Generalized Functions for Physics

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Abstract

In these notes, we provide a practical introduction to generalized functions (often called distributions) that are particularly useful in many mathematical physics applications. We focus primarily on identifying the most common generalized functions employed in physics and methods for manipulating them. Proofs of the Riemann-Lebesgue Lemma and the Poisson sum formula are also provided.

1 Examples of Generalized Functions

The examples of generalized functions discussed in this section are based on the treatments given in Refs. [1–8].

1.1 The Heavyside step function and the Dirac delta function

The Heavyside step function is defined as,

$$\Theta(k) = \begin{cases} 1\,, & \text{if } k > 0\ , \\ 0\,, & \text{if } k < 0\,. \end{cases} \tag{1}$$

Although the value of $\Theta(k)$ is not defined at $k = 0$, we shall nevertheless demand that[1]

$$\Theta(k) + \Theta(-k) = 1\,, \tag{2}$$

should be satisfied for all real values of k, *including* the origin, $k = 0$. The Heavyside step function is related to the Dirac delta function by differentiation,

$$\delta(k) = \frac{d\Theta(k)}{dk}\,. \tag{3}$$

The delta function $\delta(x)$ is not a function at all; instead it is a generalized function that only makes formal sense when first multiplied by a function $f(x)$ that is smooth and non-singular in a neighborhood of the origin, and then integrated over a range of x that may or may not

[1] In eq. (2), we allow for the possibility that $\Theta(0^+) \neq \Theta(0^-)$, which is equivalent to the statement that $\lim_{\epsilon\to 0}\big[\Theta(\epsilon) - \Theta(-\epsilon)\big] \neq 0$, in which case, $\Theta(0)$ remains undetermined. Thus, it is more precise to rewrite eq. (2) as $\lim_{\epsilon\to 0^+}\big[\Theta(k+\epsilon) + \Theta(-k-\epsilon)\big] = 1$. Indeed, when $\Theta(z)$ is regarded as a generalized function, the specification of the value of $\Theta(z)$ at the origin has no significance (e.g., see p. 63 of Ref. [2]).

include the origin. We shall also assume that $f(x) \to 0$ sufficiently fast as $x \to \pm\infty$ in order that integrals evaluated over the entire real line are convergent. It then follows that surface terms at $x = \pm\infty$, which arise when integrating by parts, vanish. The allowed set of functions $f(x)$ forms the space of test functions. In the case of the delta function, we have

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = \int_{-\infty}^{\infty} \frac{d\Theta}{dx} f(x)\, dx = f(x)\Theta(x)\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \Theta(x) \frac{df}{dx}\, dx = -f(x)\bigg|_{0}^{\infty} = f(0)\,. \tag{4}$$

By a similar computation, one can verify that the generalized function, $x\delta(x) = 0$.

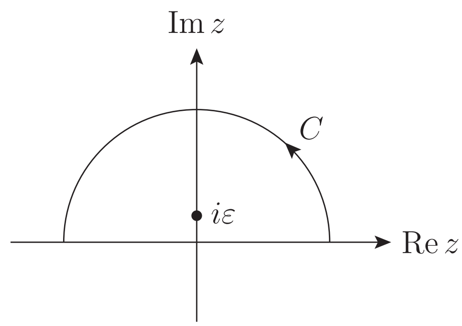
The integral representation of the Heavyside step function can be derived by considering the integral

$$I(k, \varepsilon) \equiv \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dx\, \frac{e^{ikx}}{x - i\varepsilon}\,,$$

where ϵ is a real positive infinitesimal quantity. This integral can be evaluated by considering a semicircular contour in the complex z plane. Two cases will now be treated.

Case 1: $k > 0$. Then it follows that

$$I(k, \varepsilon) = \frac{1}{2\pi i} \int_C dz\, \frac{e^{ikz}}{z - i\varepsilon}$$



where C is the closed contour shown above, and the radius of the contour is taken to infinity. Note that because $k > 0$, the integrand is exponentially damped along the semicircular part of the contour C and thus the contribution to the integral along the semicircular arc goes to zero as the radius of the semicircle is taken to infinity.

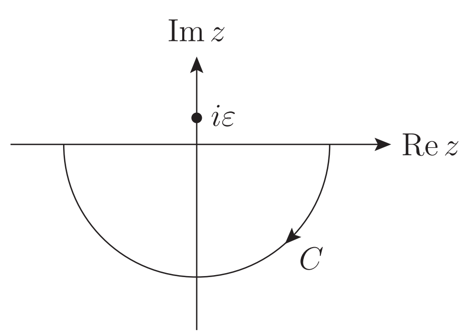
Inside the counterclockwise contour C there exists a simple pole at $z = i\varepsilon$ (since by assumption, $\varepsilon > 0$). Thus, by the residue theorem of complex analysis,

$$I(k, \varepsilon) = 2\pi i\, \frac{1}{2\pi i}\, \mathrm{Res}\left(\frac{e^{ikz}}{z - i\varepsilon}\right) = e^{-k\varepsilon}\,,$$

where $\mathrm{Res} f(z) = \lim_{z\to z_0}(z - z_0) f(z)$ is the residue due to a simple pole at $z = z_0$.

Case 2: $k < 0$. Then it follows that

$$I(k, \varepsilon) = \frac{1}{2\pi i} \int_C dz\, \frac{e^{ikz}}{z - i\varepsilon}$$



where the contour C is now closed in the lower half plane. Since in this case $k < 0$, the integrand is again exponentially damped along the semicircular part of the contour C and thus the contribution to the integral along the semicircular arc goes to zero as the radius of the semicircle is taken to infinity. But, now the pole lies outside the closed contour C. Hence, by Cauchy's Theorem of complex analysis, it follows that $I(k,\varepsilon)=0$ for $k<0$.

Taking the limit as $\varepsilon\to 0$, we conclude that

$$\lim_{\varepsilon\to 0} I(k,\varepsilon)=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}dx\,\frac{e^{ikx}}{x-i\varepsilon}=\begin{cases}1\,, & \text{if } k>0,\\ 0\,, & \text{if } k<0.\end{cases}$$

In light of eq. (1), we have verified the integral representation of the Heavyside step function:

$$\Theta(k)=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}dx\,\frac{e^{ikx}}{x-i\varepsilon}\,. \tag{5}$$

A more explicit derivation of eq. (5) is given in Appendix A.

The integral representation of the Dirac delta function is given by,

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\,dk\,. \tag{6}$$

Strictly speaking, the above integral does not converge since the integrand oscillates at infinity with a constant amplitude. Nevertheless, this integral can be interpreted in *the sense of distributions*.[2] There are two different ways to provide meaning to the integral representation of the delta function. First, one can insert a convergence factor, $e^{-\epsilon k^2}$, to render the integral convergent. Then,

$$\begin{aligned}\delta(x)&=\lim_{\epsilon\to 0^+}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx-\epsilon k^2}\,dk=\lim_{\epsilon\to 0^+}\frac{1}{2\pi}e^{-x^2/(4\epsilon)}\int_{-\infty}^{\infty}\exp\left[-\epsilon\left(k-\frac{ix}{2\epsilon}\right)^2\right]dk\\ &=\lim_{\epsilon\to 0^+}\frac{1}{2\sqrt{\pi\epsilon}}e^{-x^2/(4\epsilon)}\,, \end{aligned}\tag{7}$$

which has the property that $\delta(x)$ is an even function of x that satisfies $\delta(x)=0$ for any $x\neq 0$, and the area under the delta function is unity,

$$\int_{-\infty}^{\infty}\delta(x)\,dx=\lim_{\epsilon\to 0^+}\frac{1}{2\sqrt{\pi\epsilon}}\int_{-\infty}^{\infty}e^{-x^2/(4\epsilon)}\,dx=1\,. \tag{8}$$

Moreover one can use eq. (7) to rederive the result previously obtained in eq. (4). In particular, under the assumption that $f(x)$ is well-behaved at $x=0$ (equivalently, assuming that $f(x)$ can be expanded in a Taylor series around the origin),

$$\begin{aligned}\int_{-\infty}^{\infty}f(x)\delta(x)\,dx&=\lim_{\epsilon\to 0^+}\frac{1}{2\sqrt{\pi\epsilon}}\int_{-\infty}^{\infty}f(x)e^{-x^2/(4\epsilon)}\,dx\\ &=\lim_{\epsilon\to 0^+}\frac{1}{2\sqrt{\pi\epsilon}}\sum_{n=0}^{\infty}f^{(n)}(0)\int_{-\infty}^{\infty}x^n e^{-x^2/(4\epsilon)}\,dx\,,\\ &=f(0)+\lim_{\epsilon\to 0^+}\frac{1}{2\sqrt{\pi\epsilon}}\sum_{n=1}^{\infty}f^{(n)}(0)\int_{-\infty}^{\infty}x^n e^{-x^2/(4\epsilon)}\,dx\,, \end{aligned}\tag{9}$$

[2] Here, the term *distributions* is being employed as a synonym for generalized functions.

where $f^{(n)}(0) \equiv (d^n f/dx^n)_{x=0}$, and the Taylor series of $f(x)$ has been employed. For odd n, the integrand in eq. (9) is an odd function of x and thus the corresponding integral vanishes. For even n, the integrand is an even function of x and thus the integral is can be evaluated by taking the limits of integration from 0 to ∞ and multiplying by 2. After defining $y \equiv x/(\sqrt{2}\,\epsilon)$, we are left with the task of evaluating

$$\mathcal{I}_n \equiv \int_0^\infty y^n e^{-y^2}\,dy\,, \qquad \text{for } n = 0, 2, 4, \ldots \tag{10}$$

An integration by parts yields the recursion relation $\mathcal{I}_n = \frac{1}{2}(n-1)\mathcal{I}_{n-2}$. Using the well-known result, $\mathcal{I}_0 = \frac{1}{2}\sqrt{\pi}$, one easily derives a closed form expression for $\mathcal{I}_n$. The end result is:

$$\frac{1}{2\sqrt{\pi\epsilon}}\int_{-\infty}^\infty x^n e^{-x^2/(4\epsilon)}\,dx = \begin{cases} (2\epsilon)^{n/2}(n-1)!!\,, & \text{for } n = 0, 2, 4, \ldots, \\ 0\,, & \text{for } n = 1, 3, 5, \ldots, \end{cases} \tag{11}$$

where $(n-1)!! = (n-1)(n-3)\cdots 5\cdot 3\cdot 1$ for positive even integers n. By convention, $(-1)!! \equiv 1$. Taking the $\epsilon \to 0$ limit, it follows that the second term on the right hand side of eq. (9) vanishes, which establishes the result obtained in eq. (4).

One can also employ an alternative method to identify the integral representation of the delta function as follows. Consider the generalized function,

$$\mathcal{J}(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^0 e^{ikx}\,dk + \frac{1}{2\pi}\int_0^\infty e^{ikx}\,dk = \frac{1}{\pi}\int_0^\infty \cos(kx)\,dk\,, \tag{12}$$

after performing a variable change, $k \to -k$, in the second integral above. As expected, the integral of $\cos(kx)$ is not convergent. Nevertheless, one can employ the well known conditionally convergent integral,[3]

$$\int_0^\infty \frac{\sin(kx)}{k}\,dk = \tfrac{1}{2}\pi\,\mathrm{sgn}(x)\,, \tag{13}$$

where $\mathrm{sgn}(x)$ is the sign of the real number x,[4]

$$\mathrm{sgn}(x) = \Theta(x) - \Theta(-x) = \begin{cases} +1\,, & \text{for } x > 0, \\ -1\,, & \text{for } x < 0, \end{cases} \tag{14}$$

which satisfies

$$\frac{d}{dx}\,\mathrm{sgn}(x) = \frac{d}{dx}\big[\Theta(x) - \Theta(-x)\big] = 2\,\delta(x)\,, \tag{15}$$

in light of eq. (3). As a result, we can identify $\mathcal{J}(x)$ given by eq. (12) as the derivative of a conditionally convergent integral,

$$\mathcal{J}(x) = \frac{1}{\pi}\int_0^\infty \frac{\partial}{\partial x}\left\{\frac{\sin(kx)}{k}\right\}dk = \frac{d}{dx}\left\{\frac{1}{\pi}\int_0^\infty \frac{\sin(kx)}{k}\,dk\right\} = \frac{1}{2}\frac{d}{dx}\,\mathrm{sgn}(x) = \delta(x)\,, \tag{16}$$

in agreement with the integral representation of the delta function [cf. eqs. (6) and (12)].

The delta function arises in many applications in physics. In Appendix B, we discuss two important identities involving the delta function that arise in the study of electromagnetism.

[3] Eq. (13) is well-known and is often obtained using the residue theorem of complex analysis by integrating e^{ikx}/k over a suitably chosen closed contour. Eq. (13) is also a consequence of eq. (A.37) [since the integrand is an even function of k], which is derived in Appendix A.

[4] Some books define $\mathrm{sgn}(0) = 0$, in which case, eq. (13) would be valid at $x = 0$. However, when $\mathrm{sgn}(x)$ is regarded as a generalized function, the specification of the value at the origin has no significance (cf. footnote 1).

1.2 Additional examples of generalized functions

The generalized function, $\mathrm{P}(1/x)$, can be defined by the following equation,

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)}{x}\,dx=\int_0^{\infty}\frac{f(x)-f(-x)}{x}\,dx\,, \tag{17}$$

where $f(x)$ is regular in a neighborhood of the real axis and vanishes as $|x|\to\infty$. It is shown in Appendix C that eq. (17) is equivalent to the definition of the Cauchy principal value,

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x}\equiv\lim_{\delta\to 0^+}\left\{\int_{-\infty}^{-\delta}\frac{f(x)\,dx}{x}+\int_{\delta}^{\infty}\frac{f(x)\,dx}{x}\right\}\,. \tag{18}$$

where $\delta\to 0^+$ indicates that the limit is taken for positive values of δ.

