Exponentiating the Lie algebra of the Lorentz group

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Abstract

An explicit form for the 4-dimensional matrix representation of the most general proper orthochronous Lorentz transformation matrix $\Lambda^\mu{}_\nu$ can be obtained by exponentiating a 4×4 matrix that is a general element of the Lie algebra of the Lorentz group. An alternative method exploits the assertion that the spinor product $\eta^\dagger\overline{\sigma}^\mu\chi$ transforms as a Lorentz four-vector, where χ and η are two-component spinors. The latter result yields an expression for $\Lambda^\mu{}_\nu$ that only requires the exponentiation of 2×2 matrices. We provide an explicit demonstration that both computations yield precisely the same expression for $\Lambda^\mu{}_\nu$.

1 Proper orthochronous Lorentz transformations

Under an *active* Lorentz transformation, the spacetime coordinates $x^\mu = (ct\,;\,\vec{\boldsymbol{x}})$ transform as $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. As usual, there is an implied sum over any repeated upper/lower index pair. The condition that $g_{\mu\nu}x^\mu x^\nu$ is invariant under Lorentz transformations implies that[1]

$$\Lambda^\mu{}_\nu g_{\mu\rho}\Lambda^\rho{}_\lambda = g_{\lambda\nu}\,. \tag{1}$$

The Lie group $\mathrm{O}(1,3)$ is the set of all 4×4 matrices Λ that satisfy eq. (1). Note that $\det\Lambda=\pm 1$. Similarly, the Lie group $\mathrm{SO}(1,3)$ is the group of *proper* Lorentz transformations, which satisfy $\det\Lambda = +1$. The elements of the subgroup of $\mathrm{SO}(1,3)$ that additionally satisfy $\Lambda^0{}_0\geq 1$ are continuously connected to the identity element (the 4×4 identity matrix, denoted by $\boldsymbol{I}_4$) and constitute the proper orthochronous Lorentz transformations.

The Lie algebra of the Lorentz group is obtained by considering infinitesimal Lorentz transformations,

$$\Lambda = \boldsymbol{I}_4 + A\,, \tag{2}$$

where A is a 4×4 matrix that depends on infinitesimal Lorentz group parameters, and terms that are quadratic or of higher order in the infinitesimal group parameters are neglected. Inserting eq. (2) into eq. (1), and denoting G to be the 4×4 matrix whose matrix elements are $g_{\mu\nu}$, it follows that

$$\big(\boldsymbol{I}_4 + A^{\mathsf{T}}\big)G\big(\boldsymbol{I}_4 + A\big) = G\,. \tag{3}$$

Keeping only terms up to linear order in the infinitesimal group parameters, we conclude that

$$A^{\mathsf{T}}G = -GA\,. \tag{4}$$

[1] We employ the mostly minus convention, $g_{\mu\nu} = \mathrm{diag}(1;-1,-1,-1)$, where $\mu,\,\nu\in\{0,1,2,3\}$.

Since G is diagonal, eq. (4) implies that GA is a real antisymmetric 4×4 matrix. That is, we have shown that the Lie algebra of the Lorentz group consists of all real 4×4 matrices with the property that GA is antisymmetric.

To construct a general proper orthochronous Lorentz transformation, one can choose any real 4×4 matrix A such that GA is antisymmetric, and consider a large positive integer n such that A/n is an infinitesimal quantity. Then, a general proper orthochronous Lorentz transformation can be obtained by applying a sequence of n infinitesimal Lorentz transformations in the limit as $n \to \infty$,

$$\Lambda = \lim_{n\to\infty} \left(1 + \frac{A}{n}\right)^n = \exp A\,. \tag{5}$$

Thus, we have demonstrated that the set of proper orthochronous Lorentz transformations consists of matrices of the form $\exp A$, where GA is a real antisymmetric 4×4 matrix.

Hence, the most general proper orthochronous Lorentz transformation matrix Λ, characterized by a rotation angle θ about an axis $\hat{\boldsymbol{n}}$ [$\vec{\boldsymbol{\theta}} \equiv \theta\hat{\boldsymbol{n}}$] and a boost vector $\vec{\boldsymbol{\zeta}} \equiv \hat{\boldsymbol{v}}\tanh^{-1}\beta$ [where $\hat{\boldsymbol{v}} \equiv \vec{\boldsymbol{v}}/|\vec{\boldsymbol{v}}|$ is the unit velocity vector and $\beta \equiv |\vec{\boldsymbol{v}}|/c$],[2] is a 4×4 matrix given by[3]

$$\Lambda = \exp\left(-\tfrac{1}{2}i\theta_{\rho\lambda}s^{\rho\lambda}\right) = \exp\left(-i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{s}} - i\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{k}}\right)\,, \tag{6}$$

where $\theta_{\rho\lambda} = -\theta_{\lambda\rho}$ and $s^{\rho\lambda} = -s^{\lambda\rho}$. In particular, $\theta^i \equiv \frac{1}{2}\epsilon^{ijk}\theta^{jk}$, $\zeta^i \equiv \theta^{i0} = -\theta^{0i}$, $s^i \equiv \frac{1}{2}\epsilon^{ijk}s^{jk}$, $k^i \equiv s^{0i} = -s^{i0}$, and

$$(s^{\rho\lambda})^{\mu}{}_{\nu} = i\left(g^{\rho\mu}\delta^{\lambda}_{\nu} - g^{\lambda\mu}\delta^{\rho}_{\nu}\right). \tag{7}$$

Here, the indices $i, j, k \in \{1, 2, 3\}$ and the Levi-Civita symbol $\epsilon^{123} = +1$. More explicitly,

$$\Lambda = \exp A\,, \quad \text{where} \quad A \equiv -\tfrac{1}{2}i\theta_{\rho\lambda}s^{\rho\lambda} = \begin{pmatrix} 0 & \zeta^1 & \zeta^2 & \zeta^3 \\ \zeta^1 & 0 & -\theta^3 & \theta^2 \\ \zeta^2 & \theta^3 & 0 & -\theta^1 \\ \zeta^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix}. \tag{8}$$

As anticipated above, GA is the most general real antisymmetric 4×4 matrix.

Note that the $s^{\rho\lambda} = -s^{\lambda\rho}$ are six independent real antisymmetric 4×4 matrices that satisfy the commutation relations of the real Lie algebra of SO(1,3), henceforth denoted by $\mathfrak{so}(1, 3)$,

$$[s^{\alpha\beta}, s^{\rho\lambda}] = i(g^{\beta\rho}\, s^{\alpha\lambda} - g^{\alpha\rho}\, s^{\beta\lambda} - g^{\beta\lambda}\, s^{\alpha\rho} + g^{\alpha\lambda}\, s^{\beta\rho}). \tag{9}$$

In particular, A is a real linear combination of the Lie algebra generators $is^{\rho\lambda}$ and thus constitutes a general element of the real Lie algebra $\mathfrak{so}(1, 3)$.

The spin-1/2 representation of the Lorentz group is a two dimensional matrix representation of $\mathrm{SL}(2, \mathbb{C})$ represented by the complex 2×2 matrices of unit determinant,[4]

$$M = \exp\left(-\tfrac{1}{2}i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} - \tfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right)\,. \tag{10}$$

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[2] It is convenient to employ units where we set the speed of light $c = 1$.

[3] We follow the conventions of Ref. [1]. Note that in the notation of Ref. [2], $\vec{\boldsymbol{k}} = i\vec{\boldsymbol{K}}$ and $\vec{\boldsymbol{s}} = i\vec{\boldsymbol{S}}$, where the 4×4 matrix representations of $\vec{\boldsymbol{K}}$ and $\vec{\boldsymbol{S}}$ are given in eq. (11.91) of Jackson, which yields $\Lambda = \exp(\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{S}} + \vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{K}})$. The argument of exp differs by an overall sign with Jackson's eq. (11.93) where a *passive* Lorentz transformation is employed, which amounts to replacing $\{\vec{\boldsymbol{\theta}}, \vec{\boldsymbol{\zeta}}\}$ with $\{-\vec{\boldsymbol{\theta}}, -\vec{\boldsymbol{\zeta}}\}$.

