right hand side of eq. (B.2) has been obtained after changing the integration variable, $x \rightarrow -x$. Since δ is an arbitrary positive number, we can take the limit as $\delta \rightarrow 0$ (from the positive side), which yields

$$\text{P} \int_{-\infty}^{\infty} \frac{f(x) dx}{x} = \lim_{\delta \rightarrow 0^+} \left\{ \int_{-\infty}^{-\delta} \frac{f(x) dx}{x} + \int_{\delta}^{\infty} \frac{f(x) dx}{x} + \int_0^{\delta} \frac{f(x) - f(-x)}{x} dx \right\}. \quad (\text{B.3})$$

By assumption, the test function $f(x)$ is smooth and vanishes as $x \rightarrow \pm\infty$. Hence, one can Taylor expand $f(x)$ around the origin to obtain $f(x) = f(0) + xf'(0) + \mathcal{O}(x^2)$. It follows that for $\delta \ll 1$,

$$\int_0^{\delta} \frac{f(x) - f(-x)}{x} dx = 2f'(0)\delta + \mathcal{O}(\delta^3), \quad (\text{B.4})$$

which vanishes as $\delta \rightarrow 0$. Hence, eq. (B.3) yields,

$$\text{P} \int_{-\infty}^{\infty} \frac{f(x) dx}{x} = \lim_{\delta \rightarrow 0^+} \left\{ \int_{-\infty}^{-\delta} \frac{f(x) dx}{x} + \int_{\delta}^{\infty} \frac{f(x) dx}{x} \right\}, \quad (\text{B.5})$$

which establishes the equivalence of eqs. (14) and (15).

Next, we start from the definition of the Cauchy principal value given in eq. (B.5) and integrate by parts to obtain

$$\begin{aligned} \int_{-\infty}^{-\delta} \frac{f(x)}{x} dx &= f(x) \ln |x| \Big|_{-\infty}^{-\delta} - \int_{-\infty}^{-\delta} f'(x) \ln |x| dx = f(-\delta) \ln \delta - \int_{-\infty}^{-\delta} f'(x) \ln |x| dx, \\ \int_{\delta}^{\infty} \frac{f(x)}{x} dx &= f(x) \ln |x| \Big|_{\delta}^{\infty} - \int_{\delta}^{\infty} f'(x) \ln |x| dx = -f(\delta) \ln \delta - \int_{\delta}^{\infty} f'(x) \ln |x| dx, \end{aligned}$$

where $f'(x) \equiv df/dx$ and we have assumed that $f(x) \rightarrow 0$ sufficiently fast as $x \rightarrow \pm\infty$ so that the surface terms at $\pm\infty$ vanish. Hence,

$$\text{P} \int_{-\infty}^{\infty} \frac{f(x) dx}{x} = \lim_{\delta \rightarrow 0^+} \left\{ [f(-\delta) - f(\delta)] \ln \delta - \int_{-\infty}^{-\delta} f'(x) \ln |x| dx - \int_{\delta}^{\infty} f'(x) \ln |x| dx \right\}. \quad (\text{B.6})$$

Since $f(x)$ is differentiable and well behaved, we can define

$$g(x) \equiv \int_0^1 f'(xt) dt = \frac{f(x) - f(0)}{x},$$

which implies that $g(x)$ is smooth and non-singular and

$$f(x) = f(0) + xg(x). \quad (\text{B.7})$$

Inserting eq. (B.7) back into eq. (B.6) then yields

$$\begin{aligned} \text{P} \int_{-\infty}^{\infty} \frac{f(x) dx}{x} &= \lim_{\delta \rightarrow 0^+} \left\{ [-2g(\delta)\delta \ln \delta - \int_{-\infty}^{-\delta} f'(x) \ln |x| dx - \int_{\delta}^{\infty} f'(x) \ln |x| dx] \right\} \\ &= - \int_{-\infty}^{\infty} f'(x) \ln |x| dx. \end{aligned}$$

Note that $\ln|x|$ is integrable at $x = 0$, so that the last integral is well-defined. Finally, we integrate by parts and drop the surface terms at $\pm\infty$ (under the usual assumption that $f'(x) \rightarrow 0$ sufficiently fast as $x \rightarrow \infty$). The end result is

$$\text{P} \int_{-\infty}^{\infty} \frac{f(x) dx}{x} = \int_{-\infty}^{\infty} f(x) \frac{d}{dx} \ln|x| dx. \quad (\text{B.8})$$

That is, we have rederived the generalized function identity previous obtained in eqs. (19) and (20),

$$\frac{d}{dx} \ln|x| = \text{P} \frac{1}{x}. \quad (\text{B.9})$$

In particular, eq. (B.8) provides a definition of the Cauchy principal value prescription that is equivalent to the definitions provided by eqs. (B.1) and (B.5).