There is an alternate definition of $\mathrm{P}(1/x)$ that is also quite useful. Noting that

$$\frac{d}{dx}\ln|x|=\frac{1}{x}\,,\qquad\text{for } x\neq 0\,, \tag{19}$$

one can extend this result to $x=0$ by treating $d\ln|x|/dx$ as a generalized function. For any well-behaved test function $f(x)$ that vanishes sufficiently fast as $x\to\pm\infty$, it follows from an integration by parts that

$$\int_{-\infty}^{\infty}f(x)\frac{d}{dx}\ln|x|\,dx=-\int_{-\infty}^{\infty}\ln|x|f'(x)\,dx=-\lim_{\delta\to 0^+}\int_{|x|\geq\delta}\ln|x|f'(x)\,dx\,. \tag{20}$$

where $f'(x)\equiv df/dx$, and the boundary terms vanish due to the behavior of $f(x)$ at $\pm\infty$. Note that the limiting process above is smooth, since the integral above exists for all values of $\delta\geq 0$. To complete the calculation, we integrate by parts once more to obtain,

$$\int_{-\infty}^{\infty}f(x)\frac{d}{dx}\ln|x|\,dx=\lim_{\delta\to 0^+}\left[\int_{|x|\geq\delta}\frac{f(x)}{x}\,dx-\big[f(\delta)-f(-\delta)\big]\ln\delta\right]\,. \tag{21}$$

However, $\big[f(\delta)-f(-\delta)\big]\ln\delta=\mathcal{O}(\delta\ln\delta)$ which vanishes as $\delta\to 0$. Thus, we end up with

$$\int_{-\infty}^{\infty}f(x)\frac{d}{dx}\ln|x|\,dx=\lim_{\delta\to 0^+}\int_{|x|\geq\delta}\frac{f(x)}{x}\,dx=\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)}{x}\,dx\,, \tag{22}$$

in light of eq. (18). Hence, we can identify the generalized function,

$$\mathrm{P}\,\frac{1}{x}=\frac{d}{dx}\ln|x|\,, \tag{23}$$

which now attains meaning at $x=0$ via eq. (22). An alternative derivation of eq. (23) is given in Appendix C. Note that $\mathrm{P}(1/x)$ is an odd function of x.

In Ref. [4], two additional functions, x_+^{-1} and x_-^{-1} are defined via

$$x_+^{-1}=\Theta(x)\frac{1}{x}=\begin{cases}x^{-1}\,, & \text{for } x>0\,,\\ 0\,, & \text{for } x<0\,.\end{cases} \tag{24}$$

$$x_-^{-1}=-\Theta(-x)\frac{1}{x}=\begin{cases}0\,, & \text{for } x>0\,,\\ |x|^{-1}\,, & \text{for } x<0\,.\end{cases} \tag{25}$$

As in the case of $\mathrm{P}(1/x)$, we would like to extend the definition of x_+^{-1} and x_-^{-1} such that they yield finite results when integrated against a test function over the real axis. The corresponding generalized functions are defined by[5]

$$\frac{1}{x_+} \equiv \lim_{\mu\to 0}\left\{\Theta(x)\frac{1}{x^{1-\mu}} - \frac{1}{\mu}\delta(x)\right\}, \tag{26}$$

$$\frac{1}{x_-} \equiv \lim_{\mu\to 0}\left\{\Theta(-x)\frac{1}{(-x)^{1-\mu}} - \frac{1}{\mu}\delta(x)\right\}, \tag{27}$$

where μ is a real and positive infinitesimal. It then follows that

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{f(x)}{x_+}\,dx &= \lim_{\mu\to 0}\left\{\int_0^\infty \frac{f(x)}{x^{1-\mu}}\,dx - \frac{1}{\mu}\int_{-\infty}^{\infty}\delta(x)f(x)\,dx\right\}\\
&= \lim_{\mu\to 0}\left\{\int_0^1 \frac{f(x)}{x^{1-\mu}}\,dx - \frac{1}{\mu}f(0)\right\} + \int_1^\infty \frac{f(x)}{x}\,dx\\
&= \int_0^1 \frac{f(x)-f(0)}{x}\,dx + \int_1^\infty \frac{f(x)}{x}\,dx + \lim_{\mu\to 0}\left\{f(0)\int_0^1 \frac{dx}{x^{1-\mu}}\,dx - \frac{1}{\mu}f(0)\right\}. \qquad (28)
\end{aligned}$$

In the last step above, we wrote $f(x) = f(x) - f(0) + f(0)$ and took the $\mu \to 0$ limit in the first term, which is allowed since the corresponding integral converges for any smooth function $f(x)$ that vanishes sufficiently fast as $x \to \pm\infty$. Finally, the last two terms in eq. (28) cancel exactly, and we end up with a well-defined and finite result,

$$\int_{-\infty}^{\infty}\frac{f(x)}{x_+}\,dx = \int_0^\infty \frac{f(x)-f(0)\Theta(1-x)}{x}\,dx = \int_0^1 \frac{f(x)-f(0)}{x}\,dx + \int_1^\infty \frac{f(x)}{x}\,dx\,. \tag{29}$$

A similar computation yields,

$$\int_{-\infty}^{\infty}\frac{f(x)}{x_-}\,dx = \int_0^\infty \frac{f(-x)-f(0)\Theta(1-x)}{x}\,dx = \int_0^1 \frac{f(-x)-f(0)}{x}\,dx + \int_1^\infty \frac{f(-x)}{x}\,dx\,. \tag{30}$$

It is easy to verify that the generalized functions $1/x_\pm$ satisfy,

$$\mathrm{P}\frac{1}{x} = \frac{1}{x_+} - \frac{1}{x_-}\,. \tag{31}$$

Note that eq. (31) is true for $x \neq 0$ in light of the definitions of $x_\pm$ given by eqs. (24) and (25). In addition, one can check that subtracting eq. (30) from eq. (29) yields eq. (17). One can also use eq. (31) to obtain another definition of $\mathrm{P}(1/x)$ by employing eqs. (26) and (27),

$$\mathrm{P}\frac{1}{x} = \lim_{\mu\to 0}\left\{\Theta(x)\frac{1}{x^{1-\mu}} - \Theta(-x)\frac{1}{(-x)^{1-\mu}}\right\}. \tag{32}$$

[5]The conditions, $xx_+^{-1} = \Theta(x)$ and $xx_-^{-1} = -\Theta(-x)$, do not yield unique generalized functions. For example, $x\big[x_+^{-1} + C\delta(x)\big] = \Theta(x)$ for any constant C. The definitions given in eqs. (26) and (27) are taken from Ref. [2] and are motivated by the desire that the integrals given in eqs. (29) and (30) should be well-defined and finite. Some books write $\mathrm{Pf}(1/x_\pm)$ in eqs. (26) and (27) to distinguish these generalized functions from the functions defined in eqs. (24) and (25), where Pf stands for *pseudofunction* [3, 5, 6]. However, we choose to follow Refs. [2, 4] in omitting the symbol Pf when employing the generalized functions $x_\pm^{-1}$ in these notes.

In light of $|x| = -x$ for $x < 0$, it follows that

$$\Theta(x)\frac{1}{x^{1-\mu}} - \Theta(-x)\frac{1}{(-x)^{1-\mu}} = \big[\Theta(x) - \Theta(-x)\big]\frac{1}{|x|^{1-\mu}} = \frac{\operatorname{sgn}(x)}{|x|^{1-\mu}}\,, \tag{33}$$

where $\operatorname{sgn}(x)$ is the sign function defined in eq. (14). Hence, eq. (32) yields

$$\mathrm{P}\frac{1}{x} = \lim_{\mu\to 0}\frac{\operatorname{sgn}(x)}{|x|^{1-\mu}}\,. \tag{34}$$

It is a simple exercise to verify that multiplying the right hand side of eq. (34) by $f(x)$ and integrating over the real line yields eq. (17),

$$\begin{aligned}\mathrm{P}\int\frac{f(x)}{x}\,dx &= \lim_{\mu\to 0}\int_{-\infty}^{\infty}\frac{f(x)}{|x|^{1-\mu}}\operatorname{sgn}(x)\,dx = \lim_{\mu\to 0}\left\{\int_0^{\infty}\frac{f(x)}{x^{1-\mu}}\,dx - \int_{-\infty}^{0}\frac{f(x)}{x^{1-\mu}}\,dx\right\}\\ &= \lim_{\mu\to 0}\int_0^{\infty}\frac{f(x)-f(-x)}{x^{1-\mu}}\,dx = \int_0^{\infty}\frac{f(x)-f(-x)}{x}\,dx\,,\end{aligned} \tag{35}$$

after changing the integration variable $x \to -x$ in the second integral in the penultimate step above. In the final step, we can set $\mu = 0$ since the resulting integral is well-defined and finite, under the assumption that $f(x)$ is a smooth function that vanishes as $x \to \pm\infty$.

Next, we consider the function,

$$\frac{1}{|x|} = \big[\Theta(x) - \Theta(-x)\big]\frac{1}{x} = \begin{cases} x^{-1}\,, & \text{for } x > 0\,,\\ -x^{-1}\,, & \text{for } x < 0\,.\end{cases} \tag{36}$$

We again propose to extend the definition of $|x|^{-1}$ such that it yields a finite result when integrated against a test function over the real axis. The corresponding generalized function, denoted by $\mathrm{Pf}(1/|x|)$, is defined by,[6]

$$\mathrm{Pf}\frac{1}{|x|} = \lim_{\mu\to 0}\left\{\frac{1}{|x|^{1-\mu}} - \frac{2}{\mu}\delta(x)\right\}. \tag{37}$$

In order to see the relation between $\mathrm{Pf}(1/|x|)$ and the generalized functions $x_{\pm}^{-1}$, we employ eq. (2) to obtain the identity,

$$\frac{1}{|x|^{1-\mu}} = \frac{1}{|x|^{1-\mu}}\big[\Theta(x) + \Theta(-x)\big] = \Theta(x)\frac{1}{x^{1-\mu}} + \Theta(-x)\frac{1}{(-x)^{1-\mu}}\,. \tag{38}$$

Hence, one can re-express eq. (37) as,

$$\mathrm{Pf}\frac{1}{|x|} = \lim_{\mu\to 0}\left\{\Theta(x)\frac{1}{x^{1-\mu}} + \Theta(-x)\frac{1}{(-x)^{1-\mu}} - \frac{2}{\mu}\delta(x)\right\}. \tag{39}$$

[6]A generalized function that coincides with $1/|x|$ for $x \neq 0$ not unique; see footnote 5. However, all possible choices for defining such a generalized function have the property that they are even functions of x. One particularly convenient choice is $\mathrm{Pf}(1/|x|)$, which in the notation of Refs. [2, 4] is denoted as $1/|x|$. In contrast to footnote 5, we prefer to keep the Pf (pseudofunction) symbol in this case [3, 5, 6].

Comparing with eqs. (26) and (27), we see that

$$\mathrm{Pf}\frac{1}{|x|} = \frac{1}{x_+} + \frac{1}{x_-}\,. \tag{40}$$

Note that eq. (40) is clearly true for $x \neq 0$ in light of the initial definitions of $x_\pm^{-1}$ and $|x|^{-1}$ given by eqs. (24), (25) and (36).

It is convenient to rewrite eq. (30) by changing the integration variable $x \to -x$ and writing $|x| = -x$ for $x < 0$, which yields

$$\int_{-\infty}^{\infty} \frac{f(x)}{x_-}\,dx = -\int_{-\infty}^{0} \frac{f(x) - f(0)\Theta(1+x)}{x}\,dx = \int_{-\infty}^{-1} \frac{f(x)}{|x|}\,dx + \int_{-1}^{0} \frac{f(x) - f(0)}{|x|}\,dx\,. \tag{41}$$

Using eqs. (29) and (41), it then follows that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{Pf}\frac{1}{|x|}\,dx \equiv \int_{-\infty}^{-1} \frac{f(x)}{|x|}\,dx + \int_{-1}^{1} \frac{f(x) - f(0)}{|x|}\,dk + \int_{1}^{\infty} \frac{f(x)}{|x|}\,dx\,, \tag{42}$$

which is a well-defined and finite result, assuming $f(x)$ is smooth and vanishes sufficiently fast as $x \to \pm\infty$.

A few more generalized functions are of interest. First, $\lim_{\varepsilon\to 0} 1/(x \pm i\varepsilon)$ is related to other known generalized functions via the Sokhotski-Plemelj formula (which is derived in Section 2),

$$\lim_{\varepsilon\to 0} \frac{1}{x \pm i\varepsilon} = \mathrm{P}\frac{1}{x} \mp i\pi\delta(x)\,, \tag{43}$$

where $\varepsilon > 0$ is an infinitesimal real quantity. Note that eq. (43) is actually two separate formulae, where the two choices of signs on the left hand and right hand sides of eq. (43) are correlated. Adding these two formulae yields another representation of $P(1/x)$,

$$P\frac{1}{x} = \lim_{\epsilon\to 0} \frac{x}{x^2 + \epsilon^2}\,. \tag{44}$$

Likewise, subtracting the two formulae given by eq. (43) yields a representation of the delta function,

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon\to 0} \frac{\epsilon}{x^2 + \epsilon^2}\,. \tag{45}$$

The expressions given by eqs. (44) and (45) should be understood to mean that one first integrates over the corresponding generalized function multiplied by a smooth test function before taking the limit of $\epsilon \to 0$ [e.g., see eqs. (C.10)–(C.15) in Appendix C].

Perhaps less well known is the formula for the generalized function $(x \pm i\varepsilon)^{-1}\ln(x \pm i\varepsilon)$,[7]

$$\lim_{\varepsilon\to 0} \frac{1}{x \pm i\varepsilon} \ln(x \pm i\varepsilon) = \mathrm{P}\frac{1}{x}\ln|x| \mp i\pi\frac{1}{x_-} + \tfrac{1}{2}\pi^2\delta(x)\,, \tag{46}$$

[7] The symbol for the generalized function, $(x \pm i\varepsilon)^{-1}\ln(x \pm i\varepsilon)$, should be considered as a single (unified) symbol. In particular, one should not split up this symbol into the product of two separate generalized functions. That is, one cannot derive eq. (46) by simply multiplying eqs. (43) and (47).

which, as in eq. (43), consists of two separate equations—one where the upper signs are employed and one where the lower signs are employed. In eq. (46), x is a real variable and the complex logarithm is given by its principal value,

$$\lim_{\varepsilon\to 0}\ln(x\pm i\varepsilon)=\ln|x|\pm i\pi\Theta(-x)\,, \tag{47}$$

on the cut complex plane, where the branch cut runs along the negative real axis.