[4] The matrices M defined in eq. (10) constitutes the $\left(\frac{1}{2}, 0\right)$ representation of $\mathrm{SL}(2, \mathbb{C})$. There is a second inequivalent two dimensional representation of $\mathrm{SL}(2, \mathbb{C})$ represented by the matrices $(M^{-1})^\dagger$, which constitutes the $\left(0, \frac{1}{2}\right)$ representation of $\mathrm{SL}(2, \mathbb{C})$. For further details, see Ref. [1].

It is convenient to introduce the Pauli matrices, σ^1, σ^2, σ^3, using the notation of Ref. [1],

$$\sigma^\mu = (\boldsymbol{I}_2\,;\,\vec{\boldsymbol{\sigma}})\,, \qquad\qquad \overline{\sigma}^\mu = (\boldsymbol{I}_2\,;\,-\vec{\boldsymbol{\sigma}})\,, \tag{11}$$

where $\mu \in \{0,1,2,3\}$ and $\boldsymbol{I}_2$ is the 2×2 identity matrix. Note that these sigma matrices have been defined with an upper (contravariant) index. They are related to sigma matrices with a lower (covariant) index in the usual way:

$$\sigma_\mu = g_{\mu\nu}\sigma^\nu = (\boldsymbol{I}_2\,;\,-\vec{\boldsymbol{\sigma}})\,, \qquad\qquad \overline{\sigma}_\mu = g_{\mu\nu}\overline{\sigma}^\nu = (\boldsymbol{I}_2\,;\,\vec{\boldsymbol{\sigma}})\,. \tag{12}$$

However, the use of the spacetime indices μ and ν is slightly deceptive since the sigma matrices defined above are *fixed* matrices that do not change under Lorentz transformations. If we also introduce six independent 2×2 matrices $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$, where

$$\sigma^{\mu\nu} \equiv \tfrac{1}{4}i\big(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu\big)\,, \tag{13}$$

then eq. (10) can be rewritten in the following form that is reminiscent of eq. (6),

$$M = \exp\left(-\tfrac{1}{2}i\theta_{\mu\nu}\sigma^{\mu\nu}\right)\,. \tag{14}$$

That is, the six independent $i\sigma^{\mu\nu}$ matrices are generators of the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$, henceforth denoted by $\mathfrak{sl}(2,\mathbb{C})$. It is straightforward to check that the 2×2 matrices $\sigma^{\mu\nu}$ possess the same commutation relations as the 4×4 matrices $s^{\mu\nu}$ [cf. eq. (9)], which establishes the isomorphism $\mathfrak{so}(1,3) \simeq \mathfrak{sl}(2,\mathbb{C})$.

Under an active Lorentz transformation, a two-component spinor χ_α (where $\alpha \in \{1,2\}$) transforms as,

$$\chi'_\alpha = M_\alpha{}^\beta \chi_\beta\,, \qquad \alpha,\beta \in \{1,2\}. \tag{15}$$

Suppose that χ and η are two-component spinors and consider the spinor product $\eta^\dagger\overline{\sigma}^\mu\chi$. Under a Lorentz transformation,

$$\eta^\dagger\overline{\sigma}^\mu\chi \longrightarrow (M\eta)^\dagger\overline{\sigma}^\mu(M\chi) = \eta^\dagger(M^\dagger\overline{\sigma}^\mu M)\chi\,. \tag{16}$$

We assert that the quantity $\eta^\dagger\overline{\sigma}^\mu\chi$ transforms as a Lorentz four vector,[5]

$$\eta^\dagger\overline{\sigma}^\mu\chi \longrightarrow \Lambda^\mu{}_\nu\,\eta^\dagger\overline{\sigma}^\nu\chi\,, \tag{17}$$

which implies that the following identity must be satisfied:

$$M^\dagger\overline{\sigma}^\mu M = \Lambda^\mu{}_\nu\overline{\sigma}^\nu\,. \tag{18}$$

If we multiply eq. (18) on the right by σ_ρ and use $\mathrm{Tr}(\overline{\sigma}^\nu\sigma_\rho) = 2\delta^\nu_\rho$, it follows that

$$\Lambda^\mu{}_\nu = \tfrac{1}{2}\,\mathrm{Tr}\big(M^\dagger\overline{\sigma}^\mu M\sigma_\nu\big)\,. \tag{19}$$

The goal of these notes is to show that the two expressions for Λ given by eqs. (8) and (19) coincide.

A more formal writeup of these notes can be found in Ref. [3].

[5]A similar technique can be employed to show that $\overline{\Psi}\gamma^\mu\Psi$ transforms as a four vector under a Lorentz transformation, where Ψ is a four-component spinor. For details, see Appendix A.

2 An explicit evaluation of $\Lambda^\mu{}_\nu = \frac{1}{2}\,\mathrm{Tr}\big(M^\dagger\overline{\sigma}^\mu M\sigma_\nu\big)$

It is convenient to introduce

$$\vec{z} \equiv \vec{\zeta} + i\vec{\theta}\,. \tag{20}$$

Then, using a well-known result (e.g., see pp. 383–384 of Ref. [4]),

$$M = \exp\left(-\tfrac{1}{2}\vec{z}\boldsymbol{\cdot}\vec{\sigma}\right) = \boldsymbol{I}_2\cosh\big(\tfrac{1}{2}\Delta\big) - \vec{z}\boldsymbol{\cdot}\vec{\sigma}\,\frac{\sinh\big(\frac{1}{2}\Delta\big)}{\Delta}\,, \tag{21}$$

where

$$\Delta \equiv \left(\vec{z}\boldsymbol{\cdot}\vec{z}\right)^{1/2} = \left(|\vec{\zeta}|^2 - |\vec{\theta}|^2 + 2i\vec{\theta}\boldsymbol{\cdot}\vec{\zeta}\right)^{1/2}. \tag{22}$$

Since the Pauli matrices are hermitian,

$$M^\dagger = \exp\left(-\tfrac{1}{2}\vec{z}^{\,*}\boldsymbol{\cdot}\vec{\sigma}\right) = \boldsymbol{I}_2\cosh\big(\tfrac{1}{2}\Delta^*\big) - \vec{z}^{\,*}\boldsymbol{\cdot}\vec{\sigma}\,\frac{\sinh\big(\frac{1}{2}\Delta^*\big)}{\Delta^*}\,. \tag{23}$$

We shall evaluate $\Lambda^\mu{}_\nu$ in four separate cases depending whether the spacetime index is 0 or $i\in\{1,2,3\}$. In particular, eq. (19) yields

$$\Lambda^0{}_0 = \tfrac{1}{2}\,\mathrm{Tr}(M^\dagger M)\,, \qquad \Lambda^i{}_0 = -\tfrac{1}{2}\,\mathrm{Tr}(M^\dagger\sigma^i M)\,, \tag{24}$$

$$\Lambda^0{}_i = -\tfrac{1}{2}\,\mathrm{Tr}(M\sigma^i M^\dagger)\,, \qquad \Lambda^i{}_j = \tfrac{1}{2}\,\mathrm{Tr}(M^\dagger\sigma^i M\sigma^j)\,. \tag{25}$$

Note that in obtaining eq. (25), we used $\sigma_i = -\sigma^i$. It is simply more convenient to express quantities whose indices run over the indices $i = 1,2,3$ in terms of tensors that contain only upper (contravariant) indices, even though we are evaluating $\Lambda^0{}_i$ and $\Lambda^i{}_j$ which possess one lower (covariant) index.