Appendix C: The Plemelj Formulae of the Theory of Complex Variables

The Sokhotski-Plemelj formula derived in these notes is in fact a special case of a more general result of the theory of complex variables, which is often referred to as the Plemelj formulae (and less often as the Sokhotski formulae). In this Appendix, I shall simply state the relevant results. Further details can be found in Refs. [11–15].

Consider the Cauchy type integral,

$$F(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t-z} dt, \quad (\text{C.1})$$

where z and t are complex variables, C is a smooth curve (which may be an open or a closed contour) and $f(t)$ is a function defined on C that satisfies,

$$|f(t_2) - f(t_1)| < A|t_2 - t_1|^\lambda, \quad (\text{C.2})$$

for any two points t_1 and t_2 located on the contour C , where A and λ are positive numbers. Eq. (C.2) is called the Hölder condition.²¹

For values of $z \notin C$, $F(z)$ is an analytic function, whereas $F(z)$ is not well defined for values of z on the contour C due to the singularity encountered in the integration along C . Nevertheless, $F(z)$ does have unique value that depends on how z approaches C . Indeed, there are two different possible boundary values of $F(z)$ depending on whether the contour C is approached from the left or right. We therefore introduce $F_+(z)$ and $F_-(z)$ where the former is the limit as z approaches C from the left and the latter is the limit as z approaches from the right, where left and right are defined with respect to the positive direction of the contour C .²²

²¹If $\lambda > 1$, then it follows that the derivative $f'(t)$ must vanish on C , in which case $f(t)$ is a constant. Thus, one typically assumes that $0 < \lambda \leq 1$.

²²For example, for a closed counterclockwise contour C , $F_+(z)$ is given by the limit of $F(z)$ as z approaches C from the interior of the region bounded by C and $F_-(z)$ is given by the limit of $F(z)$ as z approaches C from the exterior of the region bounded by C .

The explicit results for $F_{\pm}(z)$ are given by the Plemelj formulae,

$$F_{\pm}(z) = \pm \frac{1}{2}f(z) + \frac{1}{2\pi i} \text{P} \int_C \frac{f(t)}{t-z} dt, \quad \text{for } z \in \tilde{C}, \quad (\text{C.3})$$

and \tilde{C} consists of all points of C excluding its endpoints (in the case of a closed contour, there are no endpoints to exclude). In eq. (C.3), we employ the the Cauchy principal value prescription to treat the singularity in the integrand. In this context, the principal value is a generalization of eq. (15),

$$\text{P} \int_C \frac{f(t)}{t-z} dt = \lim_{\delta \rightarrow 0} \int_{C-C_{\delta}} \frac{f(t)}{t-z} dt, \quad (\text{C.4})$$

where the contour C_{δ} consists of the part of C with length 2δ centered symmetrically around z , and $C - C_{\delta}$ is the contour C with the part C_{δ} removed.

If $f(t)$ is analytic on C , then the proof of eq. (C.3) is a straightforward generalization of the proof given in Section 2. However, the Plemelj formulae are more general and apply to any function $f(t)$ that satisfies the Hölder condition on the contour C . In this case, the derivation of eq. (C.3) is more complicated. We have also sidestepped the case where z in eq. (C.3) is one of the endpoints of C (which is relevant if the contour C is open). The reader is referred to the references below for further details.

One can recast eq. (C.3) into another form that commonly appears in the literature,

$$F_+(z) - F_-(z) = f(z), \quad (\text{C.5})$$

$$F_+(z) + F_-(z) = \frac{1}{\pi i} \text{P} \int_C \frac{f(t)}{t-z} dt, \quad (\text{C.6})$$

for values of z located on all points of the contour C not coinciding with its endpoints. In particular, eq. (C.5) indicates that the function $F(z)$ defined in eq. (C.1), which is analytic for all complex values of $z \notin C$, has a discontinuous jump as z crosses the contour C . Moreover, the average of the two boundary values of $F(z)$ on C is given by eq. (C.1), where the singularity of the integrand is treated by the Cauchy principal value prescription.

Using the Plemelj formulae of complex variables theory, one can recover the results of Section 2 as follows. If C is a contour that runs along the real axis in the positive direction, then eq. (C.1) yields the boundary values, $F_{\pm}(z)$, of $F(z)$ as z approaches the real axis from above (i.e., from the left) or below (i.e., from the right), respectively,

$$F_{\pm}(z) = \lim_{\epsilon \rightarrow 0} F(z \pm i\epsilon) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z \mp i\epsilon} dt, \quad (\text{C.7})$$

where $\epsilon > 0$ is an infinitesimal real quantity. Hence, eqs. (C.3) and (C.7) yield

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{f(t)}{t-z \mp i\epsilon} dt = \text{P} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt \pm i\pi f(z). \quad (\text{C.8})$$

Eq. (C.8) is equivalent to the identity involving generalized functions given in eq. (58). As expected, setting $z = 0$ in eq. (C.8) reproduces eq. (57).

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