To derive eq. (46), we first take the square of eq. (47),

$$\lim_{\varepsilon\to 0}\ln^2(x\pm i\varepsilon)=\ln^2|x|-\pi^2\Theta(-x)\pm 2\pi i\Theta(-x)\ln|x|\,, \tag{48}$$

where we have used the fact that $[\Theta(-x)]^2=\Theta(-x)$. In order to derive eq. (46), we shall take the derivative of eq. (48) and divide by two. The derivative of the first term on the right hand side of eq. (48) is

$$\frac{d}{dx}\ln^2|x|=2\ln|x|\frac{d}{dx}\ln|x|=2\ln|x|\,\mathrm{P}\frac{1}{x}\,, \tag{49}$$

where we have used eq. (23) The derivative of the second term on the right hand side of eq. (48) is easily obtained after noting that

$$\frac{d}{dx}\Theta(-x)=-\frac{d}{dx}\Theta(x)=-\delta(x)\,.$$

The derivative of the third term on the right hand side of eq. (48) requires a little more effort. Under the assumption that the test function $f(x)$ is smooth and vanishes sufficiently fast as $x\to\pm\infty$, we evaluate (with the help of integration by parts),

$$\begin{aligned}
\int_{-\infty}^{\infty}f(x)\frac{d}{dx}\big[\Theta(-x)\ln|x|\big]\,dx &= -\int_{-\infty}^{0}\frac{df}{dx}\ln(-x)\,dx=-\lim_{\delta\to 0}\int_{-\infty}^{-\delta}\frac{df}{dx}\ln(-x)\,dx\\
&=-\lim_{\delta\to 0}\left\{\ln(-x)f(x)\Big|_{-\infty}^{-\delta}-\int_{-\infty}^{-\delta}\frac{f(x)}{x}\,dx\right\}\\
&=-\lim_{\delta\to 0}\left\{\ln\delta\,f(-\delta)-\int_{-\infty}^{-\delta}\frac{f(x)}{x}\,dx\right\}\\
&=-\lim_{\delta\to 0}\left\{f(0)\int_{-1}^{-\delta}\frac{dx}{x}-\int_{-\infty}^{-\delta}\frac{f(x)}{x}\,dx\right\}\\
&=\int_{-\infty}^{0}\frac{f(x)-f(0)\Theta(1+x)}{x}\,dx\\
&=-\int_{-\infty}^{\infty}\frac{f(x)}{x_-}\,dx\,.
\end{aligned} \tag{50}$$

In the derivation above, we noted that $f(-\delta) = f(0) - \delta f'(0) + \mathcal{O}(\delta^2)$ and made use of $\lim_{\delta\to 0}\delta\ln\delta = 0$. In the final step, we employed eq. (41). It then follows that

$$\frac{d}{dx}\big[\Theta(-x)\ln|x|\big] = -\frac{1}{x_-}\,. \tag{51}$$

A similar analysis (see, e.g., p. 25 of Ref. [4]) yields,

$$\frac{d}{dx}\big[\Theta(x)\ln|x|\big] = \frac{1}{x_+}\,. \tag{52}$$

We now add eqs. (51) and (52) and employ eq. (31) and $\Theta(x) + \Theta(-x) = 1$ to obtain,

$$\frac{d}{dx}\ln|x| = \mathrm{P}\frac{1}{x}\,, \tag{53}$$

in agreement with eq. (23). Likewise, if we subtract eq. (51) from eq. (52) and employ eqs. (14) and (40), we obtain,

$$\frac{d}{dx}\big[\mathrm{sgn}(x)\ln|x|\big] = \mathrm{Pf}\frac{1}{|x|}\,. \tag{54}$$

In some books, eqs. (51)–(54) are employed as the definitions of the corresponding generalized functions, x_-^{-1}, x_+^{-1}, $\mathrm{P}(1/x)$, and $\mathrm{Pf}(1/|x|)$, respectively.

Finally, taking the derivative of eq. (48) and dividing by two, we end up with

$$\lim_{\varepsilon\to 0}\frac{1}{x\pm i\varepsilon}\ln(x\pm i\varepsilon) = \mathrm{P}\frac{1}{x}\ln|x| + \tfrac{1}{2}\pi^2\delta(x) \mp \pi i\frac{1}{x_-}\,. \tag{55}$$

Thus, eq. (46) has been established.[8]

As application of eqs. (43) and (46), consider the Fourier transform of $\ln|x|$, which has been obtained in Ref. [4],[9]

$$\int_{-\infty}^{\infty}\ln|x|\,e^{ikx}\,dx = i\left\{\big[-\gamma + \tfrac{1}{2}i\pi - \ln(k+i\varepsilon)\big]\frac{1}{k+i\varepsilon} + \big[\gamma + \tfrac{1}{2}i\pi + \ln(k+i\varepsilon)\big]\frac{1}{k-i\varepsilon}\right\}, \tag{56}$$

where γ is the Euler-Mascheroni constant. In eq. (56) and in what follows, we shall always assume the $\varepsilon \to 0$ limit without explicitly indicating the limit symbol.

We can simplify eq. (56) as follows. First, we write

$$\begin{aligned}\int_{-\infty}^{\infty}\ln|x|\,e^{ikx}\,dx = -i\gamma\left(\frac{1}{k+i\varepsilon} - \frac{1}{k-i\varepsilon}\right) - \frac{\pi}{2}\left(\frac{1}{k+i\varepsilon} + \frac{1}{k-i\varepsilon}\right)\\ -i\left\{\frac{1}{k+i\varepsilon}\ln(k+i\varepsilon) - \frac{1}{k-i\varepsilon}\ln(k-i\varepsilon)\right\}.\end{aligned} \tag{57}$$

[8] An alternative proof of eq. (46) can be found on pp. 96–98 of Ref. [4].

[9] Eq. (56) is also derived on pp. 160–161 of Ref. [3]. However, one must correct a typographical error that appears in eq. (6.4.57) of this work, where $-i(u - i0)^{-1}$ should be replaced by $+i(u - i0)^{-1}$. On pp. 153–154 of Ref. [3], one can also find a derivation of eq. (60) below, although again one must correct a typographical error in eq. (6.4.33d) of this work, where $1/u$ should be replaced by $1/|u|$.

Using eqs. (43) and (46), we end up with

$$\int_{-\infty}^{\infty} \ln|x|\, e^{ikx}\, dx = -2\pi\gamma\delta(k) - \pi\left(\mathrm{P}\frac{1}{k} + 2\frac{1}{k_-}\right). \tag{58}$$

Finally, employing eqs. (31) and (40) yields

$$\mathrm{P}\frac{1}{k} + 2\frac{1}{k_-} = \mathrm{Pf}\frac{1}{|k|}. \tag{59}$$

Inserting eq. (59) into eq. (58), we end up with

$$\int_{-\infty}^{\infty} \ln|x|\, e^{ikx}\, dx = -\pi\left[\mathrm{Pf}\frac{1}{|k|} + 2\gamma\delta(k)\right], \tag{60}$$

which reproduces the result quoted below in eq. (83).

2 The Sokhotski-Plemelj Formula

The Sokhotski-Plemelj formula is a relation between the following generalized functions,

$$\lim_{\varepsilon\to 0} \frac{1}{x \pm i\epsilon} = \mathrm{P}\frac{1}{x} \mp i\pi\delta(x), \tag{61}$$

where $\epsilon > 0$ is an infinitesimal real quantity. This identity formally makes sense only when first multiplied by a function $f(x)$ that is smooth and non-singular in a neighborhood of the origin and vanish sufficiently fast as $x \to \pm\infty$ to ensure convergence when integrated along the real line. Moreover, all surface terms at $\pm\infty$ that arise when integrating by parts are assumed to vanish.

To establish eq. (61), we shall prove that

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{f(x)\, dx}{x \pm i\epsilon} = \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\, dx}{x} \mp i\pi f(0), \tag{62}$$

where the Cauchy principal value integral is defined in eq. (18), under the assumption that $f(x)$ is regular in a neighborhood of the real axis and vanishes as $|x| \to \infty$.

In this section, I will provide three different derivations of eq. (62). The first derivation is a mathematically non-rigorous proof of eq. (62), which should at least provide some insight into the origin of this result. A more rigorous derivation starts with a contour integral in the complex plane,

$$\int_C \frac{f(z)\, dz}{z}.$$

By defining C appropriately, we will obtain two different expressions for this integral. Setting the two resulting expressions equal yields eq. (62) with the upper sign. Complex conjugating this result yields eq. (62) with the lower sign. Finally, an elegant third proof makes direct use of the theory of distributions. Finally, a useful check is to consider the Fourier transform of eq. (61), which is presented in Section 3.

Eq. (61) can be generalized as follows,

$$\lim_{\varepsilon\to 0}\frac{1}{x-x_0\pm i\varepsilon}=\mathrm{P}\frac{1}{x-x_0}\mp i\pi\delta(x-x_0)\,, \tag{63}$$

where

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x-x_0}\equiv\lim_{\delta\to 0^+}\left\{\int_{-\infty}^{x_0-\delta}\frac{f(x)\,dx}{x-x_0}+\int_{x_0+\delta}^{\infty}\frac{f(x)\,dx}{x-x_0}\right\}. \tag{64}$$

Note that eq. (62) and its generalization via eqs. (63) and (64) involve integration along the real axis. These ideas generalize further to the so-called Cauchy type integrals as shown in Appendix D, and yield the Plemelj formulae of complex variables theory.

Finally, we note that by adding the two equations exhibited by eq. (63), one obtains an equivalent definition of the Cauchy principal value integral:

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x-x_0}=\lim_{\epsilon\to 0}\frac{1}{2}\left\{\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x-x_0+i\epsilon}+\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x-x_0-i\epsilon}\right\}. \tag{65}$$

In some cases, the integrals on the right hand side of eq. (65) can be evaluated by closing the contour in the upper or lower half complex plane with a semicircular arc of radius $R\to\infty$ and using Cauchy's residue theorem.

2.1 A non-rigorous derivation of the Sokhotski-Plemelj formula

We begin with the identity,

$$\frac{1}{x\pm i\epsilon}=\frac{x\mp i\epsilon}{x^2+\epsilon^2}\,,$$

where ϵ is a positive infinitesimal quantity. Thus, for any smooth function that is non-singular in a neighborhood of the origin,

$$\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x\pm i\epsilon}=\int_{-\infty}^{\infty}\frac{xf(x)\,dx}{x^2+\epsilon^2}\mp i\epsilon\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x^2+\epsilon^2}\,. \tag{66}$$

The first integral on the right had side of eq. (66),

$$\int_{-\infty}^{\infty}\frac{xf(x)\,dx}{x^2+\epsilon^2}=\int_{-\infty}^{-\delta}\frac{xf(x)\,dx}{x^2+\epsilon^2}+\int_{\delta}^{\infty}\frac{xf(x)\,dx}{x^2+\epsilon^2}+\int_{-\delta}^{\delta}\frac{xf(x)\,dx}{x^2+\epsilon^2}\,. \tag{67}$$

In the first two integrals on the right hand side of eq. (67), it is safe to take the limit $\epsilon\to 0$. In the third integral on the right hand side of eq. (67), if δ is small enough, then we can approximate $f(x)\simeq f(0)$ for values of $|x|<\delta$. Hence, eq. (67) yields,

$$\int_{-\infty}^{\infty}\frac{xf(x)\,dx}{x^2+\epsilon^2}=\lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta}\frac{f(x)\,dx}{x}+\int_{\delta}^{\infty}\frac{f(x)\,dx}{x}\right\}+f(0)\int_{-\delta}^{\delta}\frac{x\,dx}{x^2+\epsilon^2}\,. \tag{68}$$

However,

$$\int_{-\delta}^{\delta}\frac{x\,dx}{x^2+\epsilon^2}=0\,,$$

since the integrand is an odd function of x that is being integrated symmetrically about the origin. Hence, eq. (68) yields

$$\int_{-\infty}^{\infty} \frac{x f(x)\, dx}{x^2+\epsilon^2} = \mathrm{P} \int_{-\infty}^{\infty} \frac{f(x)\, dx}{x}\,, \tag{69}$$

where P denotes the Cauchy principal value prescription [cf. eq. (18)].

Next, we consider the second integral on the right hand side of eq. (66). Since ϵ is an infinitesimal quantity, the only significant contribution from

$$\epsilon \int_{-\infty}^{\infty} \frac{f(x)\, dx}{x^2+\epsilon^2}$$

can come from the integration region where $x \simeq 0$, where the integrand behaves like ϵ^{-2}. Thus, we can again approximate $f(x) \simeq f(0)$, in which case we obtain

$$\epsilon \int_{-\infty}^{\infty} \frac{f(x)\, dx}{x^2+\epsilon^2} \simeq \epsilon f(0) \int_{-\infty}^{\infty} \frac{dx}{x^2+\epsilon^2} = \pi f(0)\,, \tag{70}$$

where we have made use of

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+\epsilon^2} = \frac{1}{\epsilon} \tan^{-1}(x/\epsilon)\bigg|_{-\infty}^{\infty} = \frac{\pi}{\epsilon}\,.$$

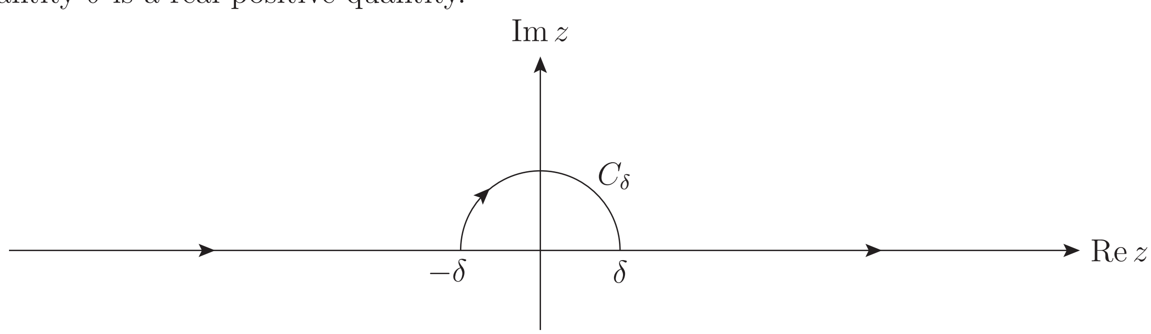
Using the results of eqs. (69) and (70), we see that eq. (66) yields,

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{f(x)\, dx}{x \pm i\epsilon} = \mathrm{P} \int_{-\infty}^{\infty} \frac{f(x)\, dx}{x} \mp i\pi f(0)\,, \tag{71}$$

which establishes eq. (62).