Plugging eqs. (21) and (23) into eq. (19) and evaluating the traces,

$$\mathrm{Tr}(\sigma^i\sigma^j) = 2\delta^{ij}\,, \tag{26}$$

$$\mathrm{Tr}(\sigma^i\sigma^j\sigma^k) = 2i\epsilon^{ijk}\,, \tag{27}$$

$$\mathrm{Tr}(\sigma^i\sigma^j\sigma^k\sigma^\ell) = 2(\delta^{ij}\delta^{k\ell} - \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk})\,, \tag{28}$$

we end up with the following expressions:

$$\Lambda^0{}_0 = |\cosh\big(\tfrac{1}{2}\Delta\big)|^2 + \left|\frac{\sinh\big(\frac{1}{2}\Delta\big)}{\Delta}\right|^2\left(|\vec{\zeta}|^2 + |\vec{\theta}|^2\right), \tag{29}$$

$$\Lambda^i{}_0 = \left(\frac{\cosh\big(\frac{1}{2}\Delta^*\big)\sinh\big(\frac{1}{2}\Delta\big)}{\Delta}\,z^i + \text{c.c.}\right) + i\left|\frac{\sinh\big(\frac{1}{2}\Delta\big)}{\Delta}\right|^2\epsilon^{ik\ell}z^{*k}z^\ell\,, \tag{30}$$

$$\Lambda^0{}_j = \left(\frac{\cosh\big(\frac{1}{2}\Delta^*\big)\sinh\big(\frac{1}{2}\Delta\big)}{\Delta}\,z^j + \text{c.c.}\right) + i\left|\frac{\sinh\big(\frac{1}{2}\Delta\big)}{\Delta}\right|^2\epsilon^{jk\ell}z^k z^{*\ell}\,, \tag{31}$$

$$\begin{aligned}\Lambda^i{}_j = {} & \left\{|\cosh\big(\tfrac{1}{2}\Delta\big)|^2 - \left|\frac{\sinh\big(\frac{1}{2}\Delta\big)}{\Delta}\right|^2\left(|\vec{\zeta}|^2 + |\vec{\theta}|^2\right)\right\}\delta^{ij} + (z^{*i}z^j + z^i z^{*j})\left|\frac{\sinh\big(\frac{1}{2}\Delta\big)}{\Delta}\right|^2 \\ & + \left(\frac{i\sinh\big(\frac{1}{2}\Delta\big)\cosh\big(\frac{1}{2}\Delta^*\big)}{\Delta}\epsilon^{ijk}z^k + \text{c.c.}\right),\end{aligned} \tag{32}$$

where c.c. means the complex conjugate of the previous term.

We can check the results of eqs. (29)–(32) in two special cases. First, consider the case of a pure boost, where $\vec{\boldsymbol{\theta}} = 0$. Then $\vec{\boldsymbol{z}} = \vec{\boldsymbol{z}}^* = \vec{\boldsymbol{\zeta}}$ and $\Delta = |\vec{\boldsymbol{\zeta}}| \equiv \zeta$. Plugging these values into eqs. (29)–(32) yields the following block matrix form,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\zeta & \dfrac{\zeta^j}{\zeta}\sinh\zeta \\ \dfrac{\zeta^i}{\zeta}\sinh\zeta & \delta^{ij} + \dfrac{\zeta^i\zeta^j}{\zeta^2}\bigl(\cosh\zeta - 1\bigr) \end{pmatrix}, \tag{33}$$

where the block matrix row and column indices run over $\mu = 0$, $i = 1,2,3$ and $\nu = 0$, $j = 1,2,3$, respectively.

Recalling that $\vec{\boldsymbol{\zeta}} \equiv \hat{\boldsymbol{v}}\tanh^{-1}\beta$ [where $\hat{\boldsymbol{v}} \equiv \vec{\boldsymbol{v}}/|\vec{\boldsymbol{v}}| = \vec{\boldsymbol{\zeta}}/\zeta$ is the unit velocity vector and $\beta \equiv |\vec{\boldsymbol{v}}|/c$], it follows that

$$\Lambda = \begin{pmatrix} \gamma & \gamma\vec{\boldsymbol{\beta}} \\ \gamma\vec{\boldsymbol{\beta}} & \delta^{ij} + (\gamma - 1)\hat{\boldsymbol{v}}^i\hat{\boldsymbol{v}}^j \end{pmatrix}, \tag{34}$$

where $\gamma \equiv (1-\beta^2)^{-1/2} = \cosh\zeta$, $\gamma\beta = \sinh\zeta$, and $\vec{\boldsymbol{\beta}} = \beta\hat{\boldsymbol{v}}$. Eq. (34) is the well-known expression for an *active* Lorentz transformation that is a pure boost.[6]

Second, consider the case of a pure rotation by an angle of $\theta \equiv |\vec{\boldsymbol{\theta}}|$ around an axis that points in the direction of $\hat{\boldsymbol{n}}$ where

$$\hat{\boldsymbol{n}} \equiv \frac{\vec{\boldsymbol{\theta}}}{\theta}\,. \tag{35}$$

In this case, $\vec{\boldsymbol{\zeta}} = 0$, and it follows that $\vec{\boldsymbol{z}} = -\vec{\boldsymbol{z}}^* = i\vec{\boldsymbol{\theta}}$ and $\Delta = i\theta$. Plugging these values into eqs. (29)–(32) yields,

$$\Lambda = \begin{pmatrix} 1 & \vec{\boldsymbol{0}} \\ \vec{\boldsymbol{0}} & R^{ij} \end{pmatrix}, \tag{36}$$

where

$$R^{ij} = \delta^{ij}\cos\theta + \hat{\boldsymbol{n}}^i\hat{\boldsymbol{n}}^j(1-\cos\theta) - \epsilon^{ijk}\hat{\boldsymbol{n}}^k\sin\theta\,. \tag{37}$$

Eq. (37) is the well-known Rodrigues' rotation formula (e.g., see p. 275 of Ref. [5]).

3 An explicit evaluation of exp A, where $A \in \mathfrak{so}(1,3)$

In this section, we shall explicitly evaluate exp A, where A is given by eq. (8). First, we compute the characteristic polynomial of A,

$$p(x) \equiv \det(A - x\boldsymbol{I}_4) = x^4 + \bigl(|\vec{\boldsymbol{\theta}}|^2 - |\vec{\boldsymbol{\zeta}}|^2\bigr)x^2 - (\vec{\boldsymbol{\theta}}\cdot\vec{\boldsymbol{\zeta}})^2 \equiv (x^2+a^2)(x^2-b^2)\,, \tag{38}$$

where

$$a^2b^2 = (\vec{\boldsymbol{\theta}}\cdot\vec{\boldsymbol{\zeta}})^2\,, \qquad a^2 - b^2 = |\vec{\boldsymbol{\theta}}|^2 - |\vec{\boldsymbol{\zeta}}|^2\,. \tag{39}$$

[6]As noted in footnote 3, Jackson employs a passive Lorentz transformation in which the coordinate axes (reference frame) transform(s) while the four-vectors remains fixed, which corresponds to replacing $\vec{\boldsymbol{\beta}}$ with $-\vec{\boldsymbol{\beta}}$ in eq. (34). After making this change, one immediately recovers eq. (11.19) of Ref. [2].

Solving eq. (39) for a^2 and b^2 yields,

$$a^2 = \frac{1}{2}\left[|\vec{\boldsymbol{\theta}}|^2 - |\vec{\boldsymbol{\zeta}}|^2 + \sqrt{\left(|\vec{\boldsymbol{\theta}}|^2 - |\vec{\boldsymbol{\zeta}}|^2\right)^2 + 4(\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}})^2}\,\right], \tag{40}$$

$$b^2 = \frac{1}{2}\left[|\vec{\boldsymbol{\zeta}}|^2 - |\vec{\boldsymbol{\theta}}|^2 + \sqrt{\left(|\vec{\boldsymbol{\theta}}|^2 - |\vec{\boldsymbol{\zeta}}|^2\right)^2 + 4(\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}})^2}\,\right]. \tag{41}$$