2.2 A more rigorous derivation of the Sokhotski-Plemelj formula

Consider a path of integration in the complex plane, denoted by C (and exhibited below), which is a contour along the real axis from $-\infty$ to $-\delta$, followed by a semicircular path C_δ (of radius δ), followed by a contour along the real axis from δ to ∞, where the infinitesimal quantity δ is a real positive quantity.



Integrating the function $f(x)/x$ along the path C yields:

$$\int_C \frac{f(x)}{x}\, dx = \mathrm{P} \int_{-\infty}^{\infty} \frac{f(x)}{x} + \int_{C_\delta} \frac{f(x)}{x}\, dx\,, \tag{72}$$

where the principal value integral is defined in eq. (18). In the limit of $\delta \to 0$, we can approximate $f(x) \simeq f(0)$ in the last integral on the right hand side of eq. (72). Noting that the contour C_δ can be parametrized as $x = \delta e^{i\theta}$ for $0 \le \theta \le \pi$, we end up with

$$\lim_{\delta\to 0} \int_{C_\delta} \frac{f(x)}{x}\, dx = f(0) \lim_{\delta\to 0} \int_\pi^0 \frac{i\delta e^{i\theta}}{\delta e^{i\theta}}\, d\theta = -i\pi f(0)\,.$$

Hence,

$$\int_C \frac{f(x)}{x}\, dx = \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)}{x} - i\pi f(0)\,. \tag{73}$$

We can also evaluate the left hand side of eq. (73) by deforming the contour C to a contour C' that consists of a straight line that runs from $-\infty + i\varepsilon$ to $\infty + i\varepsilon$, where ε is a positive infinitesimal (of the same order of magnitude as δ). Assuming that $f(x)$ has no singularities in an infinitesimal neighborhood around the real axis, we are free to deform the contour C into C' without changing the value of the integral. It follows that

$$\int_C \frac{f(x)}{x}\, dx = \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{f(x)}{x}\, dx = \int_{-\infty}^{\infty} \frac{f(y+i\varepsilon)}{y+i\varepsilon}\, dy\,, \tag{74}$$

where in the last step we have made a change of the integration variable.

Since ε is infinitesimal, we can approximate $f(y+i\varepsilon) \simeq f(y)$.[10] Thus, after relabeling the integration variable y as x, eq. (74) yields

$$\int_C \frac{f(x)}{x}\, dx = \int_{-\infty}^{\infty} \frac{f(x)}{x+i\varepsilon}\, dx\,. \tag{75}$$

Inserting this result back into eq. (73) yields

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{f(x)}{x+i\varepsilon}\, dx = \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)}{x} - i\pi f(0)\,. \tag{76}$$

Eq. (76) is also valid if $f(x)$ is replaced by $f^*(x)$. We can then take the complex conjugate of the resulting equation. The end result is[11]

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{f(x)}{x \pm i\varepsilon}\, dx = \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)}{x} \mp i\pi f(0)\,,$$

in agreement with eq. (71).

[10] More precisely, we can expand $f(y+i\varepsilon)$ in a Taylor series about $\varepsilon = 0$ to obtain $f(y+i\varepsilon) = f(y) + \mathcal{O}(\varepsilon)$. At the end of the calculation, we may take $\varepsilon \to 0$, in which case the $\mathcal{O}(\varepsilon)$ terms vanish.

[11] Alternatively, we can repeat the above derivation where the contour C_δ is replaced by a semicircle of radius δ in the lower half complex plane, which yields eq. (73) with $-i$ replaced by i. Finally, after deforming the contour of integration to a new contour that consists of a straight line that runs from $-\infty - i\varepsilon$ to $\infty - i\varepsilon$, one obtains eq. (75) with i replaced by $-i$.

2.3 An elegant derivation of the Sokhotski-Plemelj formula

We can employ $\mathrm{P}(1/x) = d\ln|x|/dx$ obtained in eq. (23) to provide a very elegant derivation of eq. (61). We begin with the definition of the principal value of the complex logarithm,

$$\ln\, z = \ln|z| + i\arg z\,,$$

where arg z is the principal value of the argument (or phase) of the complex number z, with the convention that $-\pi < \arg z \leq \pi$. In particular, for real x and a positive infinitesimal ϵ,

$$\lim_{\epsilon\to 0}\ln(x \pm i\epsilon) = \ln|x| \pm i\pi\,\Theta(-x)\,, \tag{77}$$

as previously noted in eq. (47).

Differentiating eq. (77) with respect to x and employing eqs. (3) and (23) immediately yields the Sokhotski-Plemelj formula,

$$\lim_{\varepsilon\to 0}\frac{1}{x \pm i\epsilon} = \mathrm{P}\frac{1}{x} \mp i\pi\delta(x)\,. \tag{78}$$

In this derivation, we have made use of the fact that the derivative of the principal value of the complex logarithm is $d\ln z/dz = 1/z$ for all complex values of z that do not lie on the branch cut along the negative real axis.

3 Fourier transforms of generalized functions

The generalized functions specified in eqs. (23) and (61), which we repeat below,

$$\frac{d}{dx}\ln|x| = \mathrm{P}\frac{1}{x}\,, \tag{79}$$

$$\lim_{\varepsilon\to 0}\frac{1}{x \pm i\varepsilon} = \mathrm{P}\frac{1}{x} \mp i\pi\delta(x)\,, \tag{80}$$

are only meaningful when multiplied by a test function $f(x)$ and integrated over a region of the real line that may or may not include the origin. Here, we shall require that the test functions are infinitely differentiable and vanish at $\pm\infty$ faster than any inverse power of x.[12] Clearly, e^{ikx} does not satisfy this requirement for a test function. Nevertheless, one can define Fourier transforms of generalized functions by using the well known property of the Fourier transform,

$$\int_{-\infty}^{\infty}\widetilde{f}(k)g(k)\,dk = \int_{-\infty}^{\infty}f(k)\widetilde{g}(k)\,dx\,, \tag{81}$$

where

$$\widetilde{f}(k) \equiv \int_{-\infty}^{\infty}f(x)e^{ikx}\,dx\,.$$

If $f(x)$ is a generalized function and $g(x)$ is a test function, then if follows that $\widetilde{g}(x)$ exists and is well defined. The Fourier transform of $g(x)$, denoted by $\widetilde{g}(k)$, is *defined* via eq. (81).

[12]Generalized functions that are defined by employing test functions with these characteristics are called tempered distributions.

One can now check the validity of eqs. (79) and (80) by computing their Fourier transforms. To compute the Fourier transform of eq. (79), we make use of the property of Fourier transforms that

$$\int_{-\infty}^{\infty} \frac{df(x)}{dx} e^{ikx}\, dx = -ik\widetilde{f}(k)\,.$$

Hence,

$$\int_{-\infty}^{\infty} \frac{d}{dx} \ln|x|\, e^{ikx}\, dx = -ik \int_{-\infty}^{\infty} \ln|x|\, e^{ikx}\, dx\,. \tag{82}$$

The calculation of the right-hand side of eq. (82) is rather involved, since it only exists in the sense of distributions. One can show that[13]

$$\int_{-\infty}^{\infty} \ln|x|\, e^{ikx}\, dx = -\pi \left[\mathrm{Pf}\frac{1}{|k|} + 2\gamma\delta(k)\right]\,, \tag{83}$$

where γ is the Euler-Mascheroni constant, and the generalized function $\mathrm{Pf}(1/|k|)$ was defined in eq. (37) [see also eq. (42)].

Inserting the result of eq. (56) into eq. (82) and using $k\delta(k) = 0$ and[14]

$$k\left(\mathrm{Pf}\frac{1}{|k|}\right) = \frac{k}{|k|} = \mathrm{sgn}(k)\,, \tag{84}$$

the end result is given by,

$$\int_{-\infty}^{\infty} \frac{d}{dx} \ln|x|\, e^{ikx}\, dx = i\pi\, \mathrm{sgn}(k)\,. \tag{85}$$

Next, we consider

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{e^{ikx}}{x}\, dx = \mathrm{P}\int_{-\infty}^{\infty} \frac{\cos(kx)}{x}\, dx + i\mathrm{P}\int_{-\infty}^{\infty} \frac{\sin(kx)}{x}\, dx\,. \tag{86}$$

Since $\cos(kx)/x$ is an odd function of x (i.e., it changes sign under $x \to -x$), it immediately follows from the definition of the Cauchy principle value that

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{\cos(kx)}{x}\, dx = 0\,. \tag{87}$$

Moreover, we observe that $\lim_{x\to 0} \sin(kx)/x = k$; that is, $\sin(kx)/x$ is regular at $x = 0$. Thus,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{\sin(kx)}{x}\, dx = \int_{-\infty}^{\infty} \frac{\sin(kx)}{x}\, dx = \mathrm{sgn}(k)\int_{-\infty}^{\infty} \frac{\sin y}{y}\, dy = \pi\, \mathrm{sgn}(k)\,. \tag{88}$$

Note that the P symbol has no effect on the integral given by eq. (88), since the integrand is regular at $x = 0$. The factor of $\mathrm{sgn}(k)$ arises after changing the integration variable, $y = kx$.

[13] For example, see pp. 153–154 and pp. 160–161 of Ref. [3]. There are two typographical errors on these pages. In eq. (6.4.33d), $1/u$ should be $1/|u|$ and in the last term in eq. (6.4.57), $-i(u - i0)^{-1}$ should be $+i(u - i0)^{-1}$. Eq. (56) is a consequence of the corrected eq. (6.4.57).

[14] When we multiply $\mathrm{Pf}(1/|k|)$ by k, the singularity at $k = 0$ is canceled and the prescription indicated by eq. (42) is no longer required. Noting that $k/|k|$ is equal to sign of k for $k \neq 0$, we end up with eq. (84).

When $k < 0$, the integration limits must be reversed, which then leads to the extra minus sign. Inserting eqs. (87) and (88) into eq. (86) then yields,

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x}\,dx = i\pi\,\mathrm{sgn}(k)\,. \tag{89}$$

In light of eqs. (85) and (89), we have verified that the Fourier transform of eq. (79) is satisfied.

To verify that the Fourier transform of eq. (80) is satisfied, one can employ

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x\pm i\varepsilon}\,dx = \mp 2\pi i\Theta(\mp k)\,, \tag{90}$$

which follows from the results of Section 1.1 (see also Appendix A). Then, using eqs. (2), (14) and (89), it follows that the Fourier transform of eq. (80) is

$$\mp 2\pi i\Theta(\mp k) = i\pi\big[\Theta(k)-\Theta(-k)\big] \mp i\pi\big[\Theta(k)+\Theta(-k)\big]\,. \tag{91}$$

It is a simple matter to verify that eq. (91) is satisfied for either choice of sign.

Since the Fourier transform of a generalized function and its inverse Fourier transform are unique, one can conclude that if the Fourier transforms of eqs. (79) and (80) are satisfied, then eqs. (79) and (80) are valid identities. Thus, the validity of eq. (91) provides a fourth independent derivation of the Sokhotski-Plemelj formula.

Finally, we note the following Fourier transforms of the generalized functions $x_\pm^{-1}$ [defined in eqs. (26) and (27)], which are obtained in Corollary 4.6 of Ref. [2],

$$\int_{-\infty}^{\infty}\frac{1}{x_\pm}\,e^{ikx}\,dx = -\gamma - \ln|k| \pm \tfrac{1}{2}i\pi\,\mathrm{sgn}(k)\,, \tag{92}$$

where γ is the Euler-Mascheroni constant. Subtracting or adding the two equations given by eq. (92) yields

$$\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{P}\frac{1}{x}\,dx = i\pi\,\mathrm{sgn}(k)\,, \tag{93}$$

$$\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{Pf}\frac{1}{|x|}\,dx = -2\big[\gamma+\ln|k|\big]\,, \tag{94}$$

after making use of eqs. (31) and (40). Note that eq. (93) is equivalent to eq. (89).

Taking the real part of eq. (94) yields an integral with an integrand that is an even function of x. Hence, we may conclude that

$$\int_0^{\infty}\mathrm{Pf}\frac{1}{|x|}\,\cos(kx)\,dx = -\gamma-\ln|k|\,. \tag{95}$$

Note that one cannot replace $\mathrm{Pf}(1/|x|)$ with $|x|^{-1}$ in eq. (95), as these two functions differ at $x = 0$ due to the delta function contribution in eq. (39). However, since $\cos(kx)$ is an even function of x, one can use eq. (42) to rewrite eq. (95) in a more useful form:

$$\int_0^1\frac{\cos(kx)-1}{x}\,dx + \int_1^{\infty}\frac{\cos(kx)}{x}\,dx = -\gamma-\ln|k|\,, \tag{96}$$

which reproduces a result that is easily derived from formula 3.782 number 1 on p. 451 of Ref. [9]. Taking the imaginary part of eq. (94) yields an integral with an integrand that is odd function of x. Moreover, $\mathrm{Pf}(1/|x|)\sin(kx) = |x|^{-1}\sin(kx)$, since the latter has a finite limit as $x \to 0$. It then follows that

$$\int_{-\infty}^{\infty} \frac{\sin(kx)}{|x|}\, dx = 0\,, \tag{97}$$

as one would expect.

4 The Riemann-Lebesgue Lemma

The Riemann-Lebesgue Lemma is one of the most important results of Fourier analysis and asymptotic analysis. It has many physics applications, especially in studies of wave phenomena. In this short note, I will provide a simple proof of the Riemann-Lebesgue lemma which will be adequate for most cases that arise in physical applications.

The simplest form of the Riemann-Lebesgue lemma states that for a function $f(x)$ for which the integral

$$\int_a^b |f(x)|\, dx < \infty\,, \tag{98}$$

where a and b are real numbers, we have

$$\lim_{k\to\infty} \int_a^b f(x)\, e^{ikx}\, dx = 0\,. \tag{99}$$

Sometimes, the result of eq. (99) appears in the form,

$$\lim_{k\to\infty} e^{ikx} = 0\,. \tag{100}$$

Of course eq. (100) makes no sense when interpreted as a standard limit in mathematical analysis. However, if one interprets the limit of eq. (100) in the *sense of distributions*, i.e. by treating e^{ikx} as a generalized function, then eq. (100) can be assigned a useful meaning [8].

If we further assume that $f(x)$ has certain "nice" properties [e.g., a sufficient (but not necessary) condition is that $f(x)$ is continuously differentiable for $a \leq x \leq b$], then it follows that

$$\int_a^b f(x)\, e^{ikx}\, dx = \mathcal{O}\left(\frac{1}{k}\right)\,, \qquad \text{as } k \to \infty. \tag{101}$$

Moreover, eqs. (99) and (101) continue to hold if $a \to -\infty$ and/or $b \to \infty$, assuming that eq. (98) holds over the infinite interval.

We will present a proof of eq. (99) under the assumption that $f(x)$ is continuous over the closed interval $a \leq x \leq b$. The origin of eq. (99) in this case is not too difficult to understand. In the limit of $k \to \infty$, the factor e^{ikx} oscillates faster and faster such that $f(x)\, e^{ikx}$ averages out to zero over any finite region of x inside the interval.

If we can assume that $f(x)$ is N-times differentiable in the region $a \leq x \leq b$, one can derive eq. (101) simply by a repeated integration by parts. Namely,

$$\int_a^b f(x)\, e^{ikx}\, dx = \frac{f(x)}{ik}\, e^{ikx}\bigg|_a^b - \frac{1}{ik}\int_a^b f'(x)\, e^{ikx}\, dx$$
$$= \frac{e^{ikb}f(b) - e^{ika}f(a)}{ik} - \frac{f'(x)}{(ik)^2}\, e^{ikx}\bigg|_a^b + \frac{1}{(ik)^2}\int_a^b f''(x)\, e^{ikx}\, dx$$
$$= \sum_{n=0}^{N-1}(-1)^n\, \frac{e^{ikb}f^{(n)}(b) - e^{ika}f^{(n)}(a)}{(ik)^{n+1}} + \mathcal{O}\left(\frac{1}{k^{N+1}}\right), \tag{102}$$

where $f^{(0)}(x) \equiv f(x)$ and $f^{(n)}(x) \equiv d^n f/dx^n$. For example $f'(b)$ is equal to the first derivative of $f(x)$ evaluated at $x = b$, etc. Taking $N = 1$ in eq. (102), we see that the leading term that survives is of $\mathcal{O}(1/k)$ as asserted by eq. (101).

More generally, we can prove eq. (99) without the assumption $f(x)$ is differentiable in the interval. We do this by writing the integral

$$\mathcal{I}(k) = \int_a^b f(x)\, e^{ikx}\, dx\,,$$

in two different but equivalent ways:[15]

$$\mathcal{I}(k) = \int_a^{a+\pi/k} f(x)\, e^{ikx}\, dx + \int_{a+\pi/k}^b f(x)\, e^{ikx}\, dx \tag{103}$$

and

$$\mathcal{I}(k) = \int_a^{b-\pi/k} f(x)\, e^{ikx}\, dx + \int_{b-\pi/k}^b f(x)\, e^{ikx}\, dx\,. \tag{104}$$

By a change of variables, $x' = x - \pi/k$, it is straightforward to verify that

$$\int_{a+\pi/k}^b f(x)\, e^{ikx}\, dx = -\int_a^{b-\pi/k} f\left(x + \frac{\pi}{k}\right)\, e^{ikx}\, dx\,, \tag{105}$$

after using $e^{-i\pi} = -1$ and dropping the primes from the x in the second integral. Thus, writing $\mathcal{I}$ as one half the sum of eqs. (103) and (104), and employing eq. (105), it follows that

$$\mathcal{I}(k) = \tfrac{1}{2}\int_a^{a+\pi/k} f(x)\, e^{ikx}\, dx \;+\; \tfrac{1}{2}\int_{b-\pi/k}^b f(x)\, e^{ikx}\, dx$$
$$+\, \tfrac{1}{2}\int_a^{b-\pi/k} \left[f(x) - f\left(x + \frac{\pi}{k}\right)\right] e^{ikx}\, dx\,. \tag{106}$$

We now take the limit of $k \to \infty$. The mean value theorem of for integrals states that if $f(x)$ is continuous and bounded over a closed interval, $a \leq x \leq b$, then

$$\int_a^b f(x)\, dx = f(c)(b - a)\,,$$

[15]This proof is taken from pp. 39–40 of Ref. [10].

for some real number c that lies in the interval $a \leq c \leq b$. Applying this to the first two integrals in eq. (106), we immediately conclude that

$$\int_a^{a+\pi/k} f(t)\, e^{ikx}\, dx = \mathcal{O}\left(\frac{1}{k}\right)\,, \qquad\qquad \int_{b-\pi/k}^{b} f(k)\, e^{ikx}\, dx = \mathcal{O}\left(\frac{1}{k}\right)\,,$$

which vanish in the limit of $k \to \infty$.

Finally, under the assumption that $f(x)$ is continuous at all points in the closed interval $a \leq x \leq b$, it follows that

$$\lim_{k\to\infty} \int_a^{b-\pi/tk} \left[f(x) - f\left(x + \frac{\pi}{k}\right)\right] e^{ikx}\, dx = 0\,. \tag{107}$$

This is true because a function that is continuous at all points in a closed, bounded interval is *uniformly* continuous over the interval.[16] Hence, one can make the integrand in eq. (107) arbitrarily small by choosing k sufficiently large. The limit of eq. (107) is thus established, and the proof of Riemann-Lebesgue lemma stated in eq. (99) is complete.

Note that the argument above does *not* necessarily imply that

$$\int_a^{b-\pi/k} \left[f(x) - f\left(x + \frac{\pi}{k}\right)\right] e^{ikx} = \mathcal{O}\left(\frac{1}{k}\right)\,, \tag{108}$$

as $k \to \infty$. However, if the function $f(x)$ is continuously differentiable in the interval, then we can employ the mean value theorem for differentiable functions, which states that

$$f(b) - f(a) = f'(c)(b-a)\,, \qquad \text{for some } c \text{ between } a \text{ and } b\,.$$

It follows that

$$f(x) - f\left(x + \frac{\pi}{k}\right) = -\frac{\pi}{k} f'(x+c)\,, \qquad \text{for } 0 \leq c \leq \frac{\pi}{k}\,.$$

Hence, in this case eq. (108) does hold, in which case eq. (101) is satisfied.[17]

The extension to cases where $a \to -\infty$ and/or $b \to \infty$ is straightforward. For example, suppose that

$$\int_a^\infty |f(x)| dx < \infty\,. \tag{109}$$

Then, noting that one can write

$$\int_a^\infty f(x) e^{ikx}\, dx = \int_a^b f(x) e^{ikx}\, dx + \epsilon\,,$$

[16]For further details, see e.g. pp. 228–229 of Ref. [11].

[17]We also obtain eq. (108) under slightly weaker conditions in which the function $f(x)$ satisfies the so-called *Lipschitz condition*, $|f(x) - f(y)| \leq M|x-y|$ for all x and y in the interval for some positive finite bound M. Indeed a Lipschitz continuous function is uniformly continuous (although the converse is not necessarily true). A Lipschitz continuous function need not be differentiable. On the other hand a differentiable function whose derivative is bounded on the interval satisfies the Lipschitz condition. Thus, to establish eq. (108) it is sufficient to require that $f(k)$ is Lipschitz continuous in the interval.

where $\epsilon < \int_b^\infty |f(x)| dx$, it follows [in light of eq. (109)] that it is always possible to make ϵ arbitrarily small by a suitable (finite) choice of b. Hence, we can use eq. (99) to conclude that

$$\lim_{k\to\infty} \int_a^\infty f(x)\, e^{ikx}\, dx = 0\,. \tag{110}$$

Finally, it should be noted that taking the real and imaginary parts of eq. (99) under the assumption that $f(x)$ is a real valued function yields:

$$\lim_{k\to\infty} \int_a^b f(x) \sin(kx)\, dx = 0\,, \tag{111}$$

$$\lim_{k\to\infty} \int_a^b f(x) \cos(kx)\, dx = 0\,, \tag{112}$$

with a similar extension to cases where $a \to -\infty$ and/or $b \to \infty$. That is,

$$\lim_{k\to\infty} \cos(kx) = \lim_{k\to\infty} \sin(kx) = 0\,, \tag{113}$$

where the limits are interpreted in the sense of distributions [cf. discussion below eq. (100)]. If we further assume that $f(x)$ has certain "nice" properties [cf. comments above eq. (101)], then the two integrals given in eqs. (111) and (112) behave as $\mathcal{O}(1/k)$ as $k \to \infty$.

By consulting a table of Fourier transforms [12], one can see many examples of functions that satisfy eq. (98). In all cases, you will find that the corresponding Fourier transform satisfies eq. (99). It is interesting to look for cases that satisfy eq. (99) and not eq. (101). For example,

$$\int_0^\infty x^{\nu-1}\, e^{-ax}\, e^{ikx}\, dx = \frac{\Gamma(\nu)}{(a-ik)^\nu}\,, \qquad \text{for } a > 0 \text{ and } \operatorname{Re}\,\nu > 0\,.$$

Indeed, eqs. (98) and (99) are satisfied for all Re $\nu > 0$, whereas eq. (101) is only satisfied when Re $\nu \geq 1$.

Finally, we provide some examples that contradicts eq. (111). For $f(x) = 1/x$, one obtains the well known integral [cf. eq. (13)]:

$$\int_0^\infty \frac{\sin(kx)}{x}\, dx = \tfrac{1}{2}\pi \operatorname{sgn} k\,, \tag{114}$$

which does not vanish as $k \to \infty$. This is not surprising since $\int_0^\infty x^{-1} dx$ diverges and hence eq. (109) is not satisfied. Similarly, for $f(x) = \sin(x^2)$, we employ the following result given in formula 3.691 number 5 on p. 419 of Ref. [9],

$$\int_0^\infty \sin(x^2) \sin(kx)\, dx = \tfrac{1}{2}\sqrt{\pi} \cos\left(\frac{k^2+\pi}{4}\right)\,,$$

which also does not vanish as $k \to \infty$. Again, this is not surprising since eq. (109) [with $a = 0$] is not satisfied. Indeed, even though

$$\int_0^\infty \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,,$$

is finite, one can check that

$$\int_0^\infty |\sin(x^2)| dx$$

diverges. In both examples presented above, one of the key assumptions underlying the Riemann-Lebesgue lemma is not satisfied.

5 The Poisson Sum Formula

The Poisson sum formula takes on a number of different forms in the literature. Here is one useful version:

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx} = \sum_{m=-\infty}^{\infty} \delta(x - 2\pi m) . \tag{115}$$

To prove this formula, consider the following periodic function, defined by:

$$f(x) = f(x + 2\pi) , \quad \text{where} \quad f(x) = \frac{1}{2} - \frac{x}{2\pi} \text{ for } 0 \le x \le 2\pi. \tag{116}$$

Noting that $f(x)$ is discontinuous at $x = 2\pi m$, where $m \in \mathbb{Z}$, it follows that one can expand $f(x)$ in a Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} , \tag{117}$$

where

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(x) \, dx . \tag{118}$$

Evaluating eq. (118) using $f(x)$ given in eq. (116), one easily obtains:

$$c_0 = 0 , \qquad c_n = \frac{-i}{2\pi n} , \quad (n \neq 0) . \tag{119}$$

That is,

$$f(x) = -\frac{i}{2\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{inx}}{n} . \tag{120}$$

Consider the derivative of $f(x)$, which we denote by $f'(x)$. Using the definition of $f(x)$ given in eq. (116), it follows that $f'(x) = -1/(2\pi)$ for $x \neq 2\pi m$ (for integer values of $m \in \mathbb{Z}$). At $x = 2\pi m$, the discontinuity of $f(x)$ can be described by the step function $\Theta(x)$. Specifically, in the vicinity of $x = 2\pi m$,

$$f(x) = -\tfrac{1}{2} + \Theta(x - 2\pi m) , \qquad \text{for } x \simeq 2\pi m . \tag{121}$$

That is, $f(x) = -1/2$ for $x = 2\pi m - \epsilon$ and $f(x) = 1/2$ for $x = 2\pi m + \epsilon$, where $\epsilon > 0$ is an infinitesimal quantity. Taking the derivative of eq. (121) yields:

$$f'(x) = \delta(x - 2\pi m) , \qquad \text{for } x \simeq 2\pi m .$$

We conclude that:

$$f'(x) = -\frac{1}{2\pi} + \sum_{m=-\infty}^{\infty} \delta(x - 2\pi m)\,. \tag{122}$$

We can also compute $f'(x)$ by differentiating the Fourier series of $f(x)$ term-by-term. Using eq. (120), we obtain:

$$f'(x) = \frac{1}{2\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{inx} = \frac{1}{2\pi}\left[-1 + \sum_{n=-\infty}^{\infty} e^{inx}\right]. \tag{123}$$

Equating eqs. (122) and (123) yields the desired result announced in eq. (115).