The eigenvalues of A, denoted by λ_i ($i = 1, 2, 3, 4$), are the solutions of $p(x) = 0$, which are:

$$\lambda_i = ia, -ia, b, -b\,. \tag{42}$$

To evaluate $\exp A$, we shall use of the following formula of matrix algebra based on the Lagrange interpolating polynomial. If an $n \times n$ matrix A has only distinct eigenvalues λ_i, then any function of A is given by (e.g., see eq. (1.9) of Ref. [6], eq. (5.4.17) of Ref. [7], eqs. (7.3.6) and (7.3.11) of Ref. [8], or Chapter V, Section 2.1 of Ref. [9]):

$$f(A) = \sum_{i=1}^{n} f(\lambda_i) K_i\,, \qquad \text{where} \qquad K_i = \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{A - \lambda_j \boldsymbol{I}_n}{\lambda_i - \lambda_j}\,, \tag{43}$$

and $\boldsymbol{I}_n$ is the $n \times n$ identify matrix. Applying eq. (43) to $f(A) = \exp A$, it follows that

$$\begin{aligned}\exp A = {} & e^{ia}\left(\frac{A + ai\boldsymbol{I}_4}{2ia}\right)\left(\frac{A - b\boldsymbol{I}_4}{ia - b}\right)\left(\frac{A + b\boldsymbol{I}_4}{ia + b}\right) + e^{-ia}\left(\frac{A - ai\boldsymbol{I}_4}{-2ia}\right)\left(\frac{A - b\boldsymbol{I}_4}{-ia - b}\right)\left(\frac{A + b\boldsymbol{I}_4}{-ia + b}\right) \\ & + e^{b}\left(\frac{A - ai\boldsymbol{I}_4}{b - ia}\right)\left(\frac{A + ia\boldsymbol{I}_4}{b + ia}\right)\left(\frac{A + b\boldsymbol{I}_4}{2b}\right) + e^{-b}\left(\frac{A - ai\boldsymbol{I}_4}{-b - ia}\right)\left(\frac{A + ia\boldsymbol{I}_4}{-b + ia}\right)\left(\frac{A - b\boldsymbol{I}_4}{-2b}\right).\end{aligned} \tag{44}$$

Simplifying the above expression yields,

$$\exp A = \frac{1}{a^2 + b^2}\left\{-(A^2 - b^2\boldsymbol{I}_4)\left(A\frac{\sin a}{a} + \boldsymbol{I}_4 \cos a\right) + (A^2 + a^2\boldsymbol{I}_4)\left(A\frac{\sinh b}{b} + \boldsymbol{I}_4 \cosh b\right)\right\}. \tag{45}$$

Combining terms, we end up with

$$\exp\begin{pmatrix} 0 & \zeta^1 & \zeta^2 & \zeta^3 \\ \zeta^1 & 0 & -\theta^3 & \theta^2 \\ \zeta^2 & \theta^3 & 0 & -\theta^1 \\ \zeta^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix} = \frac{1}{a^2 + b^2}\Big\{f_0(a,b)\boldsymbol{I}_4 + f_1(a,b)A + f_2(a,b)A^2 + f_3(a,b)A^3\Big\}, \tag{46}$$

where a and b are defined in eq. (39) and

$$f_0(a,b) = b^2 \cos a + a^2 \cosh b\,, \qquad f_1(a,b) = \frac{b^2}{a}\sin a + \frac{a^2}{b}\sinh b\,, \tag{47}$$

$$f_2(a,b) = \cosh b - \cos a\,, \qquad f_3(a,b) = \frac{\sinh b}{b} - \frac{\sin a}{a}\,. \tag{48}$$

The explicit formula for $\exp A$ has also been given in Refs. [10–12]. Eqs. (46)–(48) coincide precisely with the form obtained by Refs. [11, 12] using other methods.[7]

The matrix A and its powers can be conveniently written in block matrix form,

$$A = \begin{pmatrix} 0 & \zeta^j \\ \zeta^i & -\epsilon^{ijk}\theta^k \end{pmatrix}, \qquad A^2 = \begin{pmatrix} |\vec{\boldsymbol{\zeta}}|^2 & \epsilon^{jk\ell}\zeta^k\theta^\ell \\ -\epsilon^{ik\ell}\zeta^k\theta^\ell & \zeta^i\zeta^j + \theta^i\theta^j - \delta^{ij}|\vec{\boldsymbol{\theta}}|^2 \end{pmatrix}, \tag{49}$$

$$A^3 = \begin{pmatrix} 0 & \big(|\vec{\boldsymbol{\zeta}}|^2 - |\vec{\boldsymbol{\theta}}|^2\big)\zeta^j + (\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}})\theta^j \\ \big(|\vec{\boldsymbol{\zeta}}|^2 - |\vec{\boldsymbol{\theta}}|^2\big)\zeta^i + (\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}})\theta^i & (\epsilon^{jk\ell}\zeta^i - \epsilon^{ik\ell}\zeta^j)\zeta^k\theta^\ell + \epsilon^{ijk}\theta^k|\vec{\boldsymbol{\theta}}|^2 \end{pmatrix}.$$

One can simplify the ij element of A^3 by noting that the ij element of any 3×3 antisymmetric matrix must be of the form $\epsilon^{ijk}C^k$. Thus,

$$(\epsilon^{jk\ell}\zeta^i - \epsilon^{ik\ell}\zeta^j)\zeta^k\theta^\ell = \epsilon^{ijk}C^k\,. \tag{50}$$

Multiplying the above equation by ϵ^{ijm} and summing over i and j yields

$$(\delta^{i\ell}\delta^{km} - \delta^{ik}\delta^{\ell m})\zeta^i\zeta^k\theta^\ell - (\delta^{jk}\delta^{\ell m} - \delta^{j\ell}\delta^{km})\zeta^j\zeta^k\theta^\ell = 2\delta^{km}C^k\,. \tag{51}$$

It follows that,

$$C^m = (\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}})\zeta^m - |\vec{\boldsymbol{\zeta}}|^2\theta^m\,. \tag{52}$$

That is, we have derived the identity,

$$(\epsilon^{jk\ell}\zeta^i - \epsilon^{ik\ell}\zeta^j)\zeta^k\theta^\ell = \epsilon^{ijk}\big[(\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}})\zeta^k - |\vec{\boldsymbol{\zeta}}|^2\theta^k\big]\,. \tag{53}$$

One can therefore rewrite the matrix A^3 in the following form,

$$A^3 = \begin{pmatrix} 0 & \big(|\vec{\boldsymbol{\zeta}}|^2 - |\vec{\boldsymbol{\theta}}|^2\big)\zeta^j + (\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}})\theta^j \\ \big(|\vec{\boldsymbol{\zeta}}|^2 - |\vec{\boldsymbol{\theta}}|^2\big)\zeta^i + (\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}})\theta^i & \epsilon^{ijk}\big[(\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}})\zeta^k - \big(|\vec{\boldsymbol{\zeta}}|^2 - |\vec{\boldsymbol{\theta}}|^2\big)\theta^k\big] \end{pmatrix}. \tag{54}$$

It is instructive to check the two limiting cases treated previously. First, if $\vec{\boldsymbol{\theta}} = \vec{\boldsymbol{0}}$ then $a = 0$ and $b = |\vec{\boldsymbol{\zeta}}| \equiv \zeta$. It then follows that

$$A = \begin{pmatrix} 0 & \zeta^j \\ \zeta^i & \boldsymbol{0}^{ij} \end{pmatrix}, \qquad A^2 = \begin{pmatrix} |\vec{\boldsymbol{\zeta}}|^2 & \vec{\boldsymbol{0}} \\ \vec{\boldsymbol{0}} & \zeta^i\zeta^j \end{pmatrix}, \qquad A^3 = |\vec{\boldsymbol{\zeta}}|^2 A\,, \tag{55}$$

where $\boldsymbol{0}^{ij}$ is a 3×3 matrix of zeros, and

$$\Lambda = \boldsymbol{I}_4 + A + \left(\frac{\cosh\zeta - 1}{|\vec{\boldsymbol{\zeta}}|^{\boldsymbol{2}}}\right)A^2 + \frac{1}{|\vec{\boldsymbol{\zeta}}|^{\boldsymbol{2}}}\left(\frac{\sinh\zeta}{\zeta} - 1\right)A^3\,. \tag{56}$$

Using eq. (55), the above equation reduces to our previous result given in eq. (33).