The most common form for the Poisson sum formula arises in the study of Fourier analysis. Given a function $f(t)$ and its Fourier transform,

$$F(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega t} f(t) dt\,, \tag{124}$$

then the Poisson sum formula is given by:[18]

$$\sum_{m=-\infty}^{\infty} f(\alpha m) = \frac{1}{|\alpha|} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{\alpha}\right), \quad \text{for real } \alpha \neq 0. \tag{125}$$

One can derive the above result by inserting the integral expression for F on the right-hand side of eq. (125), which yields

$$\begin{aligned}
\frac{1}{|\alpha|} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{\alpha}\right) &= \frac{1}{|\alpha|} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i n t/\alpha} f(t)\, dt = \frac{1}{|\alpha|} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{2\pi i n t/\alpha} f(t)\, dt \\
&= \frac{2\pi}{|\alpha|} \int_{-\infty}^{\infty} f(t) \sum_{m=-\infty}^{\infty} \delta\left(\frac{2\pi t}{\alpha} - 2\pi m\right) dt \\
&= \int_{-\infty}^{\infty} f(t) \sum_{m=-\infty}^{\infty} \delta(t - \alpha m)\, dt \\
&= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - \alpha m)\, dt \\
&= \sum_{m=-\infty}^{\infty} f(\alpha m)\,,
\end{aligned} \tag{126}$$

after employing eq. (116) and making use of the well-known identity

$$\delta\big(\alpha(x - x')\big) = \frac{1}{|\alpha|}\delta(x - x')\,. \tag{127}$$

Thus, eq. (125) is established. For further details, see for example pp 67–71 of Ref. [1], pp. 155–159 of Ref. [2], or pp. 168–171 of Refs. [3].

[18]Note that the corresponding formula in Ref. [1] incorrectly omits the absolute value sign that appears on the right hand side of eq. (125). The corresponding formula in Ref. [2] specifies that $\alpha > 0$, and hence no absolute value sign appears. The absolute value sign appears correctly in Ref. [3].

Appendix A: An explicit derivation of the Fourier transformation of the Heavyside step function

The goal of this Appendix is to express the step function as a Fourier transform,

$$\Theta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} f(x)\, dx\,, \tag{A.1}$$

where the function $f(x)$ is to be determined.[19]

The function $f(x)$ is determined by the inverse Fourier transform,

$$f(x) = \int_{-\infty}^{\infty} e^{-ikx}\, \Theta(k)\, dk\,. \tag{A.2}$$

The integral exhibited in eq. (A.2) is not well defined. However it can be reinterpreted in *the sense of distributions*. What this phrase really means is that quantities are treated as generalized functions (also called distributions), which make sense only when integrated against test functions that are smooth, regular, and vanish sufficiently fast at $\pm\infty$.

We can evaluate $f(x)$ using the following trick. Note that

$$1 = \int_{-\infty}^{\infty} e^{-ikx}\, \delta(k)\, dk = \int_{-\infty}^{\infty} e^{-ikx}\, \frac{d\Theta(k)}{dk}\, dk\,. \tag{A.3}$$

We now integrate by parts. We can set the surface term to zero by employing

$$\lim_{k\to\pm\infty} e^{-ikx} = 0\,, \tag{A.4}$$

where the limit is interpreted in the sense of distributions [cf. eq. (100)]. It then follows that

$$1 = -\int_{-\infty}^{\infty} \Theta(k)\frac{d}{dk}e^{-ikx}\, dk = ix\int_{-\infty}^{\infty} \Theta(k) e^{-ikx}\, dk = ixf(x)\,. \tag{A.5}$$

To solve eq. (A.5), let us define $h(x) \equiv if(x)$ and consider the equation

$$xh(x) = 1\,. \tag{A.6}$$

The solution to this equation for $x \neq 0$ is clearly $h(x) = 1/x$. But, how should we deal with $x = 0$? The answer is again to appeal to generalized functions. In particular, eq. (A.6) should be interpreted as

$$\int_{-\infty}^{\infty} xh(x)g(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx\,, \tag{A.7}$$

for any smooth regular test function $g(x)$ that vanishes sufficiently fast at $\pm\infty$.

The most general solution to the inhomogeneous equation, $xh(x) = 1$, must be of the form,

$$h(x) = h_p(x) + h_h(x)\,, \tag{A.8}$$

[19]This Appendix is based on a derivation given on p. 151 of Ref. [3].

where $h_p(x)$ is a particular solution that satisfies $xh_p(x) = 1$ and $h_h(x)$ is the solution to the homogeneous equation, $xh_h(x) = 0$. I claim that one choice for the particular solution to eq. (A.6) is,

$$h_p(x) = \mathrm{P}\frac{1}{x}\,, \tag{A.9}$$

where P indicates the Cauchy principal value prescription defined in eq. (18) [see also Appendix C].

Let us check that $h(x) = h_p(x)$ given by eq. (A.9) provides a solution to eq. (A.7). It is sufficient to observe that,

$$\mathrm{P}\int_{-\infty}^{\infty} x\,\frac{1}{x}\,g(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dx\,, \tag{A.10}$$

where the P symbol can be dropped on the right hand side of eq. (A.10) since the corresponding integral is now well defined. Hence, it follows that (in the sense of distributions),

$$x\,\mathrm{P}\frac{1}{x} = 1\,, \tag{A.11}$$

and eq. (A.9) is verified.

We now turn to the most general solution to the homogeneous equation,

$$xh_h(x) = 0\,. \tag{A.12}$$

We shall solve eq. (A.12) using a Fourier transform technique. We first write

$$h_h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} q(k)\,dk\,. \tag{A.13}$$

Inverting the Fourier transform yields

$$q(k) = \int_{-\infty}^{\infty} e^{-ikx} h_h(x)\,dx\,. \tag{A.14}$$

We now take the derivative of $q(k)$ with respect to k to obtain,

$$\frac{dq}{dk} = -i\int_{-\infty}^{\infty} e^{-ikx} x h_h(x)\,dx = 0\,. \tag{A.15}$$

where we have used eq. (A.12) in the final step.

The most general solution to the differential equation $dq/dk = 0$ is $q(k) = C$, where C is an arbitrary constant.[20] Inserting this solution back into eq. (A.13), we end up with

$$h_h(x) = \frac{C}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk = C\,\delta(x)\,, \tag{A.16}$$

after employing the integral representation of the delta function [cf. eq. (6)].

[20] This statement is trivial if solutions are restricted to the space of ordinary functions. Nevertheless, $q(k) = C$ is still the unique solution of $dq/dk = 0$ even if the solution space is expanded to included generalized functions. A proof of this assertion can be found on pp. 39–41 of Ref. [4].

It is simple to check the validity of eq. (A.16). In particular, in light of eq. (4),

$$\int_{-\infty}^{\infty} x\delta(x)g(x)\,dx = xg(x)\bigg|_{x=0} = 0\,, \tag{A.17}$$

where again we have used the fact that $g(x)$ is a smooth regular function. It then follows that

$$x\delta(x) = 0\,, \tag{A.18}$$

in the sense of distributions. Hence, $xh_h(x) = Cx\delta(x) = 0$ as required. Combining the results of eqs. (A.9) and (A.16), one obtains the most general solution of eq. (A.6),

$$h(x) = h_p(x) + h_h(x) = \mathrm{P}\frac{1}{x} + C\delta(x)\,. \tag{A.19}$$

Note that eq. (A.19) implies that $\mathrm{P}(1/x)$ is the unique choice for $h(x)$ that is an odd function of x [i.e., it satisfies $h(-x) = -h(x)$]. Other possible choices for $h(x)$ treated in Section 1.2 such as $1/x_\pm$ and $1/(x \pm i\varepsilon)$ are neither even nor odd functions of x.

Returning to eq. (A.5), it follows in light of eq. (A.19) that

$$h(x) = if(x) = i\int_{-\infty}^{\infty} \Theta(k)e^{-ikx}\,dk = \mathrm{P}\frac{1}{x} + C\delta(x)\,, \tag{A.20}$$

where the constant C is still yet to be determined. To fix the constant C we proceed as follows. Replacing $x \to -x$ in eq. (A.20) yields,

$$-\mathrm{P}\frac{1}{x} + C\delta(x) = i\int_{-\infty}^{\infty} \Theta(k)e^{ikx}\,dk = i\int_{-\infty}^{\infty} \Theta(-k)e^{-ikx}\,dk\,, \tag{A.21}$$

after noting that $\delta(-x) = \delta(x)$ and changing the integration variable from k to $-k$. Adding eqs. (A.20) and (A.21) and using eq. (2), we end up with

$$2C\delta(x) = i\int_{-\infty}^{\infty} \big[\Theta(k) + \Theta(-k)\big]e^{-ikx}\,dk = i\int_{-\infty}^{\infty} e^{-ikx}\,dk\,. \tag{A.22}$$

Finally, using the integral representation of the delta function [cf. eq. (6)], we conclude that $C = i\pi$. We now insert this value of C into eq. (A.20) and employ the Sokhotski-Plemelj formula [cf. eq. (43)],

$$\lim_{\varepsilon\to 0}\frac{1}{x - i\epsilon} = \mathrm{P}\frac{1}{x} + i\pi\delta(x)\,. \tag{A.23}$$

The end result is

$$if(x) = i\int_{-\infty}^{\infty} \Theta(k)e^{-ikx}\,dk = \lim_{\varepsilon\to 0}\frac{1}{x - i\varepsilon}\,. \tag{A.24}$$

Returning to eq. (A.1), we conclude that

$$\Theta(k) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x - i\varepsilon}\,dx\,, \tag{A.25}$$

which reconfirms the result exhibited in eq. (5).

It is instructive to revisit eq. (A.20) with $C = i\pi$, which yields the noteworthy result,

$$i\int_{-\infty}^{\infty}\Theta(k)e^{-ikx}\,dk = \mathrm{P}\frac{1}{x} + i\pi\delta(x)\,. \tag{A.26}$$

In particular, in light of eq. (1), the complex conjugate of eq. (A.26) yields,

$$\int_0^{\infty} e^{ikx}\,dk = i\,\mathrm{P}\frac{1}{x} + \pi\delta(x)\,. \tag{A.27}$$

Equivalently, we can employ eq. (A.23) and rewrite eq. (A.27) in the form,

$$\int_0^{\infty} e^{ikx}\,dk = \lim_{\varepsilon\to 0}\frac{i}{x+i\varepsilon}\,. \tag{A.28}$$

Eqs. (A.27) and (A.28) must be interpreted in the sense of distributions (since the integrals do not converge in the usual sense[21]). Moreover, taking real and imaginary parts of eq. (A.27) yield,

$$\int_0^{\infty}\cos(kx)\,dk = \pi\delta(x)\,, \tag{A.29}$$

$$\int_0^{\infty}\sin(kx)\,dk = \mathrm{P}\frac{1}{x}\,. \tag{A.30}$$

Once again, the integrals of eqs. (A.29) and (A.30) must be interpreted in the sense of distributions. For example, we have already obtained eq. (A.29) in light of eqs. (12) and (16). Moreover eq. (A.30) provides yet another representation of $\mathrm{P}(1/x)$. Interpreting the improper integral in the standard way,

$$\mathrm{P}\frac{1}{x} = \lim_{R\to\infty}\int_0^R \sin(kx)\,dk = \lim_{R\to\infty}\left\{-\frac{1}{x}\cos(kx)\right\}\bigg|_0^R = \lim_{R\to\infty}\frac{1-\cos(Rx)}{x}\,. \tag{A.31}$$

One additional consequence of eq. (A.26) can be extracted if we invert the Fourier transform,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\big[\mathrm{P}\frac{1}{x} + i\pi\delta(x)\big]\,dx = i\Theta(k)\,. \tag{A.32}$$

Using the Cauchy principal value prescription and integrating over the delta function yields

$$\frac{1}{2\pi}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x}\,dx = i\big[\Theta(k) - \tfrac{1}{2}\big]\,. \tag{A.33}$$

Using eq. (2), it follows that

$$\Theta(k) - \tfrac{1}{2} = \Theta(k) - \tfrac{1}{2}\big[\Theta(k)+\Theta(-k)\big] = \tfrac{1}{2}\big[\Theta(k)-\Theta(-k)\big] = \tfrac{1}{2}\,\mathrm{sgn}(k)\,. \tag{A.34}$$

[21]Compare this with the integral in eq. (6), which does not converge in the usual sense, but nevertheless provides an integral representation of the delta function in the sense of distributions.

Inserting this result into eq. (A.33) yields,

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x}\,dx = i\pi\,\mathrm{sgn}(k)\,. \tag{A.35}$$

Taking the real and imaginary parts of eq. (A.35) yields,

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{\cos(kx)}{x}\,dx = 0\,, \tag{A.36}$$

$$\int_{-\infty}^{\infty}\frac{\sin(kx)}{x}\,dx = \pi\,\mathrm{sgn}(k)\,. \tag{A.37}$$

Note that the vanishing of the integral in eq. (A.36) is due to the fact that the integrand is an odd function of x, which integrates to zero due to the Cauchy principal value prescription. The P symbol is not needed in eq. (A.37) since the corresponding integrand has a finite limit as $k \to 0$.

As one final check, let us take the derivative of eq. (A.35) with respect to k and employ eq. (15). This yields (once again) the integral representation of the delta function (where the P symbol can be dropped as the resulting integrand is not singular at $x = 0$).

Appendix B: The delta function and Electrostatics

In the theory of electrostatics, a point charge q located at the origin has a charge density given by (e.g., see Ref. [13]):

$$\rho(\vec{\boldsymbol{x}}) = q\,\delta^3(\vec{\boldsymbol{x}})\,, \tag{B.1}$$

where $\delta^3(\vec{\boldsymbol{x}}) \equiv \delta(x)\delta(y)\delta(z)$. Indeed, by taking the volume integral over all space and using

$$\int_V \delta^3(\vec{\boldsymbol{x}})\,d^3x = \int_{-\infty}^{\infty}\delta(x)\,dx\int_{-\infty}^{\infty}\delta(y)\,dy\int_{-\infty}^{\infty}\delta(z)\,dz = 1\,, \tag{B.2}$$

one obtains the expected result,

$$\int_V \rho(\vec{\boldsymbol{x}})\,d^3x = q\,. \tag{B.3}$$

The electrostatic field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})$ can be expressed in terms of the electrostatic potential $\Phi(\vec{\boldsymbol{x}})$ via $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = -\vec{\boldsymbol{\nabla}}\Phi(\vec{\boldsymbol{x}})$. Using Coulomb's law, the electrostatic potential due to a point charge q located at the origin is given (in gaussian units) by

$$\Phi(\vec{\boldsymbol{x}}) = \frac{q}{r}\,, \tag{B.4}$$

where $r \equiv |\vec{\boldsymbol{x}}|$. Using one of Maxwell's equations, $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}} = 4\pi\rho$, one immediately obtains the Poisson equation,

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\Phi = -4\pi\rho\,. \tag{B.5}$$

It then follows from eqs. (B.1) and (B.4) that

$$\vec{\boldsymbol{\nabla}}^{\mathbf{2}}\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{\boldsymbol{x}})\,. \tag{B.6}$$

To mathematically prove eq. (B.6), we first observe that an explicit calculation yields

$$\vec{\nabla}^{2}\left(\frac{1}{r}\right)=0\,,\quad \text{for } r\neq 0. \tag{B.7}$$

To pick up the delta function contribution at the origin, we shall integrate eq. (B.20) over a volume V, which we take to be a solid sphere of radius R centered at the origin. Using the divergence theorem (also known as Gauss' theorem) of vector calculus,

$$\int_V \vec{\nabla}^{2}\left(\frac{1}{r}\right)\,d^3x=\int_V \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right)\,d^3x=\oint_S \hat{\boldsymbol{n}}\cdot\vec{\nabla}\left(\frac{1}{r}\right)\,da \tag{B.8}$$

where $\hat{\boldsymbol{n}}\equiv\vec{\boldsymbol{x}}/r$ is a unit vector that is (outwardly) normal to the surface S of the sphere of radius R that caps the closed spherical volume V, and da is is the infinitesimal area element on S. Next we make use of

$$\nabla_j\left(\frac{1}{r}\right)=\frac{\partial}{\partial x_j}\left(x_1^2+x_2^2+x_3^2\right)^{-1/2}=-\frac{x_j}{r^3}=-\frac{n_j}{r^2}\,, \tag{B.9}$$

where we are now employing the notation $\vec{\boldsymbol{x}}=(x_1,x_2,x_3)$ and $r\equiv(x_1^2+x_2^2+x_3^2)^{1/2}$. In light of eq. (B.8), and noting that $\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}=1$ and $da=R^2d\Omega$, it follows that

$$\int_V \vec{\nabla}^{2}\left(\frac{1}{r}\right)\,d^3x=-\oint_S d\Omega=-4\pi\,, \tag{B.10}$$

That is $\vec{\nabla}^{2}(r^{-1})$ is a "function" that vanishes everywhere except for the origin, and its integral over all space is -4π. Consequently, we can identify $\vec{\nabla}^{2}(r^{-1})=-4\pi\delta^3(\vec{\boldsymbol{x}})$ in agreement with eq. (B.6).

It is instructive to generalize the result obtained in eq. (B.6). Here, we shall establish the following identity (e.g., see Ref. [14]),

$$\nabla_i\nabla_j\left(\frac{1}{r}\right)=\frac{3x_ix_j-r^2\delta_{ij}}{r^5}-\frac{4\pi}{3}\delta_{ij}\,\delta^3(\vec{\boldsymbol{x}})\,. \tag{B.11}$$

We shall prove eq. (B.11) using the covariance under rotations of Cartesian tensors.[22] First, consider the case of $r\neq 0$. Because $\nabla_i\nabla_j(r^{-1})$ is a second rank symmetric Cartesian tensor that depends on the vector $\vec{\boldsymbol{x}}$, it immediately follows that

$$\nabla_i\nabla_j\left(\frac{1}{r}\right)=A\delta_{ij}+Bn_in_j\,, \tag{B.12}$$

where $n_i\equiv x_i/r$ and A and B are constants to be determined, since $A\delta_{ij}+Bn_in_j$ is the most general second rank symmetric Cartesian tensor that can be constructed out of $\vec{\boldsymbol{x}}$ (as there are no other vectors in this problem). We now multiply eq. (B.12) by δ_{ij} and sum over i and j. This yields,

$$\vec{\nabla}^{2}\left(\frac{1}{r}\right)=3A+B\,, \tag{B.13}$$

after noting that $\hat{\boldsymbol{n}}$ is a unit vector. in light of eq. (B.7), it then follows that

$$B=-3A\,. \tag{B.14}$$

[22] All Cartesian tensors of the same rank possess the same transformation law under rotations. This property is called covariance under rotations.

Next, we multiply eq. (B.12) by $n_i n_j$ and sum over i and j. This yields,

$$(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}})^2\left(\frac{1}{r}\right)=A+B\,. \tag{B.15}$$

Note that,

$$\begin{aligned}\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\left(\frac{1}{r}\right)&=\frac{1}{r}\,\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\left(\frac{1}{r}\right) \qquad\qquad (\mathrm{B.16})\\ &=\frac{1}{r}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)=-\frac{x^2+y^2+z^2}{r^4}=-\frac{1}{r^2}\,.\end{aligned}$$

Applying $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}$ for a second time, we end up with,

$$\begin{aligned}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}})^2\left(\frac{1}{r}\right)&=-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\left(\frac{1}{r^2}\right)=-\frac{1}{r}\,\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\left(\frac{1}{r^2}\right)\\ &=-\frac{1}{r}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\left(\frac{1}{x^2+y^2+z^2}\right)=\frac{2(x^2+y^2+z^2)}{r^5}=\frac{2}{r^3}\,. \qquad (\mathrm{B.17})\end{aligned}$$

Inserting this result back into eq. (B.15) yields,

$$A+B=\frac{2}{r^3}\,. \tag{B.18}$$

Combining this equation with eq. (B.14), we conclude that,

$$A=-\frac{1}{r^3}\,,\qquad\qquad B=\frac{3}{r^3}\,. \tag{B.19}$$

Plugging these results back into eq. (B.12) and writing $n_i=x_i/r$, we end up with

$$\nabla_i\nabla_j\left(\frac{1}{r}\right)=\frac{3x_ix_j-r^2\delta_{ij}}{r^5}\,,\qquad\text{for } r\neq 0. \tag{B.20}$$

To pick up the delta function contribution at $r=0$, we shall integrate eq. (B.20) over a volume V, where V is a solid sphere of radius R centered at the origin. Using the divergence theorem of vector calculus,

$$\int_V \nabla_i\nabla_j\left(\frac{1}{r}\right)\,d^3x=\oint_S \nabla_j\left(\frac{1}{r}\right)n_i\,da\,, \tag{B.21}$$

where S is the surface of the sphere of radius R, so that $da=R^2d\Omega$. Using eq. (B.9), it follows that

$$\int_V \nabla_i\nabla_j\left(\frac{1}{r}\right)\,d^3x=-\oint_S n_in_j\,d\Omega\,. \tag{B.22}$$

Since the volume V and surface S are spherically symmetric, it follows that both sides of eq. (B.22) are second rank symmetric tensors that transform as Cartesian tensors under rotations. Moreover, these tensors do not depend on $\vec{\boldsymbol{x}}$ after performing the integrations. Thus, both sides of eq. (B.22) must be proportional to δ_{ij}. In particular,

$$\oint_S n_i n_j \, d\Omega = C\delta_{ij} \,, \tag{B.23}$$

for some constant C. Multiplying both sides of eq. (B.23) by δ_{ij} and summing over i and j yields,

$$4\pi = 3C \,. \tag{B.24}$$

Hence, eqs. (B.22)–(B.24) yield,

$$\int_V \nabla_i \nabla_j \left(\frac{1}{r}\right) d^3x = -\frac{4\pi}{3}\delta_{ij} \,. \tag{B.25}$$

Consider now integrating eq. (B.20) over the volume V, which is a solid sphere of radius R centered at the origin, but with the origin omitted (to avoid a potential singularity at $r = 0$). The result must be a second rank symmetric Cartesian tensor. Hence, applying the same reasoning as above, it follows that

$$\int_V \frac{3n_i n_j - \delta_{ij}}{r^3} \, d^3x = C'\delta_{ij} \,, \tag{B.26}$$

where C' is a constant to be determined. Once again, we multiply both sides by δ_{ij} and sum over i and j. But notice that

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}(3n_i n_j - \delta_{ij}) = 0 \,. \tag{B.27}$$

It then follows that $C' = 0$, which implies that

$$\int_V \frac{3n_i n_j - \delta_{ij}}{r^3} \, d^3x = 0 \,. \tag{B.28}$$

Therefore, if we wish to extend eq. (B.20) to include the point $r = 0$, we must write

$$\nabla_i \nabla_j \left(\frac{1}{r}\right) = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} + c\,\delta^3(\vec{\boldsymbol{x}}) \,. \tag{B.29}$$

To determine c, we integrate over the volume V, which is a solid sphere of radius R centered at the origin. In light of eq. (B.28), it follows that

$$\int_V \nabla_i \nabla_j \left(\frac{1}{r}\right) d^3x = c \int_V \delta^3(\vec{\boldsymbol{x}})\, d^3x = c \,. \tag{B.30}$$

Comparing with eq. (B.25), it follows that

$$c = -\frac{4\pi}{3}\delta_{ij} \,. \tag{B.31}$$

Hence, we conclude that

$$\nabla_i \nabla_j \left(\frac{1}{r}\right) = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} - \frac{4\pi}{3}\delta_{ij}\,\delta^3(\vec{\boldsymbol{x}}) \,, \tag{B.32}$$

which confirms the result quoted in eq. (B.11). Moreover, by setting $i = j$ and summing over i, we recover eq. (B.6).

In deriving eq. (B.11), we made use of eq. (B.28) under the assumption that V is a solid sphere of radius R centered at the origin, with the origin included in V. Is this a valid conclusion? Let us examine this integral more carefully. Using spherical coordinates,

$$\int_V \frac{3n_i n_j - \delta_{ij}}{r^3}\, d^3x = \int_0^R r^2\, dr \int d\Omega\, \frac{3n_i n_j - \delta_{ij}}{r^3} = \int_0^R \frac{dr}{r} \int d\Omega\, (3n_i n_j - \delta_{ij})\,. \tag{B.33}$$

Once again, we can conclude that

$$\int d\Omega\, (3n_i n_j - \delta_{ij}) = 0\,, \tag{B.34}$$

using the same analysis previously employed in deriving eq. (B.28), which makes use of eq. (B.27). However,

$$\int_0^R \frac{dr}{r}\,, \tag{B.35}$$

is logarithmically divergent! So, it looks like eq. (B.33) is equal to $\infty \times 0$, which is indeterminate. Thus, in order to conclude that eq. (B.11) holds true, we must adopt one of two possible interpretations. One possibility is to interpret the first term on the right hand side of eq. (B.11) as having meaning only for $r \neq 0$, and demand that the $r = 0$ behavior is completely contained in the delta function of the origin. A second possibility is to define

$$\int_V \frac{3n_i n_j - \delta_{ij}}{r^3}\, d^3x = \lim_{\epsilon\to 0} \int_\epsilon^R r^2\, dr \int d\Omega\, \frac{3n_i n_j - \delta_{ij}}{r^3} = \lim_{\epsilon\to 0}\, \ln(R/\epsilon) \int d\Omega\, \frac{3n_i n_j - \delta_{ij}}{r^3} = 0\,, \tag{B.36}$$

since the integration over angles vanishes. That is, we take the $\epsilon \to 0$ limit only at the very end *after* evaluating the integral over angles. For a further discussion of the caveats associated with eq. (B.11), see Ref. [15].

Eq. (B.11) has significant implications for the study of electromagnetism. The electrostatic potential of a point electric dipole $\vec{\boldsymbol{p}}$ located at the origin (in gaussian units) is given by

$$\Phi(\vec{\boldsymbol{x}}) = -\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\left(\frac{1}{r}\right) = \frac{\vec{\boldsymbol{p}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}}{r^3}\,, \tag{B.37}$$

after making use of eq. (B.9). It then follows that the components of the electric field of a point electric dipole located at the origin are given by

$$E_i = -\nabla_i \Phi = \sum_{j=1}^{3} p_j \nabla_i \nabla_j \left(\frac{1}{r}\right)\,. \tag{B.38}$$

Using eq. (B.11) to evaluate $\nabla_i \nabla_j (r^{-1})$, we end up with

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = -\vec{\boldsymbol{\nabla}}\Phi(\vec{\boldsymbol{x}}) = \frac{3\hat{\boldsymbol{n}}(\vec{\boldsymbol{p}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}) - \vec{\boldsymbol{p}}}{r^3} - \frac{4\pi}{3}\vec{\boldsymbol{p}}\,\delta^3(\vec{\boldsymbol{x}})\,, \tag{B.39}$$

where $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{x}}/r$ is a unit vector pointing in the radial direction from the origin to the point $\vec{\boldsymbol{x}}$. Some of the physical implications of the delta function contribution in eq. (B.39) can be found in Ref. [16].

Appendix C: The Cauchy Principle Value

In this Appendix, we verify that the definitions of the Cauchy principal value given in eqs. (17), (18), (23), and (44) are all equivalent. We begin with eq. (17), which we repeat here for the convenience of the reader,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)}{x}\,dx = \int_0^{\infty} \frac{f(x)-f(-x)}{x}\,dx\,. \tag{C.1}$$

For any positive real number δ, the following equation is an identity,

$$\int_0^{\infty} \frac{f(x)-f(-x)}{x}\,dx = \int_0^{\delta} \frac{f(x)-f(-x)}{x}\,dx + \int_{\delta}^{\infty} \frac{f(x)}{x}\,dx + \int_{-\infty}^{-\delta} \frac{f(x)}{x}\,dx\,, \tag{C.2}$$

where the last integral on the right hand side of eq. (C.2) has been obtained after changing the integration variable, $x \to -x$. Since δ is an arbitrary positive number, we can take the limit as $\delta \to 0$ (from the positive side), which yields

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} = \lim_{\delta\to 0^+}\left\{\int_{-\infty}^{-\delta} \frac{f(x)\,dx}{x} + \int_{\delta}^{\infty} \frac{f(x)\,dx}{x} + \int_0^{\delta} \frac{f(x)-f(-x)}{x}\,dx\right\}. \tag{C.3}$$

By assumption, the test function $f(x)$ is smooth and vanishes as $x \to \pm\infty$. Hence, one can Taylor expand $f(x)$ around the origin to obtain $f(x) = f(0) + xf'(0) + \mathcal{O}(x^2)$. It follows that for $\delta \ll 1$,

$$\int_0^{\delta} \frac{f(x)-f(-x)}{x}\,dx = 2f'(0)\delta + \mathcal{O}(\delta^3)\,, \tag{C.4}$$

which vanishes as $\delta \to 0$. Hence, eq. (C.3) yields,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} = \lim_{\delta\to 0^+}\left\{\int_{-\infty}^{-\delta} \frac{f(x)\,dx}{x} + \int_{\delta}^{\infty} \frac{f(x)\,dx}{x}\right\}, \tag{C.5}$$

which establishes the equivalence of eqs. (17) and (18).