[7]Since a matrix always satisfies its characteristic equation, we know that $p(A) = 0$. Employing eq. (38), one obtains $A^4 = a^2b^2\boldsymbol{I}_4 - (a^2 - b^2)A^2$. Using $\exp A = \sum_{n=0}^{\infty} A^n/n!$, it follows that $\exp A$ can be expressed as a linear combination of $\boldsymbol{I}_4$, A, A^2 and A^3. Resumming the corresponding coefficients yields eqs. (47) and (48). For further details of this approach, see Appendix 4.7 of Ref. [13].

Second, if $\vec{\boldsymbol{\zeta}} = \vec{\mathbf{0}}$, then $a = |\vec{\boldsymbol{\theta}}| \equiv \theta$ and $b = 0$. It then follows that

$$A = \begin{pmatrix} 0 & \vec{\mathbf{0}} \\ \vec{\mathbf{0}} & -\epsilon^{ijk}\theta^k \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 0 & \vec{\mathbf{0}} \\ \vec{\mathbf{0}} & \theta^i\theta^j - \delta^{ij}|\vec{\boldsymbol{\theta}}|^2 \end{pmatrix}, \qquad A^3 = -|\vec{\boldsymbol{\theta}}|^2 A\,. \tag{57}$$

and

$$\Lambda = \boldsymbol{I}_4 + A + \left(\frac{1-\cos\theta}{|\vec{\boldsymbol{\theta}}|^2}\right) A^2 + \frac{1}{|\vec{\boldsymbol{\theta}}|^2}\left(1 - \frac{\sin\theta}{\theta}\right) A^3\,. \tag{58}$$

Using eq. (57) and defining $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{\theta}}/\theta$, the above equation reduces to Rodrigues' rotation formula given in eqs. (36) and (37).

In the general case, we shall now demonstrate that eq. (46) is equivalent to the previous results obtained in eqs. (29)–(32). First, it is convenient to rewrite eqs. (40) and (41) as follows:

$$a^2 = \tfrac{1}{2}\left(|\vec{\boldsymbol{\theta}}|^2 - |\vec{\boldsymbol{\zeta}}|^2 + |\Delta|^2\right), \qquad b^2 = \tfrac{1}{2}\left(|\vec{\boldsymbol{\zeta}}|^2 - |\vec{\boldsymbol{\theta}}|^2 + |\Delta|^2\right), \tag{59}$$

where Δ is defined in eq. (22). In particular, $a^2 + b^2 = |\Delta|^2$. Thus, eqs. (46), (49) and (54) yield:

$$\begin{aligned} \Lambda^0{}_0 &= \frac{1}{|\Delta|^2}\left[\left(b^2 - |\vec{\boldsymbol{\zeta}}|^2\right)\cos a + \left(a^2 + |\vec{\boldsymbol{\zeta}}|^2\right)\cosh b\right] \\ &= \tfrac{1}{2}(\cosh b + \cos a) + \frac{|\vec{\boldsymbol{\zeta}}|^2 + |\vec{\boldsymbol{\theta}}|^2}{2|\Delta|^2}(\cosh b - \cos a)\,, \end{aligned} \tag{60}$$

after making use of eq. (59). Note that

$$\cosh b + \cos a = \cosh b + \cosh(ia) = 2\cosh\left(\frac{b+ia}{2}\right)\cosh\left(\frac{b-ia}{2}\right) = 2\left|\cosh\left(\frac{b+ia}{2}\right)\right|^2, \tag{61}$$

$$\cosh b - \cos a = \cosh b - \cosh(ia) = 2\sinh\left(\frac{b+ia}{2}\right)\sinh\left(\frac{b-ia}{2}\right) = 2\left|\sinh\left(\frac{b+ia}{2}\right)\right|^2. \tag{62}$$

Hence, eq. (60) yields

$$\Lambda^0{}_0 = \left|\cosh\left(\frac{b+ia}{2}\right)\right|^2 + \frac{|\vec{\boldsymbol{\zeta}}|^2 + |\vec{\boldsymbol{\theta}}|^2}{|\Delta|^2}\left|\sinh\left(\frac{b+ia}{2}\right)\right|^2. \tag{63}$$

Using eq. (39), it follows that

$$(b+ia)^2 = b^2 - a^2 + 2iab = |\vec{\boldsymbol{\zeta}}|^2 - |\vec{\boldsymbol{\theta}}|^2 + 2i\vec{\boldsymbol{\theta}}\cdot\vec{\boldsymbol{\zeta}} = \Delta^2\,, \tag{64}$$

where we have made use of eq. (22) in the final step. Thus, we may put

$$\Delta = b + ia \tag{65}$$

in eq. (63) [since any overall sign will cancel out], which yields

$$\Lambda^0{}_0 = |\cosh\left(\tfrac{1}{2}\Delta\right)|^2 + \left|\frac{\sinh\left(\tfrac{1}{2}\Delta\right)}{\Delta}\right|^2\left(|\vec{\boldsymbol{\zeta}}|^2 + |\vec{\boldsymbol{\theta}}|^2\right), \tag{66}$$

in agreement with eq. (29).

Next, eqs. (46), (49) and (54) yield:

$$\Lambda^i{}_0 = \frac{1}{|\Delta|^2}\biggl\{\left(\frac{b^2}{a}\sin a + \frac{a^2}{b}\sinh b\right)\zeta^i - (\cosh b - \cos a)\epsilon^{ik\ell}\zeta^k\theta^\ell + \left(\frac{\sinh b}{b} - \frac{\sin a}{a}\right)\left[(|\vec{\boldsymbol{\zeta}}|^2 - |\vec{\boldsymbol{\theta}}|^2)\zeta^i + (\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}})\theta^i\right]\biggr\}. \tag{67}$$

In light of eq. (39), it follows that $|\vec{\boldsymbol{\zeta}}|^2 - |\vec{\boldsymbol{\theta}}|^2 = b^2 - a^2$ and $\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}} = ab$. Inserting these results into the above equation, we end up with

$$\Lambda^i{}_0 = \frac{1}{|\Delta|^2}\left[(b\sinh b + a\sin a)\zeta^i + (a\sinh b - b\sin a)\theta^i - (\cosh b - \cos a)\epsilon^{ik\ell}\zeta^k\theta^\ell\right]. \tag{68}$$

We can rewrite this result with the help of some identities. First,

$$\begin{aligned} b\sinh b + a\sin a &= \mathrm{Re}\bigl\{(b-ia)(\sinh b + i\sin a)\bigr\} = \mathrm{Re}\bigl\{(b-ia)[\sinh b + i\sinh(ia)]\bigr\} \\ &= \mathrm{Re}\bigl\{(b-ia)\bigl(\sinh\bigl[\tfrac{1}{2}(b+ia) + \tfrac{1}{2}(b-ia)\bigr] + \sinh\bigl[\tfrac{1}{2}(b+ia) - \tfrac{1}{2}(b-ia)\bigr]\bigr)\bigr\} \\ &= 2\,\mathrm{Re}\bigl\{(b-ia)\sinh\bigl[\tfrac{1}{2}(b+ia)\bigr]\cosh\bigl[\tfrac{1}{2}(b-ia)\bigr]\bigr\} \\ &= (b-ia)\sinh\bigl[\tfrac{1}{2}(b+ia)\bigr]\cosh\bigl[\tfrac{1}{2}(b-ia)\bigr] + \text{c.c.} \\ &= \Delta^*\sinh\bigl(\tfrac{1}{2}\Delta\bigr)\cosh\bigl(\tfrac{1}{2}\Delta^*\bigr) + \text{c.c.}\,, \end{aligned} \tag{69}$$

after using eq. (65). A similar computation yields

$$\begin{aligned} a\sinh b - b\sin a &= -\,\mathrm{Im}\{(b-ia)(\sinh b + i\sin a)\bigr\} \\ &= -2\,\mathrm{Im}\bigl\{(b-ia)\sinh\bigl[\tfrac{1}{2}(b+ia)\bigr]\cosh\bigl[\tfrac{1}{2}(b-ia)\bigr]\bigr\} \\ &= i\Delta^*\sinh\bigl(\tfrac{1}{2}\Delta\bigr)\cosh\bigl(\tfrac{1}{2}\Delta^*\bigr) + \text{c.c.} \end{aligned} \tag{70}$$