Next, we start from the definition of the Cauchy principal value given in eq. (C.5) and integrate by parts to obtain

$$\int_{-\infty}^{-\delta} \frac{f(x)}{x}\,dx = f(x)\ln|x|\bigg|_{-\infty}^{-\delta} - \int_{-\infty}^{-\delta} f'(x)\ln|x|\,dx = f(-\epsilon)\ln\epsilon - \int_{-\infty}^{-\delta} f'(x)\ln|x|\,dx\,,$$

$$\int_{\delta}^{\infty} \frac{f(x)}{x}\,dx = f(x)\ln|x|\bigg|_{-\infty}^{-\delta} - \int_{\delta}^{\infty} f'(x)\ln|x|\,dx = -f(\epsilon)\ln\epsilon - \int_{\delta}^{\infty} f'(x)\ln|x|\,dx\,,$$

where $f'(x) \equiv df/dx$ and we have assumed that $f(x) \to 0$ sufficiently fast as $x \to \pm\infty$ so that the surface terms at $\pm\infty$ vanish. Hence,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} = \lim_{\delta\to 0^+}\left\{\big[f(-\delta)-f(\delta)\big]\ln\delta - \int_{-\infty}^{-\delta} f'(x)\ln|x|\,dx - \int_{\delta}^{\infty} f'(x)\ln|x|\,dx\right\}. \tag{C.6}$$

Since $f(x)$ is differentiable and well behaved, we can define

$$g(x) \equiv \int_0^1 f'(xt)dt = \frac{f(x)-f(0)}{x}\,,$$

which implies that $g(x)$ is smooth and non-singular and

$$f(x) = f(0) + xg(x)\,. \tag{C.7}$$

Inserting eq. (C.7) back into eq. (C.6) then yields

$$\begin{aligned}
\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x} &= \lim_{\delta\to 0^+}\left\{\left[-2g(x)\delta\ln\delta - \int_{-\infty}^{-\delta} f'(x)\ln|x|\,dx - \int_{\delta}^{\infty} f'(x)\ln|x|\,dx\right\}\right. \\
&= -\int_{-\infty}^{\infty} f'(x)\ln|x|\,dx\,.
\end{aligned}$$

Note that $\ln|x|$ is integrable at $x=0$, so that the last integral is well-defined. Finally, we integrate by parts and drop the surface terms at $\pm\infty$ (under the usual assumption that $f'(x)\to 0$ sufficiently fast as $x\to\infty$). The end result is

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x} = \int_{-\infty}^{\infty} f(x)\frac{d}{dx}\ln|x|\,dx\,. \tag{C.8}$$

That is, we have rederived the generalized function identity previously obtained in eqs. (22) and (23),

$$\frac{d}{dx}\ln|x| = \mathrm{P}\frac{1}{x}\,. \tag{C.9}$$

In particular, eq. (C.8) provides a definition of the Cauchy principal value prescription that is equivalent to the definitions provided by eqs. (C.1) and (C.5).

Finally, in light of eq. (44), for any smooth test function that vanishes sufficiently fast as $x\to\pm\infty$,

$$\begin{aligned}
\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x} &= \lim_{\delta\to 0}\int_{-\infty}^{\infty}\frac{x}{x^2+\delta^2}\,f(x)\,dx \\
&= \lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta}\frac{x}{x^2+\delta^2}\,f(x)\,dx + \int_{\delta}^{\infty}\frac{x}{x^2+\delta^2}\,f(x)\,dx + \int_{-\delta}^{\delta}\frac{x}{x^2+\delta^2}\,f(x)\,dx\right\}.
\end{aligned} \tag{C.10}$$

To evaluate the last integral in eq. (C.10), we expand $f(x)$ in a Taylor series around $x=0$ to obtain:

$$\lim_{\delta\to 0}\int_{-\delta}^{\delta}\frac{x}{x^2+\delta^2}\,f(x)\,dx = \lim_{\delta\to 0}\int_{-\delta}^{\delta}\frac{x}{x^2+\delta^2}\left[f(0)+xf'(0)+\cdots\right]dx = 0\,. \tag{C.11}$$

To verify the result of eq. (C.11), note that the first term of the Taylor expansion yields

$$f(0)\lim_{\delta\to 0}\int_{-\delta}^{\delta}\frac{x}{x^2+\delta^2}\,dx = 0\,, \tag{C.12}$$

since the integrand of the last integral in eq. (C.12) is an odd function. The second term of the Taylor expansion yields

$$f'(0)\lim_{\delta\to 0}\int_{-\delta}^{\delta}\frac{x^2}{x^2+\delta^2}\,dx = f'(0)\lim_{\delta\to 0}\left[x-\delta\tan^{-1}\left(\frac{x}{\delta}\right)\right]\Bigg|_{-\delta}^{\delta} = \lim_{\delta\to 0}2\delta\left(1-\tfrac{1}{4}\pi\right)=0\,. \tag{C.13}$$

It is straightforward to verify that the contributions from all terms in the Taylor expansion in eq. (C.11) vanish in the $\delta\to 0$ limit.

Consequently, eq. (C.10) yields

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x} = \lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta}\frac{x}{x^2+\delta^2}\,f(x)\,dx + \int_{\delta}^{\infty}\frac{x}{x^2+\delta^2}\,f(x)\,dx\right\}. \tag{C.14}$$

One can now Taylor expand $(x^2+\delta^2)^{-1}$ around $\delta = 0$. Only the leading term survives, and we end up with

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x} = \lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta}\frac{f(x)}{x}\,dx + \int_{\delta}^{\infty}\frac{f(x)}{x}\,dx\right\}, \tag{C.15}$$

in agreement with eq. (C.1).

Appendix D: The Plemelj Formulae of the Theory of Complex Variables

The Sokhotski-Plemelj formula derived in these notes is in fact a special case of a more general result of the theory of complex variables, which is often referred to as the Plemelj formulae (and less often as the Sokhotski formulae). In this Appendix, I shall simply state the relevant results. Further details can be found in Refs. [17–25].

Consider the Cauchy type integral,

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt\,, \tag{D.1}$$

where z and t are complex variables, C is a smooth curve (which may be an open or a closed contour, but does *not* contain a corner or cusp) and $f(t)$ is a function defined on C that satisfies,

$$|f(t_2) - f(t_1) < A|t_2 - t_1|^{\lambda}\,, \tag{D.2}$$

for any two points t_1 and t_2 located on the contour C, where A and λ are positive numbers. Eq. (D.2) is called the Hölder condition.[23]

For values of $z \notin C$, $F(z)$ is an analytic function, whereas $F(z)$ is not well defined for values of z on the contour C due to the singularity encountered in the integration along C. Nevertheless, $F(z)$ does have unique value that depends on how z approaches C. Indeed, there are two different possible boundary values of $F(z)$ depending on whether the contour C is approached from the left or right. We therefore introduce $F_+(z)$ and $F_-(z)$ where the former is the limit as z approaches C from the left and the latter is the limit as z approaches from the

[23] If $\lambda > 1$, then it follows that the derivative $f'(t)$ must vanish on C, in which case $f(t)$ is a constant. Thus, one typically assumes that $0 < \lambda \leq 1$.

right, where left and right are defined with respect to the positive direction of the contour C.[24]

The explicit results for $F_\pm(z)$ are given by the Plemelj formulae,

$$F_\pm(z) = \pm\tfrac{1}{2}f(z) + \frac{1}{2\pi i}\,\mathrm{P}\int_C \frac{f(t)}{t-z}\,dt\,, \quad \text{for } z \in \widetilde{C}\,, \tag{D.3}$$

and $\widetilde{C}$ consists of all points of C excluding its endpoints (in the case of a closed contour, there are no endpoints to exclude). In eq. (D.3), the Cauchy principal value prescription is used to treat the singularity in the integrand. In this context, the principal value is a generalization of eq. (18),

$$\mathrm{P}\int_C \frac{f(t)}{t-z}\,dt = \lim_{\delta\to 0}\int_{C-C_\delta}\frac{f(t)}{t-z}\,dt\,, \tag{D.4}$$

where the contour C_δ consists of the part of C with length 2δ centered symmetrically around z, and $C - C_\delta$ is the contour C with the part C_δ removed.

If $f(t)$ is analytic on C, then the proof of eq. (D.3) is a straightforward generalization of the proof given in Section 2. However, the Plemelj formulae are more general and apply to any function $f(t)$ that satisfies the Hölder condition on the contour C. In this case, the derivation of eq. (D.3) is more complicated. We have also sidestepped the case where z in eq. (D.3) is one of the endpoints of C (which is relevant if the contour C is open). The reader is referred to the references below for further details.

One can recast eq. (D.3) into another form that commonly appears in the literature,

$$F_+(z) - F_-(z) = f(z)\,, \tag{D.5}$$

$$F_+(z) + F_-(z) = \frac{1}{\pi i}\,\mathrm{P}\int_C \frac{f(t)}{t-z}\,dt\,, \tag{D.6}$$

for values of z located on all points of the contour C not coinciding with its endpoints. In particular, eq. (D.5) indicates that the function $F(z)$ defined in eq. (D.1), which is analytic for all complex values of $z \notin C$, has a discontinuous jump as z crosses the contour C. Moreover, the average of the two boundary values of $F(z)$ on C is given by eq. (D.1), where the singularity of the integrand is treated by the Cauchy principal value prescription.

Using the Plemelj formulae of complex variables theory, one can recover the results of Section 2 as follows. If C is a contour that runs along the real axis in the positive direction, then eq. (D.1) yields the boundary values, $F_\pm(z)$, of $F(z)$ as z approaches the real axis from above (i.e., from the left) or below (i.e., from the right), respectively,

$$F_\pm(z) = \lim_{\epsilon\to 0} F(z \pm i\epsilon) = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z\mp i\epsilon}\,dt\,, \tag{D.7}$$

where $\epsilon > 0$ is an infinitesimal real quantity. Hence, eqs. (D.3) and (D.7) yield

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{f(t)}{t-z\mp i\epsilon}\,dt = \mathrm{P}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}\,dt \pm i\pi f(z)\,. \tag{D.8}$$

Eq. (D.8) is equivalent to the identity involving generalized functions given in eq. (63). As expected, setting $z = 0$ in eq. (D.8) reproduces eq. (62).

[24] For example, for a closed counterclockwise contour C, $F_+(z)$ is given by the limit of $F(z)$ as z approaches C from the interior of the region bounded by C and $F_-(z)$ is given by the limit of $F(z)$ as z approaches C from the exterior of the region bounded by C.

References

[1] M.J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions* (Cambridge University Press, Cambridge, UK, 1958) pp. 67–71.

[2] D.S. Jones, *The Theory of Generalised Functions* (Cambridge University Press, Cambridge, UK, 1982).

[3] Ram P. Kanwal, *Generalized Functions—Theory and Application*, 3rd Edition (Birkhäuser, Boston, MA, 2004).

[4] I.M. Gel'fand and G.E. Shilov, *Generalized Functions–Properties and Operations*, Volume 1 (Academic Press, Inc., New York, 1964).

[5] Ricardo Estrada and Ram P. Kanwal, *A Distributional Approach to Asymptotics–Theory and Applications*, Second Edition (Birkhäuser, Boston, MA, 2002).

[6] A.H. Zemanian, *Distribution Theory and Transform Analysis–An Introduction to Generalized Functions, with Applications* (Dover Publications, Inc., Mineola, NY, 1987).

[7] V.S. Vladimirov, *Generalized Functions in Mathematical Physics* (Mir Publishers, Moscow, 1979).

[8] J. Campos Ferreira, *Introduction to the Theory of Distributions* (Addison Wesley Longman Limited, Essex, UK, 1997).

[9] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Eighth Edition, edited by Daniel Zwillinger and Victor Moll (Academic Press, Waltham, MA, 2015).

[10] Brian Davies, *Integral Transforms and Their Applications*, 3rd edition (Springer-Verlag, New York, 2002).

[11] David Bressoud, *A Radical Approach to Real Analysis*, 2nd edition (The Mathematics Association of America, Washington, DC, 2007).

[12] F. Oberhettinger, *Table of Fourier Transforms and Fourier Transforms of Distributions* (Springer-Verlag, Berlin, 1990).

[13] John David Jackson, *Classical Electrodynamics*, 2nd Edition (John Wiley & Sons, New York, 1975).

[14] Charles P. Frahm, Am. J. Phys. **51**, 826 (1983).

[15] V. Hnizdo, Eur. J. Phys. **32**, 287 (2011).

[16] John David Jackson, *The nature of intrinsic magnetic dipole moments*, CERN Yellow Report, CERN-1977-017.

[17] Mark J. Ablowitz and Athanassios S. Fokas, *Complex Variables: Introduction and Applications* (Cambridge University Press, Cambridge, UK, 1997), Chapter 7.

[18] Jian-Ke Lu, *Boundary Value Problems for Analytic Functions* (World Scientific, Singapore, 1993).

[19] Peter Henrici, *Applied and Computational Complex Analysis*–Volume 3 (John Wiley & Sons, New York, 1986).

[20] Frederick W. King, *Hilbert Transforms*–Volume 1 (Cambridge University Press, Cambridge, UK, 2009),

[21] A.I. Markushevich, *Theory of Functions of a Complex Variable*, Part I (AMS Chelsea Publishing, Providence, RI, 2005), Chapter 14.

[22] F.D. Gakhov, *Boundary Value Problems* (Dover Publications, Inc., New York, NY, 1990), Chapters 1–5.

[23] N.I. Muskhelishvili, *Singular Integral Equations: Boundary problems of functions theory and their applications to mathematical physics*, 2nd edition (Dover Publications, Inc., New York, NY, 2008), Chapter 2.

[24] Alexander O. Gogolin, *Lectures on Complex Integration* (Springer International Publishing, Cham, Switzerland, 2014), Chapters 1.4 and 3.4.

[25] Ricardo Estrada and Ram P. Kanwal, *Singular Integral Equations* (Birkhäuser, Boston, MA, 2000).