Moreover, we can employ eqs. (62) and (65) to obtain

$$(\cosh b - \cos a)\epsilon^{ik\ell}\zeta^k\theta^\ell = 2\left|\sinh\left(\frac{b+ia}{2}\right)\right|^2\epsilon^{ik\ell}\zeta^k\theta^\ell = -i\left|\sinh\bigl(\tfrac{1}{2}\Delta\bigr)\right|\epsilon^{ik\ell}z^{*k}z^\ell\,, \tag{71}$$

after making use of eq. (20). Collecting the results obtained above, we end up with

$$\Lambda^i{}_0 = \left(\frac{\sinh\bigl(\tfrac{1}{2}\Delta\bigr)\cosh\bigl(\tfrac{1}{2}\Delta^*\bigr)}{\Delta}(\zeta^i + i\theta^i) + \text{c.c.}\right) + i\left|\frac{\sinh\bigl(\tfrac{1}{2}\Delta\bigr)}{\Delta}\right|^2\epsilon^{ik\ell}z^{*k}z^\ell\,, \tag{72}$$

in agreement with the result of eq. (30). The computation of $\Lambda^0{}_j$ is nearly identical. The only change is due to the change in the sign multiplying the term proportional to the Levi-Civita tensor. Consequently, it is convenient to replace eq. (71) with

$$(\cosh b - \cos a)\epsilon^{jk\ell}\zeta^k\theta^\ell = 2\left|\sinh\left(\frac{b+ia}{2}\right)\right|^2\epsilon^{jk\ell}\zeta^k\theta^\ell = i\left|\sinh\bigl(\tfrac{1}{2}\Delta\bigr)\right|\epsilon^{jk\ell}z^kz^{*\ell}\,. \tag{73}$$

Hence, we end up with

$$\begin{aligned}\Lambda^0{}_j &= \frac{1}{|\Delta|^2}\Big[(b\sinh b + a\sin a)\zeta^j + (a\sinh b - b\sin a)\theta^j + (\cosh b - \cos a)\epsilon^{jk\ell}\zeta^k\theta^\ell\Big] \\ &= \left(\frac{\sinh\left(\tfrac{1}{2}\Delta\right)\cosh\left(\tfrac{1}{2}\Delta^*\right)}{\Delta}\,(\zeta^j + i\theta^j) + \text{c.c.}\right) + i\left|\frac{\sinh\left(\tfrac{1}{2}\Delta\right)}{\Delta}\right|^2 \epsilon^{jk\ell}z^k z^{*\ell}\,, \qquad (74)\end{aligned}$$

in agreement with the result of eq. (31).

Finally, we use eqs. (46), (49) and (54) to obtain:

$$\begin{aligned}\Lambda^i{}_j = \frac{1}{|\Delta|^2}\Bigg\{&(b^2\cos a + a^2\cosh b)\delta^{ij} - \left(\frac{b^2}{a}\sin a + \frac{a^2}{b}\sinh b\right)\epsilon^{ijk}\theta^k \\ &+(\cosh b - \cos a)(\zeta^i\zeta^j + \theta^i\theta^j - \delta^{ij}|\vec{\boldsymbol{\theta}}|^2) \\ &+\left(\frac{\sinh b}{b} - \frac{\sin a}{a}\right)\Big[\epsilon^{ijk}\big[(\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}})\zeta^k + \big(|\vec{\boldsymbol{\theta}}|^2 - |\vec{\boldsymbol{\zeta}}|^2\big)\theta^k\big]\Big]\Bigg\}\,. \qquad (75)\end{aligned}$$

First, we examine the terms in eq. (75) that are proportional to δ^{ij}. The following identity is noteworthy:

$$b^2\cos a + a^2\cosh b - (\cosh b - \cos a)|\vec{\boldsymbol{\theta}}|^2 = \tfrac{1}{2}(a^2+b^2)(\cos a + \cosh b) - \tfrac{1}{2}(\cosh b - \cos a)\big(b^2 - a^2 + 2|\vec{\boldsymbol{\theta}}|^2\big)\,. \qquad (76)$$

Using eq. (39), $b^2 - a^2 + 2|\vec{\boldsymbol{\theta}}|^2 = |\vec{\boldsymbol{\zeta}}|^2 + |\vec{\boldsymbol{\theta}}|^2$. Applying eqs. (61), (62) and (65) yields the coefficient of δ_{ij},

$$\frac{1}{|\Delta|^2}\Big[b^2\cos a + a^2\cosh b - (\cosh b - \cos a)|\vec{\boldsymbol{\theta}}|^2\Big] = |\cosh\left(\tfrac{1}{2}\Delta\right)|^2 - \left|\frac{\sinh\left(\tfrac{1}{2}\Delta\right)}{\Delta}\right|^2\big(|\vec{\boldsymbol{\zeta}}|^2 + |\vec{\boldsymbol{\theta}}|^2\big)\,. \qquad (77)$$

Next, we note that in light of eq. (20), $z^{*i}z^j + z^i z^{*j} = 2(\zeta^i\zeta^j + \theta^i\theta^j)$. Thus,

$$\frac{1}{|\Delta|^2}(\cosh b - \cos a)(\zeta^i\zeta^j + \theta^i\theta^j) = \big(z^{*i}z^j + z^i z^{*j}\big)\left|\frac{\sinh\left(\tfrac{1}{2}\Delta\right)}{\Delta}\right|^2\,. \qquad (78)$$

We now examine the terms in eq. (75) that are proportional to $\epsilon^{ijk}\theta^k$. In light of eq. (39),

$$\begin{aligned}&\frac{1}{|\Delta|^2}\left\{\left(\frac{\sinh b}{b} - \frac{\sin a}{a}\right)\big(|\vec{\boldsymbol{\theta}}|^2 - |\vec{\boldsymbol{\zeta}}|^2\big) - \left(\frac{b^2}{a}\sin a + \frac{a^2}{b}\sinh b\right)\right\} \\ &\qquad = \frac{1}{|\Delta|^2}\left\{(a^2 - b^2)\left(\frac{\sinh b}{b} - \frac{\sin a}{a}\right) - \left(\frac{b^2}{a}\sin a + \frac{a^2}{b}\sinh b\right)\right\} \\ &\qquad = -\frac{1}{|\Delta|^2}\big[b\sinh b + a\sin a\big] = -\left\{\frac{\sinh\left(\tfrac{1}{2}\Delta\right)}{\Delta}\cosh\left(\tfrac{1}{2}\Delta^*\right) + \text{c.c.}\right\}, \qquad (79)\end{aligned}$$

after employing eq. (69). Finally, we examine the terms in eq. (75) that are proportional to $\epsilon^{ijk}\zeta^k$. In light of eq. (39),

$$\frac{1}{|\Delta|^2}\left(\frac{\sinh b}{b}-\frac{\sin a}{a}\right)\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\zeta}}=\frac{ab}{|\Delta|^2}\left(\frac{\sinh b}{b}-\frac{\sin a}{a}\right)=\frac{1}{|\Delta|^2}\big[a\sinh b-b\sin a\big]$$
$$=i\,\frac{\sinh\left(\tfrac{1}{2}\Delta\right)}{\Delta}\cosh\left(\tfrac{1}{2}\Delta^*\right)+\text{c.c.} \tag{80}$$

Note that the terms proportional to ϵ^{ijk} combine nicely and yield,

$$\frac{i\sinh\left(\tfrac{1}{2}\Delta\right)\cosh\left(\tfrac{1}{2}\Delta^*\right)}{\Delta}\,\epsilon^{ijk}z^k+\text{c.c.}\,, \tag{81}$$

after using eq. (20).

Collecting the results of eqs. (77), (78) and (81), we end up with

$$\Lambda^i{}_j=\left\{|\cosh\left(\tfrac{1}{2}\Delta\right)|^2-\left|\frac{\sinh\left(\tfrac{1}{2}\Delta\right)}{\Delta}\right|^2\left(|\vec{\boldsymbol{\zeta}}|^2+|\vec{\boldsymbol{\theta}}|^2\right)\right\}\delta^{ij}+(z^{*i}z^j+z^iz^{*j})\left|\frac{\sinh\left(\tfrac{1}{2}\Delta\right)}{\Delta}\right|^2$$
$$+\left(\frac{i\sinh\left(\tfrac{1}{2}\Delta\right)\cosh\left(\tfrac{1}{2}\Delta^*\right)}{\Delta}\epsilon^{ijk}z^k+\text{c.c.}\right)\,, \tag{82}$$

in agreement with eq. (32).

We have therefore verified by an explicit computation that

$$\Lambda^\mu{}_\nu=\exp\begin{pmatrix}0 & \zeta^1 & \zeta^2 & \zeta^3\\ \zeta^1 & 0 & -\theta^3 & \theta^2\\ \zeta^2 & \theta^3 & 0 & -\theta^1\\ \zeta^3 & -\theta^2 & \theta^1 & 0\end{pmatrix}=\tfrac{1}{2}\operatorname{Tr}\left(M^\dagger\overline{\sigma}^\mu M\sigma_\nu\right)\,, \tag{83}$$

where $M=\exp\left\{-\tfrac{1}{2}(\vec{\boldsymbol{\zeta}}+i\vec{\boldsymbol{\theta}})\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right\}$.

4 A proof using infinitesimal Lorentz transformations

Most textbooks proofs of eq. (83) demonstrate that both sides of eq. (83) agree to first order in $\vec{\boldsymbol{\zeta}}$ and $\vec{\boldsymbol{\theta}}$. It is convenient to introduce the six independent 2×2 matrices $\overline{\sigma}^{\mu\nu}=-\overline{\sigma}^{\nu\mu}$, where

$$\overline{\sigma}^{\mu\nu}=\tfrac{1}{4}i(\overline{\sigma}^\mu\sigma^\nu-\overline{\sigma}^\nu\sigma^\mu)\,, \tag{84}$$

which differ from the $\sigma^{\mu\nu}$ matrices introduced in eq. (13).[8]

It then follows that

$$M^\dagger=\exp\left(\tfrac{1}{2}i\theta_{\rho\lambda}\overline{\sigma}^{\rho\lambda}\right)=\exp\left(\tfrac{1}{2}i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}-\tfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right)\,. \tag{85}$$

[8]The six independent $i\sigma^{\mu\nu}$ matrices are generators in the $(\frac{1}{2},0)$ representation of $\mathfrak{sl}(2,\mathbb{C})$ as noted below eq. (14), whereas the six independent $i\overline{\sigma}^{\mu\nu}$ matrices are generators in the $(0,\frac{1}{2})$ representation of $\mathfrak{sl}(2,\mathbb{C})$ [cf. footnote 4].

Working to first order in the parameters $\theta^{\mu\nu}$ and making use of eqs. (7), (14) and (85)

$$\Lambda^\mu{}_\nu \simeq \delta^\mu_\nu + \tfrac{1}{2}\left(\theta_{\lambda\nu}g^{\lambda\mu} - \theta_{\nu\rho}g^{\rho\mu}\right)\,, \tag{86}$$

$$M \simeq \boldsymbol{I}_2 - \tfrac{1}{2}i\theta_{\rho\lambda}\sigma^{\rho\lambda}\,, \tag{87}$$

$$M^\dagger \simeq \boldsymbol{I}_2 + \tfrac{1}{2}i\theta_{\rho\lambda}\overline{\sigma}^{\rho\lambda}\,. \tag{88}$$

It follows that

$$M^\dagger\overline{\sigma}^\mu M \simeq \big(\boldsymbol{I}_2 + \tfrac{1}{2}i\theta_{\rho\lambda}\overline{\sigma}^{\rho\lambda}\big)\overline{\sigma}^\mu\big(\boldsymbol{I}_2 - \tfrac{1}{2}i\theta_{\rho\lambda}\sigma^{\rho\lambda}\big) \simeq \overline{\sigma}^\mu + \tfrac{1}{2}i\theta_{\rho\lambda}\big(\overline{\sigma}^{\rho\lambda}\overline{\sigma}^\mu - \overline{\sigma}^\mu\sigma^{\rho\lambda}\big)\,. \tag{89}$$

One can easily derive the following identity [1],

$$\overline{\sigma}^{\rho\lambda}\overline{\sigma}^\mu - \overline{\sigma}^\mu\sigma^{\rho\lambda} = i\big(g^{\lambda\mu}\overline{\sigma}^\rho - g^{\rho\mu}\overline{\sigma}^\lambda\big)\,. \tag{90}$$

Hence eq. (89) yields,

$$\begin{aligned} M^\dagger\overline{\sigma}^\mu M &\simeq \overline{\sigma}^\mu - \tfrac{1}{2}\theta_{\rho\lambda}\big(g^{\lambda\mu}\overline{\sigma}^\rho - g^{\rho\mu}\overline{\sigma}^\lambda\big) \simeq \big[\delta^\mu_\nu - \tfrac{1}{2}\theta_{\rho\lambda}\big(g^{\lambda\mu}\delta^\rho_\nu - g^{\rho\mu}\delta^\lambda_\nu\big)\big]\overline{\sigma}^\nu \\ &\simeq \big[\delta^\mu_\nu - \tfrac{1}{2}\big(\theta_{\nu\lambda}g^{\lambda\mu} - \theta_{\rho\nu}g^{\rho\mu}\big)\big]\overline{\sigma}^\nu \simeq \big[\delta^\mu_\nu + \tfrac{1}{2}\big(\theta_{\lambda\nu}g^{\lambda\mu} - \theta_{\nu\rho}g^{\rho\mu}\big)\big]\overline{\sigma}^\nu\,, \end{aligned} \tag{91}$$

after using the antisymmetry of $\theta_{\nu\lambda}$ in the final step. After employing eq. (86) on the right hand side of eq. (91), we arrive at[9]

$$M^\dagger\overline{\sigma}^\mu M = \Lambda^\mu{}_\nu\overline{\sigma}^\nu\,, \tag{92}$$

thereby confirming the result of eq. (18) to first order in $\theta_{\rho\lambda}$.

Of course, the derivation of eq. (92) is much simpler than the explicit proof of eq. (83), which requires the exact evaluation of all the relevant matrix exponentials. However, we can now assert that having derived eq. (92) to first order in $\theta_{\rho\lambda}$, this result must be true for arbitrary $\theta_{\rho\lambda}$. The reason that a derivation based on the infinitesimal forms of Λ, M and $M^\dagger$ is sufficient is due to the strong constraints imposed by the group multiplication law of the Lorentz group near the identity element, which implies via eq. (5) that a proper orthochronous Lorentz transformation can be expressed as an exponential of an element of the corresponding Lie algebra.

Having derived eqs. (29)–(32), it is quite simple to demonstrate that eq. (83) is true without an explicit computation of the exponential of the 4×4 matrix A, in light of the comments above. First, we expand Λ to linear order in the boost and rotation parameters,

$$\Lambda^\mu{}_\nu = \exp\begin{pmatrix} 0 & \zeta^1 & \zeta^2 & \zeta^3 \\ \zeta^1 & 0 & -\theta^3 & \theta^2 \\ \zeta^2 & \theta^3 & 1 & -\theta^1 \\ \zeta^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix} \simeq \begin{pmatrix} 1 & \zeta^1 & \zeta^2 & \zeta^3 \\ \zeta^1 & 1 & -\theta^3 & \theta^2 \\ \zeta^2 & \theta^3 & 0 & -\theta^1 \\ \zeta^3 & -\theta^2 & \theta^1 & 1 \end{pmatrix}. \tag{93}$$

This is to be compared with the evaluation of eqs. (29)–(32) to linear order in $\vec{\boldsymbol{\zeta}}$ and $\vec{\boldsymbol{\theta}}$, where $\Delta \simeq 0$ in light of eq. (22). The end result is

$$\Lambda^0{}_0 \simeq 1\,, \tag{94}$$

$$\Lambda^i{}_0 \simeq \zeta^i\,, \tag{95}$$

$$\Lambda^0{}_j \simeq \zeta^j\,, \tag{96}$$

$$\Lambda^i{}_j \simeq \delta^{ij} - \epsilon^{ijk}\theta^k\,, \tag{97}$$

which coincides with the right hand side of eq. (93).

[9] Eq. (92) is a statement of the well-known isomorphism SO(1,3) $\cong$ SL(2,$\mathbb{C}$)/$\mathbb{Z}_2$, since the SL(2,$\mathbb{C}$) matrices M and $-M$ correspond to the same Lorentz transformation Λ.

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Appendix A A four-component spinor product that transforms as a four vector

One can construct four-component spinors [1],

$$\Psi \equiv \begin{pmatrix} \chi \\ \eta^\dagger \end{pmatrix} , \tag{A.1}$$

in terms of a pair of two-component spinors χ and η. Gamma matrices can be expressed in the chiral representation in terms of σ^μ and $\overline{\sigma}^\mu$,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} . \tag{A.2}$$

It is convenient to introduce

$$\tfrac{1}{2}\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix} , \tag{A.3}$$

where $[\gamma^\mu, \gamma^\nu] \equiv \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$. The Dirac adjoint spinor is defined by

$$\overline{\Psi}(x) \equiv \Psi^\dagger(x)\gamma^0 = \begin{pmatrix} \eta & \chi^\dagger \end{pmatrix} . \tag{A.4}$$

The matrix γ^0 satisfies

$$\gamma^0\gamma^\mu(\gamma^0)^{-1} = (\gamma^\mu)^\dagger , \tag{A.5}$$

$$\gamma^0\Sigma^{\mu\nu}(\gamma^0)^{-1} = (\Sigma^{\mu\nu})^\dagger . \tag{A.6}$$

Four-component spinors transform under an active Lorentz transformation in the $(\frac{1}{2}, 0)\oplus(0, \frac{1}{2})$ representation of the Lorentz group,

$$\Psi' = \mathbb{M}\Psi , \tag{A.7}$$

where

$$\mathbb{M} = \begin{pmatrix} M & 0 \\ 0 & (M^{-1})^\dagger \end{pmatrix} = \exp\left(-\tfrac{1}{4}i\theta_{\mu\nu}\Sigma^{\mu\nu}\right) , \tag{A.8}$$

and

$$M = \exp\left(-\tfrac{1}{2}i\theta_{\rho\lambda}\sigma^{\rho\lambda}\right) = \exp\left(-\tfrac{1}{2}i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} - \tfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) , \tag{A.9}$$

$$(M^{-1})^\dagger = \exp\left(-\tfrac{1}{2}i\theta_{\rho\lambda}\overline{\sigma}^{\rho\lambda}\right) = \exp\left(-\tfrac{1}{2}i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} + \tfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) . \tag{A.10}$$

To compute matrix inverses, simply change the overall sign of the parameters $\theta_{\mu\nu}$. For example,

$$\mathbb{M}^{-1} = \exp\left(\tfrac{1}{4}i\theta_{\mu\nu}\Sigma^{\mu\nu}\right) . \tag{A.11}$$

Note that in light of eq. (A.6),

$$\gamma^0 \mathbb{M}(\gamma^0)^{-1} = (\mathbb{M}^{-1})^\dagger \,. \tag{A.12}$$

Using eqs. (A.4) and (A.7), it then follows that

$$\overline{\Psi}' = \Psi'^\dagger \gamma^0 = \Psi^\dagger \mathbb{M}^\dagger \gamma^0 = \overline{\Psi}(\gamma^0)^{-1}\mathbb{M}^\dagger \gamma^0 \,. \tag{A.13}$$

Finally, taking the hermitian conjugate of eq. (A.12) and using eq. (A.5) [which implies that $(\gamma^0)^\dagger = \gamma^0$], we end up with

$$\overline{\Psi}' = \overline{\Psi}\mathbb{M}^{-1} \,, \tag{A.14}$$

under an active Lorentz transformation.

It immediately follows from eqs. (A.7) and (A.14) that $\overline{\Psi}'\Psi' = \overline{\Psi}\Psi$, which we recognize as a Lorentz scalar. Next, consider the following two identities,

$$M^\dagger \overline{\sigma}^\mu M = \Lambda^\mu{}_\nu \, \overline{\sigma}^\nu \,, \tag{A.15}$$

$$M^{-1}\sigma^\mu (M^{-1})^\dagger = \Lambda^\mu{}_\nu \, \sigma^\nu \,. \tag{A.16}$$

Eq. (A.15) has already been established in these notes. Eq. (A.16) implies that

$$\Lambda^\mu{}_\nu = \tfrac{1}{2}\,\mathrm{Tr}\big[M^{-1}\sigma^\mu (M^{-1})^\dagger \overline{\sigma}_\nu\big] \,, \tag{A.17}$$

which yields

$$\Lambda^0{}_0 = \tfrac{1}{2}\,\mathrm{Tr}\big[M^{-1}(M^{-1})^\dagger\big] \,, \qquad \Lambda^i{}_0 = \tfrac{1}{2}\,\mathrm{Tr}\big[M^{-1}\sigma^i (M^{-1})^\dagger\big] \,, \tag{A.18}$$

$$\Lambda^0{}_i = \tfrac{1}{2}\,\mathrm{Tr}\big[(M^{-1})^\dagger \sigma^i M^{-1}\big] \,, \qquad \Lambda^i{}_j = \tfrac{1}{2}\,\mathrm{Tr}\big[M^{-1}\sigma^i (M^{-1})^\dagger \sigma^j\big] \,. \tag{A.19}$$

Comparing with the computation of Section 2, we see that $M \to (M^{-1})^\dagger$ and $M^\dagger \to M^{-1}$, which results in $\vec{\boldsymbol{\theta}} \to \vec{\boldsymbol{\theta}}$ and $\vec{\boldsymbol{\zeta}} \to -\vec{\boldsymbol{\zeta}}$. Hence, it follows that $\vec{\boldsymbol{z}} \to -\vec{\boldsymbol{z}}^*$ and $\Delta \to \Delta^*$. Under these replacements, the expressions for $\Lambda^\mu{}_\nu$ obtained in eqs. (29)–(32) are unchanged. Hence, eq. (A.16) is confirmed.

One can also check the validity of eq. (A.16) using the method outlined in Section 4, by using the first order expressions,

$$\Lambda^\mu{}_\nu \simeq \delta^\mu_\nu + \tfrac{1}{2}\left(\theta_{\lambda\nu} g^{\lambda\mu} - \theta_{\nu\rho} g^{\rho\mu}\right) \,, \tag{A.20}$$

$$(M^{-1})^\dagger \simeq \boldsymbol{I}_2 - \tfrac{1}{2} i\theta_{\rho\lambda}\overline{\sigma}^{\rho\lambda} \,, \tag{A.21}$$

$$M^{-1} \simeq \boldsymbol{I}_2 + \tfrac{1}{2} i\theta_{\rho\lambda}\sigma^{\rho\lambda} \,. \tag{A.22}$$

Using eqs. (A.8), (A.15) and (A.16), it then follows that

$$\mathbb{M}^{-1}\gamma^\mu \mathbb{M} = \Lambda^\mu{}_\nu \gamma^\nu \,. \tag{A.23}$$

Consequently, in light of eqs. (A.7), (A.14) and (A.23), it follows that under an active Lorentz transformation,

$$\overline{\Psi}\gamma^\mu\Psi \longrightarrow \overline{\Psi}\mathbb{M}^{-1}\gamma^\mu \mathbb{M}\Psi = \Lambda^\mu{}_\nu \overline{\Psi}\gamma^\nu\Psi \,. \tag{A.24}$$

That is, under an active Lorentz transformation, $\overline{\Psi}\gamma^\mu\Psi$ transforms as a four vector.

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